PHILOSOPHICAL LOGIC

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PREFACE

This special issue of Studies in Logic, Grammar and Rhetoric is devoted to new ideas in philosophical logic. The purpose of this special issue is to concentrate on various interrelated aspects of non-classical logics: philosophical, logical and algebraic among them. Modern philosophical logic sets up the different problems trying to

- consider a new logical solution of general philosophical problems like problems about analyticity, predication, entailment, and validity;
- theoretically explicate the notion of a truth value, analyze the relation between meaning and truth-conditions, and examine syntactic and semantic properties of many-valued logics formalized in axiomatic, sequent, tableau etc. style;
- investigate topical problems of fuzzy logics and consider possible applications of many-valued and fuzzy logics in various areas of computer science interest;
- examine syntactic and semantic properties of non-classical logics, including different versions of many-valued logics, fuzzy logics, supervaluationist logics, paraconsistent logics, intuitionistic logic, etc.;
- propose higher-order formalization of informal mathematical reasoning and consider syntactic and semantic properties of higher-order logic;
- investigate the logic of higher-order vagueness and explicate the notion of a higher-order fuzzy class;
- propose different approaches to non-well-founded vagueness and examine a non-well-founded proof theory.

Gabbay and Guenthner's 'Handbook of Philosophical Logic' is the most authoritative source dedicated to various tendencies in philosophical logic. This issue does not have pretensions to examine all the existing tendencies. The aim of this issue is to survey some novel ideas in philosophical logic, in particular in many-valued logics (Arnon Avron, Vitaly I. Levin, Vladimir A. Moshchenskii), in the p-adic case of non-well-founded probability

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theory (Andrei Khrennikov), in intuitionistic logic (Alexander Lyaletski), and in logical methods of recognition (Arkady D. Zakrevskij). This issue also contains an analysis of general problems of philosophical logic (Martin Tabakov, Alexander S. Karpenko, Andrew Schumann).

I would like to thank Prof. Kazimierz Trzęsicki and the other colleagues working at the Department of Logic, Informatics, and Science Philosophy, Białystok University for their substantial help in preparing this publication.

Andrew Schumann

Martin Tabakov

PHILOSOPHICAL LOGIC AS PHILOSOPHY OF LOGIC

The paper examines questions which are essential for philosophical interpretation of the notion "philosophical logic" (PhL): How is PhL possible and relevant? What is PhL called? For what scientific and practical reasons did the term 'PhL' become popular? The common usage of the term 'PhL' and the common objections against it. Is PhL a kind of logic? What are the relations between PhL, logic and philosophy? We consider four meaningful interpretations of the term 'PhL': PhL as a collection of logical systems in connection with philosophy; 'PhL' as logic in (of) philosophy; 'PhL' as a philosophy of logic, the latter is the main goal of the second part of this paper. The development of logic is evaluated in respect to the popular conceptions of philosophy of science (Kuhn's concept of scientific revolutions and Lacatos' concept of proliferation connected to the problem of monism and pluralism in logic). We also survey two different revolutions in modern logic: a transition from traditional to classical logic and a transition from classical to non-classical logic. The reason for both revolutions was that the development of practical applications of logic has gone ahead in relation to logical theory. We propose the idea of "logical neofundamentalism", concerning the problem of the universality of logic and classifications of logical systems.

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1. On the term 'philosophical logic'

First of all, I will discuss the following questions: Why, for what reasons did the term 'philosophical logic' (PhL) appear and become popular? How is PhL possible and relevant? What is PhL called? I will analyze the common usage of the term PhL, the common objection against its usage, problems posed by using the term PhL. And finally I will explain my conception what PhL could be.

There are two main reasons the term PhL to be used: theoretical and practical. The main question appeared is "Is there a significant field of study where there is no suitable term?" From this it follows the next main

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questions: "Where is the field of study named PhL in the philosophical map?" and "Is PhL either a kind of logic or pure philosophy?"

The introduction of a new term is possible only in the context of a newly developed and significant problem, which is not designated in a particular way. Is there such a significant problem field in logic of the 20th century and which is it? New philosophical problematics was developed because of the paradoxes in set theory and the limitative theorems (Tarski, Gödel). They necessitated the elaboration of new philosophical and conceptual investigation into the methods, nature and subject of mathematics and logic and the broad epistemological topics connected to them. But the really new research field in logic was non-classical logic. The non-classical logic and especially modal logic became a central topic of logical research in the mid-20th century and by the same token was considered as highly significant for philosophy.

In my opinion, the main arguments for use the term PhL are every-day-life reasons! Approving such a term is convenient and useful for a large group of scientists for their works and careers.

*P*1 People with a traditional, classical education without sufficient knowledge of modern logic (those who do not know well enough current logical researches), but working in the field of logic. Philosophers who have good knowledge in some other fields (ontology, epistemology, philosophy of science, history of philosophy), but for different reasons find job as logic lecturers.

*P*2 Scientists with good skills in formal (mathematical) methods, frequently having a mathematical education and working in the field of non-classical logics who find job as logic lecturers in philosophical departments.

The mathematicians often identify the concepts of 'logic' and 'logic of mathematics'. However, many non-classical logics are not reduced to logic of mathematics, they (e.g. with the exception of intuitionistic logic) cannot play the role of the basis of mathematical theory (it is the reason why mathematicians were not interested in non-classical logics for a long time). In mathematics, there is no contextual ambiguity and modality! This is the reason why such notions are not interesting for the mathematicians. Moreover, this led to the employment of logicians interested in such problematics in philosophical faculties.

Publicity PhL sounds impressive and hardly provoking objection. It helps to earn money from machinery of government, foundation, deans, rectors and other university managers. The term PhL is attractive also for students.

The *main objections* against usage of the term PhL are as follows: Which problems do belong to logic but not to PhL? Are papers of Aristotle (Frege, Hilbert, Kripke, ...) PhL? What are relations between PhL, logic and philosophy? Is PhL a logic? By the way, if PhL is philosophy but not logic, it will conflict with people ($P1$) who want to be logical lecturers. PhL is unnecessary, logic is enough in any case. In my opinion, it is reasonable. In the almost all-common usage of the term PhL without any problems we can use 'logic'!

Let us consider *different uses of 'PhL'*

- PhL is traditional logic,
- PhL is what "philosophical logicians" are studying (a vicious circle),
- PhL is metaphysics,
- PhL is philosophy of language,
- PhL is non-classical logic (e.g. modal logic),
- PhL is general semantics for different logical systems, semantical foundations of logic,
- PhL is metalogic,
- PhL is philosophy of mathematics,
- PhL is philosophy of logic,
- PhL is a philosophical examination of systems of formal logic,
- PhL is a part of philosophical elucidation of those notions that are indispensable for the proper characterization of rational thought and its contents,
- PhL is a discussion of the undefinable,
- PhL is an analysis of the implicit presuppositions,
- PhL is an ontology of the logical forms and the objects of logic (their essence and nature),
- PhL is an area where the logical concepts and problems are discussed and analyzed in a phenomenological, hermeneutical or Hegelian style, rather than by the methods of analytical philosophy. Or more generally: philosophical studies in which questions and concepts of logic are discussed, but which do not comply with the established criteria for logical investigations. And in particular they don't use formal (mathematical) methods. But actually, Aristotle used formal methods, so we can claim that its work are not PhL. Mathematical methods allow explicating also essential logical moments of the extra-mathematical logical 'empirics'. Besides, in the most contemporary studies in logic, there coexist both the content-based reasoning and the formal methods! It is started with qualitative reasoning which justify the approach, then a logical system is defined and analyzed by

formal methods, but the results are analyzed at the informal level. Are they PhL?

Actually, there are so many 'PhLs' as 'philosophical logicians'! So, is there any sense to introduce a new term if this concept proliferates?

I accept the term 'PhL', when it is motivated by practical goals. But at the theoretical level, the introduction of a special concept of 'PhL' leads to the unnecessary augmentation of essences (Ockam's razor). But as long as the term 'PhL' is quite often used, I shall expose my view on the possible reasonable use.

If logic is a part of philosophy so why we must restrict logic by the more general concept 'philosophy' (PhL looks like 'mathematical algebra'). In general there are two types of interpretation for 'logic': (i) restricted, as study of logical inference, argumentation, and the formal structure of reasoning, (ii) extended, as study including problems from other divisions of philosophy (ontology, epistemology, dialectics). The second interpretation makes inevitable the emergence of self-reflexivity and tautologies – PhL looks like 'philosophical philosophy'. This means that we have to drop the convenient term 'PhL'! The moral is: we may choose what conception of logic we want, but if we want to keep the term 'PhL', we would have to stick to the restricted interpretation of logic. And if we want to keep the extended conception of logic, we are forced to dey the term 'PhL'.

The introduction of the term 'PhL' presupposes the elaboration of a complete and consistent conception about its nature, topic and subject. In this context, I can consider the following four meaningful interpretations of the term 'PhL':

1. *PhL as (a type of) logic and studding logical systems connected to philosophy. Especially, PhL as logic investigating non-mathematical reasoning.* It encompasses those thematic fields and divisions of contemporary logic in which the treatment of inference and the formal structure of reasoning is connected to philosophical problems and specific logical systems that are constructed and founded in this way. In this view, PhL is a collection of logical systems related to philosophical problems. By the way, it becomes proliferated and gives rise to many mutually independent "PhLs". When they are interpreted as "divisions of logic", we assume the informal idea and its different formal explications. For example, using the term 'many-valued logics' we designate both the idea that there is relativity and grades of truth-values and the different formal explications of this idea as presented by Łukasiewicz, Post, etc.

But what kind of logical systems can be called 'PhLs'? Evidently that not all logics are philosophical, otherwise the term should be empty. Regrettably, Gabbay and Guenthner's famous 'Handbook of Philosophical Logic' does not solve the question which logical systems must not be treated as PhLs and encompasses all of the well-studied logical systems.

A fundamental criterion for the appellation of the term 'PhL' is a connection with philosophical problematics, the logical systems that treat specific problems should be excluded from consideration in this setting. Logical systems which aim is to provide a basis for the formal treatment of mathematics have not to be considered as PhL, too. They are an explication of logic that is used in the deductive sciences and expresses the deductive reasoning in them. In this context, the logical consequence is considered as some utterly abstract theories and as well as formal languages and presupposes as strong abstractions used primarily in mathematics. When philosophy treats too much specific topics it looses its characteristics, therefore I exclude the applied systems as deontic logic and logic of norms from the class of the PhLs. The same applies to the class of logical systems that are pure mathematical models and as well as to logical systems used as basis for mathematical theory and to systems created without philosophical justification. They are interesting at least because they allow us to set analogies with other systems interesting in philosophy. And also applied logic that does not claim to explicate logic in any general philosophical, epistemological or ontological sense, but its aim only to express logical relations involved in a particular field far from philosophy.

But the creation of many-valued, modal, intuitionistic logics is related to philosophical postulations! Logical systems that are attempts for the explication of logical inference (in philosophy) are PhL; logical systems expressing epistemological aspects and logical systems related to ontological aspects, too.

2. *'PhL' is a logic in (of) philosophy and explores the rules of the logical inference, the modes of deduction from and in philosophy.* In this sense, PhL is a study of logical consequence and argumentation in philosophy, and particularly the logical problems in epistemology, dialectics, the analysis and explication of the logical structure of specific philosophical theory or classes of such theories. As far as mathematical logic is "mathematical theory of the logical reasoning patterns that mathematicians use to prove their theorems" (Curry), PhL would be (by the same token) an analysis of those reasoning patterns that philosophers use to establish their conclusions. Then 'mathematical logic' (classical logic) is 'logic of mathematics' and 'PhL' (some non-classical logics) is 'logic of philosophy' and therefore both are different. PhL analyzes the forms of reasoning that are not mathematical. This, if PhL is confined to classical logic, would be lo-

gic of mathematics or philosophical analysis of logic of mathematics – too humble task for such a grandiose term. It is an explanation why PhL is to be non-classical logic but not a particular member of the family of non-classical logics. For instance, material implication does not show the content-relation between the two related judgments and therefore does not capture the logical consequence in philosophy. The philosophical reasoning is relevant, therefore the logical systems that embody a better explication of logical consequence have the right to be claimed using the title 'PhL'. The classical logic is founded on the principle that any two statements can be combined, no matter what are their respective content, which is unacceptable if our aim is an explication of logical consequence outside mathematics. In this way, mathematical logic studies material implication and PhL relevant implication. The same is applied to the logical systems that express different epistemological situations: modal, many-valued, paraconsistent, temporal and fuzzy logics, i.e. the relativity of truth is focal points in philosophy but cannot be coordinated with the bivalence principle. The notions of 'necessity' and 'possibility' are the basic notions of philosophy. From this point of view, modal logics are closer to PhL. Other kinds of logic namely temporal, many-valued logics with their descriptions are closely connected to PhL, too. One of the aims of logical research is to find the words whose substitution is invariant for logical inference (logical terms). The logical terms of PhL are studied in non-classical logics and can be treated as philosophical terms. The language of philosophy is closer to the language of non-classical logic than to that of classical logic.

In the wide sense, 'PhL' is not logic, but an interdisciplinary field between logic and other philosophical areas: ontology, epistemology, philosophy of science, history of philosophy.

3. *'PhL' is a philosophy in logic*, a look at contemporary logic from the point of view of some well-known philosophical schools. For example, how can we look at contemporary logic from the point of view of Hegel or Kant? How do dialectics emerge in logic? What are merits of platonism, agnosticism, realism and nominalism, sensualism and rationalism, pragmatism, hermeneutics and existentialism in logic? PhL as philosophy **in** logic must analyze the mutual interweaving of logical and philosophical ideas. It considers the relationship between the results of modern logic and some basic philosophical categories and conceptions. And it tries also to find tangent points and common studies with established philosophical approaches, in which the word 'logic' is used, therefore some questions appear in relation to logic.

4. *'PhL' is a philosophy of logic*, i.e. a part of the general philosophy of science, similar to the philosophy of physics, of biology, of mathematics. In this view, PhL is an analysis of the development of contemporary logic from the point of view of philosophy of science. It is an analysis of the most important logical results from the standpoint of methodology and discussing questions that are essential for the philosophical interpretation of modern logic. Questions that are putted to philosophy by the development of modern logic, concerning the limits of the logical, the unity and the universality of logic, about the demarcation, etc. Answering these questions is the main goal of this paper.

2. Philosophical logic as philosophy of logic

The development of logic could be evaluated from the point of view of the popular conceptions of philosophy of science, i.e. Kuhn's conception of scientific revolutions and Lacatos' conception of principles of proliferation. Were there revolutions in logic? In my opinion, there were two different revolutions in modern logic: the transition from traditional to classical logic and the transition from classical to non-classical logic. The old tradition, after a considerable amount of time, has been replaced twice by a new tradition having its own language and based on a new conception of study of logic. The new paradigm has almost completely replaced twice the old one. The reason for both revolutions was that the development of the logical empirical sphere has gone considerably ahead in relation to logical theory. In evaluating the great development in contemporary logic, from a historical and methodological point of view, the word 'revolution' is suggested as the most precise description. In a sphere of knowledge deemed almost complete, whose basic postulates (dogmas) have been viewed as doubtless and have almost been canonized, totally new conceptions emerge! 'Revolution' corresponds to the scale of change and re-evaluation of values in modern logic, comparable to important moments in the development of other fields: Newton's theory, the periodic table of the elements, quantum mechanics, the transition from Newtonian physics to Einstein's theory of relativity, non-Euclidean geometry. The transition to non-classical logic has often been compared to the transition from Euclidean to non-Euclidean geometry but this is superficial and imprecise: the analogy is done by external signs while the essential question is left in the background, namely, what sphere precisely is concerned, and that is logic! The science studying the most general laws of thinking; the science studying the correlation of thoughts by their truth-value; a sphere related to the laws and processes of cognition and knowledge, and to the question of truth. Geometry is only one of the disciplines of mathematics. In its relation to the universal philosophical concepts, to the world, to 'being'(existence), to the 'logos' and 'ontos', it cannot be compared to logic.

Such an evaluation is found in some papers. However, most authors acquire it and preserve it, without further analyzing what the revolution precisely consists in, what stages it has passed through, and whether or not this is the case of the two different revolutions. The development of contemporary logic is evaluated on a broader scale, without specifying or going into details of its separate stages. In my view, it would be more precise to speak of two revolutions, both leading to a replacement of one paradigm by another. The major cause for these revolutions is that logical empirical development has gone considerably ahead of logical theory. It has been debated many times whether logic has its 'empirical basis' or 'empirics'('empirical sphere') or whether it is a purely deductive structure. This question is important for the philosophy and methodology of logic. Some authors argue in favor that logic is a science without 'empirics'. My position is that logic does have 'empirics' and this is basically the language and methods of reasoning in scientific theories. Deduction is applied most consistently in mathematics; therefore the language and methods of reasoning in mathematics are 'empirics' for logic. However, it would be groundless to restrict the logical 'empirics' to reasoning in mathematics alone, the language and methods of reasoning in many non-mathematical theories are 'empirics' too. Logic deals with the transfer of truth from the premises to the conclusion in an objective and non-empirical way but this 'non-empirical transfer' has its own 'empirics'. It has been argued in favor that the concepts and ideas of logic are based on experience. If the 'empirics' of logic consist in the reasoning and justifiability (provability) of scientific theories, one should not forget that on their part they are the result of 'empirics'. It would seem that the 'empirics' of logic are logic itself but this is not dangerous, paradox-generating self-reflection: the demarcation is clear in reasonings and the theory that studies them. In the 18th and 19th centuries, very rich and elaborated 'empirics' were provided for logic by mathematics. This is why mathematical reasoning was taken as the basis for the construction of the first logical systems. In mathematics, 'empirics' were rich enough to be generalized and explicated into a logical system.

The 'empirics-and-theory' relationship is connected to the concepts of 'verification' and 'falsification'. In logic, 'empirics' have a more specific character; one cannot say that they provide a direct 'verification' and 'falsification' of the theory. There is an intermediate theory. However, taking into account this intermediary theory, one can speak again of 'verification' and 'falsification'. For the logical theory called 'classical logic', the mediating theory is mathematics, and its 'empirics' (the logic of mathematical reasoning) verify classical logic. However, in classical logic, the paradoxes of material implications were obtained, while for the bulk of scientific theories such a type of logical reasoning is unacceptable, and this is why, in my view, we can think that the logic of such a type of theory (i.e. its 'empirics') 'falsifies' classical logic (or, at least, casts doubt on its universality).

The lagging of the science of logic behind logical 'empirics', and especially behind the 'empirics' provided by mathematical reasoning started long ago. Euclid used, more logical forms and constructions than Aristotle: disjunction, conjunction, negation, multiple-place predicates such as 'between'. They fell outside the scope of Aristotelian syllogistic and could not be described in its language. Subsequently, logical theory had for a long time been lagging behind logical 'empirics' in mathematics. Especially, it was in the 17th to 19th centuries when mathematics underwent a rapid development. The paradigm of traditional logic exhausted its capacities long before the first revolution eventually happened. The works of Boole, De Morgan, Frege, Russell, Hilbert are the result of the generalization of logical 'empirics' of mathematics. They eliminated the lagging in question. This is an example of how logical theory generalizes logical 'empirics'. Since mathematics is related to (and is a result of) extensional theoretical structures that lie outside it, the mentioned results are the generalization of 'empirics' encompassing a larger area.

The revolution, *from traditional to classical logic*, consists mainly in the change of solutions, instruments, and methods. The system of classical logic uses a wider range of logical instruments. It allows working with multiple-place predicates, it provides the possibility to unite spheres that have therefore been considered in logical theories, in isolation, each for its own sake. In my opinion, it is exactly this possibility that presents the most significant result of the revolutionary transition.

A question is to what extent the new paradigm is commensurable with the old one. The system of classical logic does not deny traditional logic. It generalizes and essentially extends its capacities. This is why I believe that we cannot speak of a complete incommensurability. Some of the orthodox followers of Kuhn would conclude that there could be no talk of revolution here. However, in my view, a relative absence of incommensurability can be also observed in some changes that have been recognized as revolutions in science, especially in the transition from separate theses and works to a complete theory of a larger scale. Newton's theory is of that type. To a certain degree, incommensurability comes forward in the second revolution: the transition from classical to non-classical theory. Here, the incommensurability is much more often discussed and most scholars seem to accept it overtly or implicitly. I have a specific opinion concerning this point. There is a doubtless incommensurability but it could be sublimated into a suitable conception of logical neo-fundamentalism.

The *transition from classical to non-classical logic* is another revolution, and a much more important one, since it affects issues of fundamental nature. It is not a question of introducing qualitatively new methods; there are qualitatively new methods in the first revolution while here there is only a certain extension of these methods. The question is of qualitatively new conceptions, ideas that radically change our understanding of logic. In the first revolution, the introduction of new, mathematical methods makes theory clearer and all encompassing. Logical theory becomes, in style and method, closer to mathematics (the field where the 'empirics' show the greatest need for a new logical theory). Though rather later, this revolution was comparable to the works of Newton and the creation of classical physics, and also to the creation of the periodic table of the chemical elements. (The analogy comes naturally here – the transition from classical to non-classical logic can be compared to the theory of relativity and quantum mechanics.) In its spirit, it was a **fundamentalistic** revolution giving new foundations in logic. The second revolution, on the other hand (the transition from classical to non-classical logic), is strongly **anti-fundamentalistic** in its nature; it casts doubt on the basis that the first revolution propounded. It was provoked by the discrepancy between theory and 'empirics'. However, logical 'empirics' became extra-mathematical: it was the logic of scientific research. The development of contemporary logic seems to be natural and consistent: the initial breakthrough has been performed in the area where logic is used most extensively, most systematically and most coherently, in the use of its means of expression (in the most extensive, systematic, and coherent way). Being aware of the efficiency and productivity of contemporary methods, contemporary logic continues to develop but this time it holds by going deeper and wider: mathematical methods allow us to explicate essential logical moments of the extra-mathematical logical 'empirics'.

While the first revolution was a revolution concerning methods, the second revolution (the transition from classical to non-classical logic) was a revolution of ideas and postulations.

An important reason for considering the two revolutions as a one is that they are very near in time to each other and in fact occurred almost simultaneously. It seems to me that Boole's idea for the introduction of algebraic methods in logic, reducing logic to mere algebra could well have emerged considerably earlier. The works of Frege, however, are historically related to the development of science. They come in response to the need of logical analysis of the foundations of mathematics, and thus are the natural development of the creation of the set theory and its establishment as the basis of mathematics. But non-classical logic came to light very soon after them. The system of classical predicate logic had not been completely created yet when Łukasiewicz proposed his system of a three-valued logic, and Lewis's system emerged. It is natural to regard them as continuation of the works of Frege and Hilbert. The methods are similar: formal and mathematical. (By analogy, we could imagine that Newton's theory historically came much later and was created only in the late 19th century.) This was another obstacle to their methodological demarcation. It is rather easy to view them as the one whole or as the one natural process, as two stages of the same development. If there had been a transition from classical to non-classical logic and if the real construction of systems, alternative to classical logic, had had to happen, then classical logic itself would have been completed at least to some extent. Heyting's system of intuitionistic logic was created naturally after the creation of several axiomatic systems of propositional logic. However, it was not at all absolutely necessary for the classical logic to be constructed up to its total completion before its competing alternatives came into being, moreover that the idea of non-classical logic came much earlier.

Another general reason to view the two revolutions as the one whole is that the second is a continuation of the first: apart from being temporally near to each other, they also have similar causes that generated them. Both are a result of theory's lagging behind 'empirics' findings. The difference is that in the first case empirics were mathematical while in the other they were non-mathematical.

Logical paradoxes are the third general reason: the paradoxes in set theory, the semantic paradoxes, and the paradoxes of material implication. However, in classical logic they exert their influence at a later, intermediate stage. The paradoxes in set theory serve as catalysts and accelerators, they affect principles and postulations that are basic for the theory. The idea to form a set of sets is essential to Cantor's theory. In his approach, the set is an object like any other object and one can treat and manipulate it in the same way as other mathematical objects. Then it is natural to use the same logic. However, finite and infinite sets are different, they have qualitatively different properties and, since they are different, it is debatable whether the same logic may be applied. While in the first revolution it was Russell's paradox that gave the greatest impetus, semantic paradoxes and especially the paradox of material implication that played a greater role in the second one, demonstrating the incapacity of classical logic to express logical inference adequately. In part, the synthesis of the two types of rationality coexisting in contemporary logic, the philosophical and the mathematical one, is the basis of the revolutions in contemporary logic. This is especially valid for non-classical logics. The creation of many-valued, modal, intuitionistic logics is related to philosophical postulations. Unfortunately, this synthesis is sometimes partial, inconsistent and incomplete.

There is another synthesis in the revolutions. The Stoics divided propositions into simple and complex and studied how the truth in the complex ones depended on the truth of the simple ones of which they were composed and on the form and manner of their combination. Aristotle restricted himself to the analysis of some elementary subject-predicate relations. Propositional logic regards propositions as indivisible, unstructured, with a single property of being 'true' or 'false'. In the system of classical predicate logic, a successful synthesis has been achieved of the two approaches to logic which began their development in antiquity: Aristotle's subject-predicate approach studying inferences based on the subject-predicate structure, and the Megarian and Stoic approach studying inferences based on propositional relations (well-realized in the system of classical propositional logic).

The first revolution separated logic from ontology. The second provided a possibility to restore this connection. The ideological basis of the first is Leibniz's idea to present demonstration as a calculation similar to the mathematical one, and to introduce more systematically mathematical methods into logic. Retrospectively, the essence of the idea is the view that it is precisely the methods of this type that would be efficient for the logician. The ideological basis of the second revolution is provided by Aristotle's modal syllogistic, by the works of Ramos, and unexpectedly to some by the spirit of Hegel's works.

When we speak of revolutions, two views emerge immediately: the revolutionary and the evolutionary one, namely, revolutions are gradually and continuously prepared by evolutions, but as they are well prepared, revolution is revolution. We face this question both when we speak about social development and when we discuss issues of philosophy of science. My aim here is not to consider this question in detail but I still think that a synthesis between the two views is possible: revolution is a result of evolution. In revolution, the evolutionary potential becomes realized quickly, in a relatively narrow period, leading to a considerable change in a relatively short time. For a number of specialists in social philosophy, the cause of a revolution is the retarding or retention of the natural evolutionary way of development of society, and revolutions are the symptom of an unsound state of society, of essential errors in its development. If it had developed in the natural evolutionary way, there would have been no need for a revolution. I might accept this position concerning society, but I do not think that it could be mechanically applied for the revolutions of thought. I would not accept the emergence of a grandiose idea changing many basic postulations as a symptom for an 'unsound state' of the given field or for 'essential errors' in its development. However, I consider these words appropriate when such an idea has come too late and thus the respective field has lagged behind other fields related to it! The revolutions in contemporary logic have been conditioned by the considerable lagging behind in the development of logic compared to mathematics and other sciences related to it. At a certain moment, it became necessary for it to catch up within very short terms. There is a link between the development of ideas in different sciences. Many discoveries in physics have been related to results of mathematics, and vice versa. Achievements in chemistry have been related to results in physics. This internal correlation in the development of the separate sciences exists especially in fields that penetrate in a specific way into many areas of knowledge. A clear and well-known example of that is mathematics: one can discover mathematical regularities in many areas of knowledge, mathematical models have a wide application. Maybe less frequent than mentioned, logic is such a field with a considerable wide application. This is the reason for perceiving the lagging behind especially painfully.

The revolutions in logic raise questions in a similar perspective to the ones of the revolutions in physics: what would be justly deemed as a revolution; is there one or are there two revolutions? Philosophers of physics ask themselves "whether one can consider the discovery of X-rays or Mëssbauer's effect as a revolution..." A question that seems to be similar is whether Gödel's limitative theorems are revolutionary. If the emergence of classical dynamics (Galileo and Newton) is a revolution, then the emergence of the system of classical logic is, for me, an achievement in logic, similar in spirit, and this is why it should be called a revolution, too. The question whether it is a matter of one or two revolutions is not raised only in logic. A similar question has also been asked about the development of physics. The relativist and quantum revolutions: are those two separate or a common relativist-quantum revolution? Is the theory of relativity a continuation of the revolution in electrodynamics, or it is separate as a subsequent revolution? Is it one or two revolutions, of the special or general theory?

Consider another analogy: the theory of relativity does not deny or replace classical physics. Analogically, non-classical logic does not deny or replace classical logic. As the philosophy of physics accepts the thesis that the theory of relativity is more suitable to some theories related to certain entities and ontology, similarly the philosophy of logic could accept the thesis that for some theories related to certain entities and ontology, a kind of non-classical logic is more suitable.

Another moment, characteristics of revolutions, is observed in the development of logic: reconsidering universality, the scope of certain basic principles and postulations. There is no just one, but several such moments, and in each of them the sphere of validity of some basic principles or postulations undergoes a change. This is considered to be a sufficient reason to speak of a revolutionary transition, but the word 'revolution' is nevertheless too strong. I accept as the most appropriate and I use the concept of 'micro-revolution' for the reconsideration of a basic principle of classical logic. The transition from classical to non-classical logic consists of some different micro-revolutions. Micro-revolutions do not happen in a single turn, they have a more complex nature; they consist of different steps that are sometimes so small that look like evolution. Micro-revolutions are not always historically consecutive, they do not strictly follow one another. In the historical development of logic, some micro-revolutions took place parallel to others and even some came before others. The ideas of the Russian logician N. Vasiliev, the forerunner of intensional and paraconsistent logics, came before the emergence of many-valued and modal logics. The idea of extending classical logical operators by modal ones seems to be more natural and acceptable from a classical point of view and much more important and desirable from a philosophical point of view. However, there is a greater awareness of it only after Gödel's axiomatization of the systems of Lewis – later than the creation of intuitionistic and many-valued systems. Relevant logic, which is much more closely related to logical inference, emerged only after it became clear that modal logic was not suitable for that purpose.

The important moments of the second revolutionary transition (micro-revolutions) [6] are as follows:

a) *Denying the principle of the excluded middle* (tertium non datur). Intuitionistic logic encompasses the principles of logical reasoning which Brouwer used in developing his intuitionistic mathematics.

b) *Denying the bivalence principle*, one of the basic and most important principles of classical logic. The main idea is that two-valued logic cannot express the diversity of logical situations, e.g. in propositions of modality or probability. In most epistemological situations only 'truth' and 'falsity' are too coarse – it can be said of many statements that they are true only to a certain extent. But the indefiniteness of their truth does not exempt us of the need to perform logical operations on them. Depending on the epistemological situation to which the logical system is related, admitting a third value having a truth status different from 'true' and 'false' allows us to set a rich palette of interpretations: "neutral", "indefinite", "possible", "unknown", "predefined", "probable", "antinomic", "meaningless", "overloaded", etc. Abandoning the principle of two values does not stop with the introduction of three values: allowing for one intermediate value leads to the requirement for others; most interpretations of the third value allow to be set for different degrees, e.g. 'more indefinite'. The interpretations of the third value have a modal importance, and modal operations also allow to be set for degrees: "very necessary", "more necessary", "less necessary", etc.

c) *Extension of classical logical terms.*

d) *Denying the principle of extensionality* (intensional logic).

e) *Denying the universality of the law of non-contradiction* (paraconsistent logic).

f) *Denying monotonicity* (non-monotonic logic).

I accept the possibility for several other micro-revolutions to take place, but my impression is that all basic ideas for taboo breaking have already been tried.

Usually, problems in knowledge come from borderline cases. The development of science involves specifying, particularizing, and giving shape to and displacement of boundaries. In the development of the 20th-century logic, such a displacement has usually lead to enlargement. Each micro-revolution involves such a displacing and enlargement. The main moment is an enlargement of the boundaries of set of logical terms. For some time, many-valued, modal, and intuitionistic logics have been at the borderline. However, after the emergence of the semantics of possible worlds, the borders have been enlarged and modal logic has steadily come into the interior and even close to the core of contemporary logic. In my view, deontic and paraconsistent logics are currently close to the border but on the inside, while the border itself contains non-monotonic logic, erotetic logic and most of the many recently emerging and not properly established logical systems. Deontic, erotetic logic, etc. are at the borderline because, by some of the criteria, there is a discrepancy with definition of logic – in the spirit of 'Jorgensson's dilemma', there are doubts whether there can be any logical inference in them at all. In the terminology of Lakatos they are 'monsters'.

Important methodological results, as a consequence of the first revolution, were Gödel's incompleteness theorems. There is a popular thesis saying that these theorems are in themselves a revolution. However, their condition was the creation of the system of classical logic. This is an interesting point: a new theory is built and it allows to obtain an important result. If we believe that there are reasons to call this result a revolution, then we have the question: When did it start with that result or with the creation of the new theory that was its condition? I am more inclined to call a revolution the creation of classical logic. There are quite a number of important results in physics whose condition was Newton's theory but they are not called revolutions. Although historically Gödel's theorems were not expected, something of the kind could have been expected retrospectively, since Hilbert's program allowed to be situation of self-reflection similar to some semantic paradoxes. The consistency of arithmetic is proved by arithmetization: the system contains arithmetic and must be proved by it, i.e. it contains both its own syntax's and its semantics.

Contemporary logic is closely connected to the development of the philosophy of mathematics. The crises in the foundations of mathematics are among the important factors for the development of contemporary logic, especially for the revolutions discussed here. The creation of the system of classical logic was provoked by the Third Crisis in the foundations of mathematics related to the paradoxes of set theory, and at the very moment when it was hoped to become a stable fundament for the construction of mathematics. We face here an interesting point, maybe not very standard for the philosophy of science. A standard situation is: a crisis in a certain field leads to a revolution in the same field. In the case of the revolution in logic, things are somewhat different: a crisis in mathematics leads to a revolution in another, though very close and related field, i.e. in logic! An attempt to standardize the situation would be to accept that in fact that was not a crisis in mathematics, but in the logical foundations of mathematics instead. However, set theory is a 'purely' mathematical discipline. Most philosophers of mathematics talk of a 'third crisis in the development of mathematics'. The development of logic has lead to reconsidering many questions of the philosophy of mathematics. The first revolution, for instance, imposed to practicing mathematicians a "structuralistic" conception of mathematics: "With its development, mathematics is becoming more and more abstract and, from a 'science of quantitative relations and spatial forms', it has turned into a science studying abstract mathematical structures, mathematical models of theoretical systems."

The second revolution raised many other problems prior to philosophy of logic, but the main one is widely discussed in philosophy of science problem: the *proliferation* and respectively the questions about the *monism and* *pluralism* of logic. After the second revolution, logic has been subdivided into a number of logics; so which one now is a "proper logic"? And can we talk about "proper logic" at all? A number of alternatives to classical logic have been offered. Questions have been asked such as: "What is logic after all?" "Is logic two-valued or many-valued?" For some of non-classical logics, their status as "logic" is reasonably questioned. Is there, after all, a reason why all those disciplines use one and the same word "logic"? Could we say that some of them are 'more' logic than others? Which ones of them and what degree is the question of truth related to? Even if some of the logics are ruled out as 'illegitimate' logics or 'mathematical exercises with symbols', and then others are ruled out because they employ the word 'logic' only metaphorically. So, all those questions are valid nonetheless. What did then happen with Leibniz's thesis about the ontological universality of science: "logical truths are valid in all possible worlds"? The conception of "one logic" allows us to consider it as a necessary connection of thoughts according to their truth-value, valid for all real and possible worlds of theories. However, logic becomes parcelled; many logics have appeared, but which one is the only logic? Is there anything allowing us to use the same word for all of them? Is every one of those logics a separate, independent area with no connection to the rest, or can they be grouped together in some way, and by what criteria? Which of them and to what extent are the most general laws of thought studied (namely, the connection of thoughts according to their truth, the laws and process of scientific knowledge and knowledge in general)? How are they related to the question of truth? Which of them is rather a game with symbols, mathematical experiment bearing only a distant relationship to logic, and which are logics whose "logicality" can be proved by ontological and epistemological arguments? What about the a priori nature of logical laws? The philosophy of science calls this subdivision a proliferation of the theory, and considers it to be a symptom and a reason for the methodological crisis in the field. So, is the proliferation in logic the beginning of crisis or is it a sign of crisis in the development of logic? How is this related to the conceptions of monism and pluralism of logic? The problems related to the philosophical interpretation of non-classical logical systems have assumed an ever increasing actuality in modern logic. What should the aim be in constructing such systems? What is the position of a given system among the rest of logical systems? What is their role in the theory of knowledge? In current practice methodological developments considerably lag behind the formal logical investigations.

An important question is how far and to what degree the appearance of non-classical logic has led to the crisis in logic. Does their appearance mark the beginning of a crisis, or are they the symptom of an already existing, latent crisis? In that case studies of non-classical logic will be attempts to overcome this crisis.

My stand on this question is that the appearance of non-classical logics is rather a symptom, showing that traditional logic is becoming more and more unable to respond to the requirements conditioned by the development of science and cognition. In this respect, the appearance of non-classical logics is one way of trying to overcome the crisis. The way out of the crisis goes through the establishment of the concept of "logicality", which embraces non-classical logics (or at lest a great deal of them), and especially their confirmation as logics.

Maybe there is not only one crisis, but two. The first is related to the "limitedness" and the inadequacy of the traditional and classical logic, which led to the appearance of non-classical logics. The second is a result of the existence of these logics, which led to proliferation. The common ground in both cases is a crisis in the foundation of logic, related to fundamental questions such as "the subject and essence of logic and the logical", "the unity of logic, its universality and its boundaries."

There are two positions:

The first is 'logical monism' which insists on the uniqueness of logic: 'There is one logic and everything else called "logic" is just an application of this unique logic in one or another field'. This thesis often leads to the thesis that this is the traditional (or classical) logic.

The other is 'logical fundamentalism', i.e. the desire to find a doubtless fundament to logic, to demarcate logic and make it independent, to show what unites different logics: 'Many logics are types of logic, and, being such, they must necessarily have something in common'. The task to find and demonstrate what is 'common' belongs to philosophy.

A radically possible way out of the methodological situation in logic is "*logical anti-fundamentalism*": 'Many logics are independent disciplines, each with its own object and methodology, therefore there is no proliferation and no reason to talk of a crisis'. Then no logic can justly declare that it is the 'real logic'. However, the proliferation of logic provokes a natural tendency toward fundamentalism, which can be observed in many papers written by philosophers of logic. That is why I propose "*logical neo-fundamentalism*" as an alternative way out. This is a conception of the character, object and nature of the logical, viewed from both the epistemological and the ontological aspect. The conception of "logical neo-fundamentalism" must lend a support to orientation in the many logics, introduce some kind of structure, order, organization in them. At the same time, it must demonstrate their unity in their diversity, clearly state what they have in common. And it must clearly justify whether, in the presence of so many different logics, there can be universality of logic and, if it can, in what sense we are to understand it. Such a ("neo-fundamentalistic") conception must necessarily involve the following aspects:

1. It must provide a clear and argumentative conception of logic, which should be a development of the previously existing conceptions of logic. It must include a consideration of the nature and objects of logic and also must encompass non-classical logics or at least a considerable part of them, and serve as a criterion of evaluating which logical systems can be called logic and which cannot. It also must allow systems that have not been constructed yet to be evaluated. My shortly *definition for logic* is as follows: 'logic deals with the objectively ontologically predetermined dependence (relation) between thoughts according to their truth, as expressed in language'. This synthetic definition encompasses the relations of logic, thinking, truth and inference and as well as of language and ontology.

2. It must provide a consideration of the universality of logic as understood in the sense of item 1. My idea is replacing the conception of logic in genera ('logic of the world') by 'logic of a theory, or of a group of theories (about the world)', and thus the universality of logic is realized by the universality of logics and the universality of "logicality". The conception of one logic allows it to be regarded as a necessary correlation between the thoughts of truth and makes it valid for all real and possible worlds. The concept of a multitude of logics renders this impossible. This narrows the universality of each separate logic. It can only be universal as far as it is logic, that is, as far as its laws are universal for a particular field or theory. It is clear that a logic should be universal with respect to the world it describes, but then a number of different worlds are possible. 'A world' could even mean a certain aspect of another world. The universality of logic will be realized by the universality of the separate logics, inasmuch as they will be complementing each other. The common thing between them will be "logicality", that is, the very fact that they are logics and that they express a necessary correlation between the thoughts of truth valid for their corresponding theory. This position allows to discuss a number of controversial issues. Such is Hilary Putnam's thesis that "some of the necessary truths of logic can sometimes happen to be false by empirical reasons" [3]. Expressed in this form, this statement is unclear to me. It comes out that 'necessary truths' are not necessary. I think that my position allows to explain: the truths in question are 'the necessary truths of classical logic' and therefore the formulation becomes as follows: 'empirical reasons can replace classical

logic by non-classical logic'. Related to 'empirical reasons', ontology defines logic! Another unclear thesis in [3] is that "logic, in a certain sense, is a natural science" – what does precisely this mean? I perceive that logic has its 'empirics' (a somewhat specific one, to be sure) and it is in this sense that "logic is similar to natural sciences."

3. It must provide a kind of structuring of the many logical systems constructed so far; the criteria for such a structuring must be related to the points provided in 1 and 2. This idea of structuring must come from one common position and allow to be develop.

My own attempts and the analysis of many other attempts to classify logical systems (e.g. see "map" in Resher's paper [4]) have convinced that a successful orthodox classification is only possible if using internal mathematical, syntactic criteria. However, such a classification would not give much to the philosophical methodological meaning of those systems. Mathematicians classify their structures according to internal criteria, but the philosophical methodological look must evaluate them according to whether they are pure models or correspond to 'objective relations of objective things', and to which precisely. Every logical structure constructed as a mathematical system is a potential model and, depending on its interpretation, may serve for modelling in diverse areas. However, our purpose requires a classification corresponding to the philosophical interpretation and the importance of the systems rather than to their formal technical elements. This will not be a classification in the precise sense of the word, but a "*typologization*", since a system may have different interpretations and the demarcation will not be strictly exclusive.

I propose the following "typologization" of logics, corresponding to their philosophical evaluation (this is a "typologization", and perhaps even a classification of the basic, most important and crucial philosophical methodological interpretations of logical systems) [6]:

(i) *Logical systems serving as bases for a mathematical theory.* Classical logic and the intuitionistic logic are certainly here.

(ii) *Logical systems, attempts for the explication of logical inference*, e.g. Alexander Zinoviev's complex logic and naturally relevant logic.

(iii) *Logical systems expressing epistemological aspects.* Modal, many-valued, paraconsistent logics, and to some extent also temporal logic and fuzzy logic, can be viewed as logics of particular epistemological situations.

(iv) *Logical systems related to ontological aspects.* There is the conception that ontology determines logic and that entities in themselves impose the logic to the theory studying (describing) them. This argumentation is related to the idea that some properties of the objects of the micro-world require a non-classical logic for the theories studying them. I would formulate a 'thesis of Reichenbach, Birkhoff, and von Neumann': "The mathematical apparatus of quantum mechanics requires a logic different from classical logic" which I interpret: "The ontology of the objects of quantum mechanics imposes a non-classical logic upon the theories studying them." The strongest version is that the problems with the quantum mechanics are first symptoms emerging in the initial approach to the genuine non-classical science. In the future, maybe there will be much more drastic situations. In the process of cognition we face again and again new entities and theories about them in which the work with classical logic meets considerable difficulties and inconveniences and this will lead to essential complications. And for theories studying these entities, we will have to reconsider the universality of some of the laws of classical logic.

(v) *Logical systems as (only) mathematical models* and systems created without philosophical justification: they are interesting at least because they allow analogies with other systems interesting in philosophy. Such logical systems can be regarded as "uninterpreted" abstract languages.

(vi) *Applied logic*, logical systems that do not claim to explicate logic in any general philosophical, epistemological or ontological sense, but aim only to express logical relations involved in a particular field of application, such as artificial intelligence, cognitive science, linguistics, computer science, etc.

A comparative analysis of the classification according to internal mathematical criteria and the typologization according to philosophical-and-methodological criteria contributes to understanding the place and the importance of each separate logical system. Logics in which the two-value principle does not hold are closest to logical systems typologized as logics related to epistemological situations, and also to the controversial but very interesting type of logics related to non-classical ontologies. For systems typologized as related to logical inference, a non-observance of the principle of combinability and of extensionality is convenient. The historically established unification concerning the question of the logic of necessity with the question of inference (related to the paradoxes of material implication) is not necessary. The two questions could also be considered separately, with different methods and instruments. Many modal logics do not observe the principle of functionality. However, if we consider modal logics as logical systems related to epistemological situations, then modal operators will transform, in a certain manner, the truth-value of the propositions into another truth-value. Then, it is reasonable to construct also modal logics in which to retain the principle of functionality in some form, probably some kinds of many-valued functionality. The systems of Lewis are actually built on classical logic, but would the idea of logic with a different implication not be realized successfully in systems constructed differently? After the emergence of the semantics of possible worlds, this modal logic can be defined, rationalized and interpreted as logic of possible worlds. Unlike extensional interpretations, here the truth of a proposition does not depend solely on the truth of A in the particular world but also on its truth in other possible worlds. Thus possible worlds are regarded in themselves as two-valued but with a notion of truth relative to the particular possible world. This is an elegant and beautiful generalization of the classical two-valued logic: it is the logic of a universe with one only possible world. The necessity to take into account the character and degree of truth of the separate propositions of one area leads to multiple possible worlds in it, which leads to a non-classical logic for their universe!

If we analyze the possible interpretations of the third value, we will see that in most of them it is a subject to discussion whether the logical system that would corresponds to them should have precisely three values. For some interpretations, another type of many-valued logic would seem to be more appropriate. Thus, as logic for databases and computer systems, four-valued logic is more suitable. Just one additional value provides a few opportunities. This kind of analysis can lead to the conclusion that for a given interpretation there is no adequate logical system, which could lead to the construction of new systems that would take into account the specific nature of the interpretation! That is, epistemological interpretations of modal logic lead to the idea of many-valued logic, but a many-valued logic with a special interpretation of truth-values related to their modal interpretation! The main point is not to have multiple-values with truth-values defined beforehand, or multiple-values generating modality, but multiple-values generated by a modality extracted from the need for a logic considering a modal graduation of truth. I would classify many-valued logical systems having fixed matrices of truth-values, although they too have some interpretations, as mathematical models. From a philosophical-and-methodological point of view, probability and topological logics are much more interesting than the systems of Post.

The difficulties in building paraconsistent set theory are not a matter of chance. Although mathematics is said to be an ontology-free science some mathematical ontology still exist! The ontology of the micro-world determines the logic of sciences that deals with it, and so does the ontology of mathematical objects in respect to the related theories. Intuitionistic logic turns out to be compatible with mathematical ontology, but not with paraconsistent logic. The importance of paraconsistent logic for philosophy is greater than for mathematics. The aim of paraconsistent logic is to study the logic of a theory that tolerates contradictions. But the necessity of such a logic is quite different from the necessity to construct mathematics to deal with them. The construction of systems, tolerant to contradictions, can be associated with the explication of real epistemological situations. History of science could provide instances of inconsistent theories doing their job. For some time such theories do not come out of use, though scientists use them with caution. Many contemporary mathematicians who are not so easily disconcerted with the paradoxes use naive set theory in their current work. In this perspective the role of paraconsistent logic could be considered as methodological in the following sense. Paraconsistency does not open a way out of the paradoxes, as it is not the likely candidate for an underlying logic of mathematics. It rather explicates in logical terms how mathematics (or other theory) with paradoxes do their job. “If a contradiction were now actually found in arithmetic will only prove that an arithmetic with a contradiction in it could render very good service.” [7]. There is a philosophical view according to which for some fragments of reality, there can be no consistent world-picture or account. Any such account would be incomplete. This view is connected with the so-called “Hegel's thesis” that there are “true contradictions”. According to that, consistency is a sufficient, but not a necessary condition for the existence of abstract objects, and concerning the existence of concrete objects it is neither necessary nor sufficient. The “real antinomy” could not be eliminated in any normal way that would come down to a replacement of subjective elements by objective ones. Real antinomies are not fallacies but are “peculiar objective truths” [2]! This makes it necessary to produce logic, adequate for the purpose of studying contradictory, inconsistent entities. By the way, it can be argued with great certainty that the logic of individuals, as well as of social groups, is inconsistent. It is practically established that any sufficiently large databases are inconsistent. Contradictory information may be acquired through different channels (or even by one and the same channel). Paraconsistent logic is a goal oriented not in the same way as classical logic, and this makes it improper to extrapolate the classical aims in the field of paraconsistency. This fact should not distract us from the merit of a paraconsistent system, on the contrary, it should be considered an asset from a philosophical point of view. They have their place among non-classical logic, which are not close to intuitionistic logic, but rather to modal or many-valued logic. Therefore, we should regard them in the way we do the above mentioned logics. There are hardly any attempts to use modal logic as an underlying logic for mathematical theory, but this is no reason to underestimate them, to examine their proper use in this. The same holds for the paraconsistent logic. So, the purpose of paraconsistent logic is to account for a kind of epistemological situations, for the logic of a certain social or computer system and also to explicate the logic of a theory, which studies a specific sort of ontological entities. Since paraconsistent logics are not the alternative to classical logic, it is not necessary to construct them thus as systems, which serve as bases for mathematical theory they are framed [5].

The idea to construct a relevant logic as a restricted subsystem of classical logic is similar to the idea of intuitionistic logic (another restricted subsystem). However, it is an open question whether the philosophical ideas on which they are based would not be realized better in logical systems of a qualitatively different type. Classical logic is so constructed as to be a convenient basic logic for mathematical theory. But how far methods suitable for one kind of questions can be convenient in resolving quite different types of questions? The typologization helps to distinguish between logics explicating logical inference, logics of possible worlds, and logics of necessity and contingency. The modal systems of Lewis are closest to the second kind. Fuzzy logics are not only applied they are closely to questions related to epistemology and dialectics where often the terms ‘fuzzy’ and ‘rough’ are used.

The question of the philosophical and the methodological rationalization of the intuitionistic logic is still open. To what extent does intuitionistic logic correspond to the philosophy of intuitionism and are the kinds of semantics that are proposed for it in accordance with this philosophical standpoint? Is it logic of (and for) mathematics, or does it refer to other theories and fields? Does it explicate the constructive thinking only in mathematics, or does it explicate the constructive thinking in a considerably wider philosophical sense? Is it the only possible constructive logical conception, or are there other alternatives. So, what about superintuitionistic logics? If superintuitionistic logics are alternatives to intuitionistic logic as another explication of a constructive thinking, as another constructive logical conception.

Karpenko [1] claims that the presence of denumerably many superintuitionist logics is shocking and deserves philosophical and logical interpretation!

But is there a kind of logical intuition in the background of the class of superintuitionist logics or at least of some of its members? If yes, then the questions put by Karpenko: “How can we interpret the presence of denumerably many logical systems? Is each of them correct and can be called logic on the same grounds as the others? How can we interpret the presence of a continuum of logical systems that can be qualified as logics on the same grounds as the others?" have a great significance for philosophy. On the other hand, if we treat them as pure mathematical models, I do not think that their infinite number presents to us some significant problems. In my opinion, superintuitionistic logics are pure mathematical models. So they are not PhL.

Figuratively, some of the systems of modern logic justify the expression: "*Non-classical logics are not what they are.*"

The place, where a system is located in the typologization, essentially determines the criteria and methods of evaluating. For systems, which are evaluated as basis for a mathematical theory and as pure mathematical models, the basic feature is the mathematical perfection, i.e. precision, interesting theorems. In applied logical systems, it is important that the system is a successful model related to the particular questions of the specific field to which the system is directed. However, when the system is typologized as philosophical, e.g. in logical systems aiming at a better explication of logical inference, or in logics related to certain epistemological situations, or logics related to new non-standard ontology's, the situation is different. This increases the methodological importance of the system, but also it increases the philosophical requirements to it. When the basic value of the system is mathematical, the criteria are mathematical and the judges are mathematicians. When the system is closely related to methodological questions and conceptions, when it explicates notions and categories of philosophical nature, the criteria for it should be philosophical and its judges should be philosophers! The big boom in the establishment of non-classical logics came when it was demonstrated that they also have an essentially applied character, e.g. in fields like computer science. This is an important but not a fundamental argument, since my basic thesis is the philosophical and methodological importance of non-classical logics related to issues of epistemology, dialectics, and ontology.

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MODERN STUDY IN PHILOSOPHICAL LOGIC: WORLDWIDE LEVEL AND RUSSIAN SCIENCE

In this paper, we survey the development of philosophical logic in Russia within the framework of worldwide tendencies. Philosophical logic is an extremely wide area of logical studies, requiring philosophical judgement of the basic concepts, used in modern logic, and the outcomes obtained by means of mathematical logic. However, we need to remark that the term 'philosophical logic' is uncertain and has no uniform use. Even if philosophical logic is represented as a special scientific discipline, it is not possible to define its subject, limits of application, and methods. Therefore we consider the background of different directions in philosophical logic and its connection with philosophy of logic, foundations of logic, and computerization of logic.

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1. Introduction

The modern development of logic and comparative analysis of two logical journals: 'Journal of Symbolic Logic' and 'Journal of Philosophical Logic', published under aegis of the international Association for Symbolic Logic, started in 1936, shows a rapprochement of topics, methods, and results in logical papers. At the 9th International Congress of Logic, Methodology and Philosophy of Science (Uppsala, Sweden, 1991) G. von Wright stated: "The point of logic is that it has fused in diverse researches of mathematics." Nevertheless, modern researches in logic should be divided into three basic sections:

I. Mathematical (symbolic) logic.
II. Philosophical logic.
III. Non-classical (unconventional) logic.

The third section in its following directions

1. intuitionistic and superintuitionistic logics (including Markov's constructivism),

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2. modal and temporal logics,
3. many-valued and fuzzy logics,
4. relevant and paraconsistent logics

unconditionally belongs to philosophical logic by virtue of those especially philosophical premises from which these directions appeared. However, logical schema become so refined, formally developed and mathematized that they do not keep a place to philosophical gamble. But by the end of the 20th century, the question was raised at the junction of all these three sections: "what is a logical system?" and generally "what is a logic?"

In this paper, we shall be basically limited by section (**II**). However, it is important to notice that Russian logicians have scientific results of the highest worldwide level just in non-classical logics.

First of all, we shall characterize how mathematical logic should be understood. In the preface to 'Handbook of Mathematical Logic' [6], J. Barwise writes: "The mathematical logic is traditionally subdivided into four sections: model theory, set theory, recursion theory and proof theory." However, nowadays the state of affairs was changed a little, taking into account the significant role which logic plays in computer science. Hence, we observe an amplification of significance of proof theory and we see that problems of computability and complexities took the first place in recursion theory. It was regarded at the conference of Association for Symbolic Logic in Urbana-Champaign in June, 2000, namely at a special meeting "Perspectives of mathematical logic in the twenty first century." As a result of this meeting, the work of the four authors S. R. Buss, A. S. Kechris, A. Pillay, and A. Shore is published with the same title [20] in correspondence with the four main sections of mathematical logic.

Notice that in this work no directions of non-classical logic are mentioned and references to Russian logicians are extremely rare (it is typical for any western survey on logic). However, there is V. A. Uspensky's survey "Mathematical logic in the former Soviet Union: Brief history and current trends" [149].

A unique history of Russia in the 20th century has also predetermined a unique development of logic in it, in many respects this development is not clear for the western historians of science. In a totalitarian system, the "truth" becomes a subject of only ideological manipulations, and the lie and terror become the highest values. One of the features of the logic development in the former Soviet Union had consisted in the thesis about the union of logicians-mathematicians and logicians-philosophers. It is not surprising: in the Soviet political system, logicians-philosophers searched for a support of logicians-mathematicians who still had possibility for scientific publications even in foreign issues, as philosophical papers were published in the 20's and 30's years in the unique Soviet philosophical journal "Under the Banner of Marxism", where formal logic was compared to dialectics and dialectical logic. The first (formal logic) was declared bourgeois and the second (dialectic) proletarian; as a result, formal logic was betrayed to the anathema. The disastrous atmosphere of those years for philosophy and logic is circumscribed in Bazhanov's book [9], which unambiguously is named 'The Interrupted Flight' (see also [76]). Even when in 1947 formal logic was returned in the system of the secondary and higher education, its position in this system was not independent. Already in the beginning of the 50's years during the imposed controversy it has been fixed that the highest level of thinking is dialectical logic, the lowest is formal logic. Within this framework all 50's and even 60's years have passed in mutual polemic.

All this should be taken into account, if we want to conduct the comparative analysis of development of logic in our country and abroad.

2. Philosophy of logic or philosophical logic

Some tendencies of development of logic have been revealed at the end of the 20th century [75]. Here we shall be limited to philosophical logic.

Philosophical logic is extremely wide area of logical studies, requiring philosophical judgement of the basic concepts, used in modern logic, and the outcomes obtained by means of mathematical logic. However, we need to remark that the term 'philosophical logic' is uncertain and has no uniform use. In modern logic and philosophy, philosophical logic is understood by various experts variously and in their own way. Even if the philosophical logic is represented as a special scientific discipline, it is not possible to define its subject, limits of application and methods. Moreover, it is not possible to divide strictly the two different directions of researches: philosophical logic and philosophy of logic. Frequently, one is substituted another and occasionally one does not distinguish them.

The term 'philosophical logic' has appeared in the English-speaking logical-philosophical literature and has had a wide application already in the 50–60's years of the 20th century, when in the USSR logicians-philosophers have discussed what level of thinking (either formal or dialectical) they belong to. On the one hand, the crisis in the foundations of mathematics (detecting paradoxes in set theory and A. Tarski and K. Gödel's limitative theorems) has required a deep judgement of the most conceptual means of logic. On the other hand, the appearance and rapid development of non-classical logic, first of all modal logic, has drawn a wide attention of logicians with the philosophical orientation.

2.1. Philosophy of logic

Firstly, we shall sketch the area of logical studies which have the title 'philosophy of logic'. For logicians-mathematicians, philosophy of logic is a development of set theory and appropriate problems on the method of forming sets and on the nature of number. The detection of paradoxes in set theory and, in particular, Russell's paradox has raised the question about the nature of mathematics. *Logicism* tried to define the basic concepts of mathematics by means of logical terms (G. Frege in 1884 and B. Russell in 1903). It is both technical and philosophical problem: whether it is possible to infer all mathematics from some (or even one of) logical terms? In this sense, the grandiose construction, undertaken by Whitehead and Russell in 'Principia Mathematica', has appeared unsuccessful. And though in their logical-mathematical theory there were no paradoxes, but it has appeared impossible, for example, to infer the existence of infinite sets from only logical axioms. *Intuitionism*, as one more reply to the detection of paradoxes, has set up the problem about a distinction between the finite and infinite, e.g. about a difference of the potential infinity from the actual one. There was the problem of existence and substantiation of proofs; and it is the most important there was the problem of the status of classical logical laws. All this is philosophical problematics. D. Hilbert's formalistic program, in particular, his problem of *finitism*, has also caused a brisk philosophical controversy. One more method to avoid paradoxes in mathematics is *axiomatic set theory*. All these four approaches to **foundations of mathematics** require the deepest philosophical judgement (see [79], [44], [24], and also the book [110], in which one criticizes a philosophical basis of classical programs of foundations of mathematics). Chapter 7 in [57] is devoted to Hilbert's program within the framework of results of Gödel's theorems.

Actually, the aforementioned concerns more to philosophy of mathematics, than to philosophy of logic, but the problem of philosophical judgement of applying logic to a solution of various problems of mathematics remains. A convincing example here is K. Gödel's limitative incompleteness theorems (1931) which say that there is no adequate formalism, enveloping all mathematics, and in general this formalism is impossible. Philosophical corollaries of these outcomes are discussed up until now and have drawn a huge attention not only of logicians-professionals, but also of philosophers and methodologists. However, we can refer to papers of experts [32], [111], [137].

To this it is necessary to add also a philosophical controversy concerning the thesis of Church-Turing, asserting that all computers are equivalent among themselves.[1] If we consider a human brain as a computer, then there are no obstacles for a computerization of human logic.

It is interesting that mathematicians were occupied with philosophy of logic who have the deep results which have obtained in mathematics (G. Frege, B. Russell, L. Brower, K. Gödel, W. Quine, R. Carnap, etc.). Quine published in 1940 the book with the title 'Mathematical logic', and in 1970 with the title 'Philosophy of logic' [115] (it is reprinted in 1986) where logic is understood as a systematic study of logical truthes and philosophy of logic becomes a tool for the analysis of the natural language. The book contains the following sections which Quine refers to philosophy of logic: 'Meaning and Truth' (problem of sentences and propositions, sentences as the information, the theory of sense of language expressions, truth, and the semantic consent); 'Grammar' (the recursive setting of grammar, categories, the revision of the purpose of grammar, names and functors, the lexical criterion; time, events and verbs, propositional aims and modality); 'Truth' (definition of truth in Tarski's style, paradoxes in language, connection between semantic and logical paradoxes); 'Logical Truth' (in terms of structure, in terms of model, in terms of substitution, in terms of proof, in terms of grammar); 'Scope of Logic' (the word problem, set theory, the quantification); 'Deviant of logic' (namely, non-classical logics, first of all, many-valued logic, intuitionistic logic, branching quantifiers); 'Foundations of Logical Truth' (place of logic, logic and other sciences).

Thus, Quine has concentrated the work around of the main problem in philosophy of logic: what the truth is? However, only due to the development of mathematical logic, namely in A. Tarski's paper of 1933 [143] the semantic definition of truth for the big group of the formalized languages was given for the first time and the limits of such a definition were simultaneously indicated. To the further discussion concerning Tarski's definition of truth the special issue of the journal "Synthese" 126, Nos 1–2 (2001) is devoted. Problems of truth are considered in [35] in the context of philosophy of logic. The significant part of the book [138] is devoted to Tarski and Kripke's theory of truth and generally envelops a wide circle of problems of modern interest to the truth concept.

There is a special site 'Philosophy: Philosophy of Logic'[2]. Here it is

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[1] *Edit.*: More precisely, all conventional approaches to computability reduced to recursive functions or Turing machines (in Turing's words automated machines) are equivalent.

[2] http://dirt.netscape.com/Society/Philosophy/Philosophy_of_Logic/

possible to find many links concerning philosophy of logic. Also, notice the site of the Logical Sector of Institute of Philosophy of the Russian Academy of Science develops permanently too (http://iph.ras.ru/∼logic/).

Among recent monographs on philosophy of logic we remark S. Haak's books [58] and [59]. See also the monograph [117]. We pay attention to the site Factasia [70], created in 1994, where it is possible to find the universal philosophical approach to understanding of logic, to its significance and applications.

As it is noticed in the electronic 'Encyclopedia Britannica' (1994–1999): "The diversification of different logical semantics became central area of researches in philosophy of logic." Problems of logical semantics are considered in books [151] and [136], the first of them became classical. However, the main problem is a development of the uniform semantic approach, enveloping completely various logical systems, and as well as a contemporary analysis of various semantic concepts and their distribution on classes of logics. R. Epstein's monograph [36] (the 2nd edition in 1995) is devoted to this. In it so-called "set assignment semantics" are developed. The same fundamental work [37] is devoted to first-order logics. There is a site devoted to logical semantics with personalias, starting from G. Frege[3]. The existence of infinite classes of logics puts the problem on the semantic basis of logic in a new way. We underline that one of the most popular topics becomes a research of classes of semantics for which various non-classical logics are complete. As a result, model theory is transformed too. If initially it dealt with mutual relation between a formal language and its interpretation in mathematical structures, now logic becomes the tool for study of the most various structures and their classification (see [7]).

Certainly, the area of philosophy of logic is much wider. It includes the theory of the propositional form as sentences about some states of affairs in the world, generally, the doctrine about the logical form (see the monograph [121]), the doctrine about logical and semantic categories, the theory of reference and prediction, the identification of objects, the problem of existence, the doctrine about presupposition, the relation between analytical and synthetic judgments, the problem of scientific law, the informativeness of logical laws, ontological assumptions in logic and many other things. And even such, apparently, only logical problems concern to philosophy of logic: the essence and the general nature of deduction, the logical deducibility between any expressions or sets of expressions, the meaning of logical connec-

[3] http://www.phil.muni.cz/fil/logika/til/inks

tives, the significance of fundamental theorems, obtained in mathematical logic, and in this connection the careful analysis of such concepts as 'induction', 'computability', 'decidability', 'demonstrability', 'complexity' and besides 'truth'.

2.2. Philosophical logic

Initially, philosophical logic referred to modal logic, i.e. to the logical analysis of such philosophical concepts as 'possibility' and 'necessity'. Historically, these two concepts, especially since Aristotle, drew to themselves a constant attention of philosophers; and due to the development of symbolic logic, there is a unique possibility to explicate modality and their mutual relation by means of exact methods. This also concerns to such philosophical concepts as 'future' and 'past'. In modal logic one started to investigate new kinds of modalities: temporary, modal-temporary (synthesis of modal and temporary operators), physical or causal, deontic, epistemic, etc. Since the edition of Gabbay and Guenthner's 'Handbook of Philosophical Logic' [47] in the 80's years (hereinafter *HPL*), some results of development of philosophical logic are fixed. The 2nd and 3rd volumes content a consideration of various non-classical logics: in the 2nd volume one considers extensions of classical logic $\mathbf{C}_2$, for example, such as modal, temporary, deontic logic, etc., and in the 3rd volume one examines alternatives to classical logic, for example, such as many-valued, intuitionistic, relevant logic, etc. Notice that such a division of non-classical logic is not convincing. The point is not that there are logics which do not concern to one of these subdividings, for example, syllogistics, Leśniewski's logical systems, combinatory logic, infinitary logic, etc. It is a bit unexpected that logics, which originally were under construction as limitation of some classical laws and principles, actually are the extension of classical logic: for example, some many-valued logics as well as modal logics are extensions of $\mathbf{C}_2$. The matter is that the definition of non-classical logic is open yet. Therefore it is not surprising that the more neutral term 'non-standard (unconventional) logic' recently began to be used. There is a site with the short description of 29 non-standard logics and the short references to each of them [140].

Let us pay attention that in each of those logics there is an own philosophy of logic and as well as all aforementioned philosophical problems, because the definition of truth-value of formula, the logical deduction, the concept of sentence, and the meaning of logical operations are different for the majority logics. In each philosophical logic there is additional philosophical problematics. For example, in modal logics those are the problem of reference, cross-identification, i.e. identifications of objects in the various possible worlds, and in this connection there is the problem of quantification. In many-valued logics there is a very difficult philosophical problem of interpretation of the set of truth-values, usually expressed by numbers: rational, natural, real. Many philosophical problems are connected to intuitionistic logic, for example, the existence of two heterogeneous and irreducible to each other classes of semantics for it: realizedness and Kripke's models.

Philosophical logic has a language and technical means that are much reacher and more flexible than in symbolic logic; it allows to start the analysis and reconstruction of only philosophical problems and even such fundamental ones as the problem of logical and theological fatalism, determinism and contingencies, asymmetries of time, etc. (See [74]). An *introduction* to philosophical logic is contained in the monographs [159] and [56]. A unique monograph in Russian with the title 'Philosophical Logic' is written by A. Schumann [123], he understands philosophical logic as "general semantics of various logical calculi."

A modern understanding of philosophical logic is reflected in the collected works, basically representing surveys on the most important directions in modern philosophical logic [67]. It contains 46 papers in the following 14th sections:

I. "Historical development of logic";
II. "Symbolic logic and usual language";
III. "Philosophical dimensions of logical paradoxes";
IV. "Truth and the certain description in the semantic analysis";
V. "Concepts of logical deduction";
VI. "Logic, existence, and ontology";
VII. "Metatheory and orb and limits of logic";
VIII. "Logical foundations of set theory and mathematics";
IX. "Modal logic and semantics";
X. "Intuitionistic, free and many-valued logics";
XI. "Inductive, fuzzy, and quantum-probability logics";
XII. "Relevant and paraconsistent logics";
XIII. "Logic, mechanization and cognitive science";
XIV. "Mechanization of inference and detection of proofs".

Notice that the last section is especially indicative that it is referred to philosophical logic.

Let us pay attention that both philosophy of logic (see W. Quine and S. Haak's monographs) and philosophical logic study non-standard logics. In the latter case it becomes the obvious tendency exhibited already in the book of N. Rescher [118] and precisely designated, as it was already spoken, in the first *HPL*, namely referring the increasing class of non-standard logics to area of philosophical logic. This tendency amplifies in the new 18 volumes of *HPL* [48]. In these books one have already refused from the division of non-classical logics into extensions of $\mathbf{C}_2$ and alternatives to it. Another tendency consists in that one considers as philosophical logic all branches which directly do not concern the four sections of mathematical logic: model theory, set theory, recursion theory, and proof theory. Therefore it is not surprising that the paper with the title 'Algebraic logic' [2] is included in the 2nd volume of new *HPL*. The major area of researches in algebraic logic is a definition of necessary and sufficient conditions for construction of algebraic semantics, i.e. for constructions of Lindenbaum's algebra (algebra of formulas). An appropriate classical work is [14]. The fact that it is not possible for any logical calculus to construct Lindenbaum's algebra (for example, for well-known da Costa's paraconsistent logics Cn) became an additional stimulus of developing new semantic methods. Even earlier, the so-called 'valuation semantics' or 'bivalence semantics' (see [28]) has appeared in the beginning of the 70's years (N. da Costa, R. Suszko). If usually a function of value is an algebraic valuation, i.e. a homomorphism of algebra of formulas into an algebra of the same type, then now this limitation is removed and a value is a simple function, which associates one of two bivalent values with each formula, i.e. two-valued valuations are considered as characteristic functions of sets of formulas. There are some methods of the proof that any propositional logic has bivalent semantics.

Coming back to the problematics of algebraic logic, we underline that its means is a good tool for clearing up such a complicated question as mutual relation between various logical systems. In the book of P. Halmos and S. Givant "Logic as algebra" [61] is shown that standard outcomes in logic well correspond with known algebraic theorems. The famous Russian logicians A. Kuznecov predicted such a universal analysis of logic by algebraic methods in the fine article "Algebra of logic" for the Philosophical Encyclopedia [85], but even he could not foresee a wide line of applying algebraic logic. See the monograph "Algebraic methods in philosophical logic" [33], where the basic attention is concentrated on representation theorems as tool for completeness theorems. The same tool is basic for the study of formal phenomenology (!) n V. L. Vasyukov's monograph [154].

The basic present-day conclusion is as follows: *the logical laws are no other than algebraic laws*. All this happens within the framework of unreasonable revival of psychologism in logic in our country. For the last decade the big number of textbooks and handbooks on logic, and even the encyclopedic editions on philosophy containing the items on logic were published, in them one affirms (however, there are some exception, see, for example, [52] and [3]) that logic studies the laws of thinking. At the same time, not only the mathematical development of logic, but also somewhat philosophical development of logic shows that there are no more laws of thinking distinct from laws of algebra (see [27]).

Generally, the concept of philosophical logic is inconsistent. On the one hand, it includes, as it was already spoken, all logical researches which are not only mathematical. On the other hand, the modern development of modal, temporary, intuitionistic, and especially many-valued logics showed that they are no other than the sections of symbolic logic: the same methods of a symbolization and an axiomatization and in many respects the same technical problems and tasks. This also caused the construction of new theories of sets on the basis of non-classical logics, being on the origin only philosophical, namely many-valued, modal, relevant, paraconsistent theories of sets have appeared, in which one tries to deny corollaries implying from Gödel's theorems.

Notice that Zermelo-Fraenkel's system with the axiom of choice, but without the axiom of foundation is specially interesting, see, for example, [114] and [29].

3. Foundations of logic

Now we should pay attention to the main tendency of development of logic at the end of 20th century and the beginning of the 21st century. As well as the problem on foundations of mathematics has risen hundred years ago, so now there is the problem on foundations of logic. The following topics refer to:

(i) What is an inference?
(ii) What are logical concepts (operations)?
(iii) What is a logical system?
(iv) What is a logic?

In a very authoritative edition on the history and development of logic [80] (the 9th edition is in 1985) we find the following traditional definition of a subject of logic: "science which researches principles of correct or acceptable reasonings." However, such a definition does not solve the problem of exact area of the given subject, i.e. what area of applying logic is? For traditional logic it is *syllogistic* reasonings and there are equally 24 correct syllogisms. In turn, mathematical logic studies *mathematical* reasonings: "If his researches are devoted first of all to study of mathematical reasonings, the subject of his investigations can be called mathematical logic" [97]. Informal logic studies *informal* reasonings, and philosophical logic, as a result, studies *philosophical* reasonings. In order to avoid similar senselessness, it is necessary to select the nucleus or base concepts with which the given science deals.

Such a nucleus undoubtedly is the concept of 'logical inference (consequence)'. A. Tarski in 1936, as one of creators of modern logic, sketched its essence in the work with the characteristic title "On the concept of logical consequence" (see [114]). However, we can add there the methodological aspects: in what terms it is or what paradigm of the offered answer is. Approaches to the answer concerning an orb of logic, its basic concepts, which are used by the conception of logical inference, may be completely various: model-theoretic, semantic set-theoretic, proof-theoretic, constructive, combinatory, etc. As we shall see, A. Tarski's answer is within the framework of the semantic approach: "A proposition X logically follows from propositions of the class $\mathbf{K}$ if and only if each model of the class $\mathbf{K}$ also is a model of a proposition X" [141].

Nowadays Tarski's concept of logical consequence is regarded as debatable. Tarski's work has more philosophical, nontechnical character and allows to interpret it in various conflicting way, for example, there is an opinion that Tarski's definition is incorrect from the point of view of modern mathematical logic [26] or that it should be generally rejected [38] and [39]. An interesting analysis of Tarski's work is proposed in [120], where Sagöillo examines three basic concepts of logical inference, each of them envelops an important part of argument and each of them is accepted by logical community. The conclusion of the author is interesting too that Tarski does not speak, what the logical consequence is, and considers what the logical consequence is similar to G. Ray [116] tried to defend Tarski's conception in his big article (see the reply in [63]) and as well as M. Gómez-Torrente defends this conception [54] and [55].

The basic objections against Tarski's definition of the concept of logical inference are as follows. Anywhere in the given work it is not stipulated that the data domain should vary, as it is in modern logic (see [26]). Logical properties, in particular the general validity of the argument of logical inference, should be independent of a separately selected universal set of reasonings, in which language appears interpreted. Otherwise, many statements about a cardinality of data domain at a special interpretation of language can be expressed only by means of logical constants and, as result, they should appear logical true. However, Tarski himself considers the idea of the term 'logic' as excluding among logical trues any statements about a cardinality, let even of logical area. Another objection is directed against Tarski's acceptance of the ω-rule (the rule of infinite induction) at formalizing first-order arithmetic. However, actually it was only a version of this rule in the simple theory of types. In connection with these objections it is necessary to make some general notes. Tarski knew very well Gödel's works about the completeness, where the theorem is proved on the basis of truth of statements at all possible interpretations, and as well as about the incompleteness (ω-incompleteness) of first-order arithmetic. In the first case one showed a concurrence of logical consequence in the first-order classical logic (hereinafter by **PC**) with syntactic consequence, in the second case one did not. From Tarski's works it clearly follows that he considers the logical consequence and deductability as various concepts and the first as much wider than the second. The basic intention of Tarski was to define the logical inference, applied for very wide class of languages, so wide that, as we shall see further, there are the problems of already other level relating the item (iv).

For now notice that the concept of logical consequence has taken the central place in logic and therefore the following problem seems to be very important: *What does this mean for the conclusion A to be inferred from premises* Σ? The following criterion is considered conventional: A follows from premises Σ if and only if any case, when each premise in Σ is true, is the case, when A is true. Pay attention that the famous Russian logician A. A. Markov connects this principle to the definition what logic is: "Logic can be defined as a science about good methods of reasonings. As "good" methods of reasonings it is possible to understand ones, when from true premises we infer a true conclusion" [93]. As a result, the essence of logical inference is a preservation of truth in all cases. There are many ways, when using Tarski's concept of logical consequence, it is possible to represent all laws of **PC** as valid. Thus, we obtain a standard definition of classical logic together with all its logical operations. For instance, the conjunction of two formulas $A \wedge B$ is true at a situation (in a possible world) w iff A is true in w and B is true in w.

But we have there much more problems. Why we call the obtained logic classical and what does this mean? We consider this problem still. What does it mean, the standard setting of true conditions for logical connectives? Finally, what should we consider as logical operations? The concept of truth is directly connected to the understanding of logical inference, given by Tarski, and all together results in objects which we call 'logical laws': *they* *are deductions preserving the truth*. But how we can define the logical law, not having defined what we should consider as logical constants (operations), while we have a natural variability and instability of nonlogical objects of reality. If we consider all objects as logical terms: variables, numbers, etc., then a model-theoretic interpretation of each term should be fixed and, therefore, only one model should exist. It would make the concept of logical truth empty.

In the work "On the concept of logical consequence" Tarski keeps the problem open what should we consider as logical concepts (operations), and what as out-logical? Tarski writes that he does not know any objective basis for strict differentiation of these two groups of terms [141]. It is obvious that this problem did not give rest him and in thirty years he comes back to it in the lecture "What are logical notions?", read in 1966 in London Bedford College, in the same year in the Tbilisi Computer Center, and later in SUNY Buffalo in 1973. The report is published after Tarski's death in 1983 (see [142]). The basic idea consists in that logical notions (concepts, operations) should be invariant in respect to an appropriate group of transformations of reasoning area. Tarski extends an area of applying F. Klein's program, where one proposed a classification of various geometries in accordance with the space transformation, when geometrical concepts are invariant. For example, concepts of Euclid's metric geometry are invariant relatively isometric transformations. In the same way, algebra can be considered as study of concepts, invariant relatively automorphisms of such structures as rings, fields, etc. Then according to Tarski, logical concepts are invariant relatively any one-to-one transformations of the universal set onto itself, i.e. relatively any permutations of reasoning universe (data domain). Implicitly, this idea of an invariant permutability was already contained in various logical-mathematical works (for the first time [100]), in linguistic works (see [77] and [152]), in philosophical works (see [107], [94], and [131]), and as well as in the collected works with the rather actual title "The limits of logic" [126]. Tarski's thesis was a basis with some natural updating for definition of logicality in G. Sher's book [130]: the operation is logical if it is invariant relatively each bijection between areas.

Finally, in the work [96] it is shown that if Tarski's thesis is accepted, then logical operations are defined in the full infinitary language $L_{\infty,\infty}$, (in the same work there is a generalization in Sher's style, i.e. it is given a characterization of logical operations relatively isomorphic invariancy). Recall that the language $L_{\infty,\infty}$ is a language of conventional first-order logic with equality (Frege's language), but admits conjunctions and disjunctions of an arbitrary length and as well as an arbitrary length of sequence of universal and existential quantifiers. This language is very rich – it contains the whole first-order logic. The latter allows us to set a quantification on arbitrary functions, defined on areas of reasoning, as well as a usual quantification over members from this area. Since sets and relations can be represented by their characteristic functions, then the second-order logic envelops also a quantification on arbitrary sets and relations. Not only arithmetic, but also set theory are included in the second-order logic (natural numbers, sets, functions, etc. are there logical concepts), as a result, all set-theoretic problematics, including the continuum hypothesis and many other important mathematical statements, are contained in the second-order logic (see the monograph [91]). Thus, *mathematics is a part of logic.* Depending on expressive means of new logic, we come to logic of natural numbers, logic of real numbers, logic of topological spaces, etc.

In connection with these problems S. Feferman's article seems to be very interesting [40]. In this article Feferman criticizes the thesis of Tarski-Sher and one of objections is that there is an assimilation of mathematics by logic. But the main objection consists in that the thesis of Tarski-Sher does not give any natural explanation, how logical operations behave on data domains of the various cardinality. Therefore Feferman introduces the concept of operations which are homomorphic invariant on functional-type structures. Such operations, according to Feferman, are logical and, it is the most remarkable, they exactly coincide with operations of the first-order logic without equality. However, here again there is a problem whether the equality may be considered as a logical operation? See the discussion of this problem in [115], where Quine is declined to the positive answer, as a reason (among other things) he says about the deductive completeness of the first-order logic with equality. As a value of this approach, Feferman considers that the operations of **PC** are defined in terms of homomorphic invariant operations of one-place type. Thus, he refers to [78], where the central role of one-place predicates in human thinking is shown by the example of the natural language.

There would be strange, if the exact characterization of **PC** in terms of its operations appears only in 1999. Actually, already in the 60s years A. V. Kuznetsov generalized the theorem of functional completeness of propositional logic in the predicative case. Unfortunately, this proof is not published still. Much later this theorem was proved in [161], i.e. it is shown that the certain set of logical operations is adequate both for **PC** and for **PC**(=). The preference is returned **PC**(=). The author follows from the basic assumption that to be considered as logical operation, its meaning should be completely contained in axioms and inference rules. Thus, differently from the semantic approach of Tarski-Sher-Feferman the proof-theoretic approach is used for a characterization of logical operations.

The characterization of logical operations entails the characterization of the logic as a whole. However, the characterization of **PC** can be given in terms of fundamental model-theoretic properties of the theory T in the first-order language. These properties are:

The **theorem of compactness** (for countable languages). *If each finite set of propositions in T has a model, then T has a model.*

The compactness takes place, as only the finite set of premisses is used in deductions. This property was revealed by K. Gödel in his paper about the completeness of **PC** (1930). The two other properties of the first-order logic were proved earlier.

I. Löwenheim-Skolem's theorem. *If T has a model, then T has a denumerable model, too.*

II. Löwenheim-Skolem's theorem. *If T has an infinite model, then T has an uncountable model.*

Much later P. Lindström [86] showed that these properties are characteristic for *PC* in the following sense:

Lindström's theorem. *The first-order logic is the only logic closed in respect to $\wedge$, $\neg$, $\exists$ and satisfying Löwenheim-Skolem's theorems and the theorem of compactness.*

Lindström's paper began paradigmatic for the major researches in logic of the last quarter of the 20th century. In essence, Lindström's theorem defines the first-order logic, more precisely **PC**(=), in terms of its global properties. But a serious limitation on expressive means of the first-order logic follows from these properties. The most simple infinite mathematical structure is built by natural numbers and the most fundamental mathematical concept is the concept of finiteness. However, from the theorem of compactness it follows that central concepts such as a finiteness, denumerability, well-orderedness, etc. cannot be defined in first-order logic. Actually, the finiteness is not distinctive from the infiniteness. In turn, from Löwenheim-Skolem's theorem it follows that the first-order logic does not distinguish the denumerability from the uncountability and, hence, no infinite structure can be described to within isomorphism. Moreover, many linguistic concepts, distinctions and constructions are beyond applications of **PC** (see [101], [88]).

There is a lot of interesting logics, which richer than the first-order logic such as the weak logic of the second order which tries to construct the concept of finiteness in logic in the natural way (it allows to quantify over finite sets); logics with various extra-quantifiers such as 'there exists finitely many', 'there exists infinitely many', 'majority', etc.; logics with formulas of infinite length; logics of the higher-order (see [150]). However, it doesn't matter how we extend the first-order logic – in any case we lose either the property of compactness, or Löwenheim-Skolem's property, or both and as well as we lose the interpolation property and in most cases deductive completeness. However, G. Boolos [17] protecting the second-order logic, asks: Why the logic should necessarily have the property of compactness? It is interesting that we find a similar question in 1994 on pages of 'The New Encyclopedia Britannica': Why Löwenheim-Skolem's property should correspond to the internal nature of logic? (Vol. 23, p. 250).

The construction of various extensions of **PC**, especially logics with the generalized quantifiers, drew the big attention of linguists, mathematicians, philosophers, cognitivists. A total of development of this direction is reflected in the fundamental work 'Model-theoretic logics' [7], where Barwise comes to the following conclusion: "There is no back way to the point of view that logic is first-order." The authors of monographs [126] and [130] are of the same opinion too.

However, the second-order logic is too complicated. Second-order logics are not recursively enumerable deductive systems. The basic problems arise with logical trues. For example, there are statements which are logically true if and only if the generalized continuum hypothesis holds. All these difficulties and many other are an inevitable corollary of a huge potency of expressive means of second-order languages. Therefore it is no wonder that there are weak versions of second-order logic, and in new *HPL* we find the article 'Systems between first-order and second-order logics' [129]. It can be achieved due to limitative versions of understanding schemes ($\sum_1^1$-formulas and $\prod_1^1$-formulas), to limitations on the axiom of choice and limitations on the principle of induction for arithmetic, etc. Another instance is monadic second-order logic. As a rule, the majority of these languages characterizes the concept of 'finiteness' and allows a categorical characterization of natural numbers. Thus, deductive incompleteness is a characteristic property of these systems.

Probably, one of the most interesting paper belongs here to J. Hintikka [64], the paper with the title 'Revolution in logic?' [65] (and as well as the whole complex of Hintikka's works connected to application of the created by him **IF**-*logic* (Independence Friendly)) is very interesting too. The basic idea of Hintikka consists in comprehension that quantifiers of the standard first-order logic are dependent. The latter means that if we deal with expressions such as "for all x there are some y such that $R(x,y)$", then the choice of y is not independent, and it is determined by the choice of x, in other words, between x and y there is a functional dependence. The feature of **IF**-logic is its incompleteness that means impossibility to give the list of axioms from which all significant formulas of the first-order **IF**-logic can be obtained only by using formal inference rules. But at the same time it satisfies the properties of compactness and Löwenheim-Skolem (about properties of **IF**-logic see also [122], [34]).

Probably, it is necessary to agree with J. van Benthem and K. Doets [150] that there is no *sacred* logical theory. It is possible to consider it as the answer to A. Tharp's article 'Which logic is the right logic?" [146].

However, the topic of abstract logic and general-theoretical problems of the substantiation of mathematics recede into the background before new tendencies in the logic development of the end of the 20th century. The logic becomes more vital in the computer science, artificial intelligence, and programming. The similar application of logic generates the big number of new logical systems, but already aimed directly to their practical use. In particular, this entails the publication of the collected works (in England and in one year in the USA) with the title 'What is a logical system?' [49].

Generally speaking, the problem is formulated as follows: whether there is the one "true" logic and in the converse case if not, how we can limit our understanding of the logic or, more concrete, of a logical system? There are also other problems: whether there is a real distinction between syntax and semantics from the point of view of applications? And, certainly, there is a problem on traditional properties of logical systems: the completeness, elimination of cut, interpolation property, etc.

Even more problems arises with the extension or reducing of classical propositional logic. It is known that we have an infinite (*uncountable*) number of such logics (logical systems). The first outcome of a similar sort belongs to the Russian logician V. A. Jankov (1968) and concerns a cardinality of the class of extensions of intuitionistic logics. Then this fact hasn't been realized with all problems implying from here. Now the discovery of the continual class of logics is the most ordinary thing.

The unusual diversity of logical systems is generated, on the one hand, by serious criticism of "basic" and not only basic laws of propositional logic, on the other hand, by almost unlimited extensions of the concept of the logical truth (in essence this process is an inverse to the first), and also by various specifications of the concept of logical inference and by the development of computer science. All this brings us to the most important problem: **What is a logic?**

R. Jeffery's note in the book with the rather remarkable title [69] seems to be a bit interesting that Tarski's definition of logical inference does not allow us to define, what logic is, because we should take into account cases, included already in the definition of logical inference. We can consider cases as the "probable worlds" and then we have problematics connected with, what the "possible worlds" are? (see the interesting monograph [18]). Moreover, our cases can be considered as situations in Barwise and Perry's sense [8]. Situations can be regarded not only as incomplete parts of the world, but as contradictory, and also as both incomplete and contradictory. As a result, we obtain completely new logics, in essence distinct from classical such as intuitionistic, relevant, paraconsistent, paracomplete logics, etc.

If the essence of logic consists in a preservation of the designated truth-value in all cases, then various logics are obtained by various explications of these cases. So, the approach appeared in logic, called 'logical pluralism' (see [12]). The Internet project 'Logical pluralism' (http://pluralism.pitas.com/) is also created with G. Restall's participation. Actually, the logical pluralism existed in logic before the serious analysis of A. Tarski's understanding of logical deduction, namely it began in the criticism of basic laws of logic started in the beginning of the 20th century by L. Brower, N. A. Vasiljev, and J. Łukasiewicz. Furthermore, the understanding of logic had another tradition rather distinct from Tarski's and starting from G. Frege and B. Russell.

The definition of logic given by Frege is unusually beautiful: "Logic is a science about the most general laws of the existence of true" (see [45]). It seems to be a little surprising that such an understanding of logic held on almost hundred years and after a small modification come in the basis of Quine's aforementioned book 'Philosophical Logic', where a subject of logic, recall, is the "systematic study of logical trues." This almost centenary period of similar comprehension of logical subject was called by M. Dummett 'logicism's dominance'. However, the development of computer science entails the change of the paradigms in logic.

It is necessary to notice that the traditional approach to the understanding what a logic is seems to be very attractive in respect to the possibility to define logic by means of its basic laws. From the modern point of view the 'logical law' means the 'theorem of a formal system'. Without details, what a formal system is and what a proof in it is, we can consider laws of logic as preserving truths. Such an understanding of logical laws was proposed by Aristotle, but here we collide a problem of unusual complexity: What is a true? At the end, we can try to agree what we have to regard as logical laws concerning our understanding of truth. Various concepts of truth, for example, correspondent, coherent or pragmatical, apparently, do not dictate the specificity of this or that logic. But already the case of intuitionistic understanding, what a true is, speaks about irreconcilability of standpoints. In the first interpretations of intuitionistic logic (before appearance of Kripke's semantics), the concept of truth does not arise at all. However, there is again a problem, how we can define the logical laws, not defining before what logical constants (operations) are. As we saw, the extension of the classical first-order logic entails that the set of logical trues is not, certainly, axiomatizable.

Exactly in hundred years after the appearance of G. Frege's well-known work 'Concept Calculus' (1879) (see [45]), in which predicates, negation, conditional, and quantifiers are introduced as the basis of logic, and also the idea of formal system is introduced, in which demonstrating should be carried out by means of obviously formulated syntactic rules, – after hundred years of the triumphal development of logic as the independent science calling the worship, surprise, and occasionally bitter dismissal and even revenge for its former adherents and the mystical fear for the majority of others, suddenly there is J. Hacking's article under the title 'What is a logic?' [60]. Hacking highly evaluates G. Gentzen's introduction of structural rules, because the operation with them allows us to express the aspects of logical systems which have no direct relation to logical constants. This important discovery is made by G. Gentzen in 1935 (see [51]). The presentation and development of logic by the way of sequent calculus, where the principles of deduction are set by the rules, permitting to pass from one statements about the deducibility to others, allowed Hacking to define logic as science about deduction. (Pay attention that we find such an understanding of logic in V. A. Smirnov's book [133] reprinted in [135].) Therefore there are some reasons why Hacking's article is in the beginning of the collected works under the title 'What is a logical system?' [49].

J. Lambek's paper under the title 'What is a deductive system' [49] was published the same year, Lambek considers the following five styles of deductive systems: (1) Hilbert's style (deduction of the form $f :\to B$, for a formula B; (2) Lover's style ($f : A \to B$, for formulas A and B); (3) Gentzen's intuitionistic style ($f : A_1 - A_m \to B$); (4) Gentzen's classical style ($f : A_1 \dots A_m \to B_1 - B_n$), and (5) Schütte's style ($f :\to B_1 \dots B_n$). Lambek prefers Gentzen's style by virtue of introduction of structural rules. Notice that Lambek pay attention on equalities between deductions. In this connection recall that in G. Mints book [99] (now he works at the Stanford university) the deductive system **HCC** of Hilbert's type contains a definition of equivalence relation for deductions. It converts **HCC** into the closed category: formulas are objects, and equivalence classes of deductions are morphisms.

The similar approach to the proof theory became especially actual under influence of the category theory and computer science.

4. Computerization of logic

V. Carnielli in the review on [49] puts forward the basic supposition: "There are no proofs, there is no logic" [21]. The proof theory recently draws to itself much more attention (see, for example, the two-volume book [124]). At the same time, there is the site 'Proof theory on the eve of the year 2000', created by S. Feferman [41], where 10 problems are formulated and the well-known logicians, working in the given area, are proposing some solutions.

However, in the last quarter of the 20th century the proof theory has undergone significant modifications and directly began to be applied in computer science. We mean here an automatic scan of proofs. In our country the research in this area was carried out since the 60s years in Ju. M. Maslov's group. The book [25] became the classical monograph devoted to the automatic proof, translated into Russian, in this book the method of resolutions for the first-order logic develops (see also the monograph [50]).

Within two last decades many theoretical ideas of the automatic proof have been embodied in computer programs, so-called provers. These programs carry out the search of deductions in various logical calculi. So, in the middle of the 80s years in the Aragonne National Laboratory in the USA the resolutive prover OTTER was created for the first-order logic, its description is in [95]. Up until now its creators work at the development of the program and improve of speed in its separate parts. In 90s SCOTT appeared (see the report [132]) – the program of the Australian Project of Automatic Proof, the set including OTTER and permitting to use semantic limitations and therefore essentially to reduce an operating time of the program during construction of deductions.

The Russian logicians, employees of the Philosophical Faculty of the Moscow State University and employees of the Institute of Philosophy of the Russian Academy of Science, have written their interactive prover DEDUCTIO which is described in detail in [135]. The distinctive advantage of DEDUCTIO consists in the wide area of its possible usage: the axiomatic deduction, natural, analytic-tableaux. There is the site with the bibliography on provers, created in Canada (1997–2001), containing more than 3000 references[4].

[4] http://www.ora.on.ca/biblio/biblio-prover-html

Applying logic in computer science became so wide that it is possible to speak about the main phenomenon in the development of logic of the end of the 20th century. So, the term ‘computing logic’ and later ‘computer logic’ appeared in the 70s years.

The creation of artificial intelligence is a special theme. The American Association Artificial Intelligence, issuing the journal ‘Artificial Intelligence’ and organizing annual international conferences, symposiums and summer schools (http://www.aaai.org/) is started in 1979. In books [148] and [30], various non-standard logics are proposed for artificial intelligence. The two books (but with different titles) in French are devoted to the logical approach to an artificial intelligence, published in 1988 and 1989 (see their translations in Russian: [144] and [145]). About the logical-and-philosophical approach to artificial intelligence see the collected works [147]. Pay attention to multi-volume handbooks [1] and [46].

The creation of artificial intelligence (hereinafter *AI*) passed from obsession to the plane of serious discussions and became a fundamental problem: whether the logic can really become the basis of *AI*? Here it is necessary to mean that logical deduction is a discrete process, while the human thinking isn’t.

There are supporters of the ‘strong’ conception of *AI* (mechanists), asserting that the human brain (reason) can be precisely simulated by a discrete (digital) computer or a Turing machine. The most known criticism against mechanists belongs to J. Lucas [89], his philosophical article was repeatedly reprinted. Lucas uses basically Gödel’s theorems of incompleteness, asserting the existence of absolutely insolvable arithmetic propositions. According to Lucas, this essentially limits a computing sphere of computers. J. Webb in the book [158] appeals to the efficiency of Gödel’s result, concluding that Gödel for the first time has shown that from the statement “I can find limitations in any computer” it undoubtedly follows that “I am not the computer.” The known physicist R. Penrose is of the same opinion [108] (the book is translated in Russian in 2003), who, among other things, including physical arguments, is also based on the insolvability of the decision problem for mass problems, i.e. on the absence of a uniform algorithm for solution of mathematical problems (it is proved by A. Turing in 1937, and later by A. Church in 1941). Lucas and Penrose give reasons that there are human procedures (computing methods) which cannot be simulated by a Turing machine. But if the abilities of the human reason exceeds any computer, then the reason somehow comprehends trues unavailable to the computer. The same opinion belongs to Gödel in his unpublished works (see [156], [157], and also the third volume [53]). However, the problem consists in finding obvious examples of similar computing processes.

There is the big literature subjected Lucas and Penrose's viewpoint to criticism (see, for example, [105] and also D. Hofstadter's book [66], involved the significant attention and translated into Russian in 2001). After three decades Lucas [90] strengthens, or tries to strengthen, his standpoint; on the other hand, Penrose devotes more than 200 pages to the replies to the critics and as well as devotes them to the invention of rather intriguing arguments in the new book [109]. It is interesting that both mechanists and anti-mechanists understand and accept the power and universality of Gödel's limitative theorems. But there is a little bit paradoxical impression that for the first this means a limitation of human computing abilities, and for the second otherwise: computing abilities of the person are much more difficult than ones of the computer and the human reason also operates with abstract objects (see [83]). Let us especially pay attention to the recent work of the famous logicians S. Shapiro [128], where arguments of contending parties are in detail analyzed. Here it is marked that an extension of the human computing abilities implies that the human reason becomes not only *infallible*, but also *omniscient*. Recall that the fierce discussion in the Middle Ages concerning compatibility of Christian dogmas about God's omniscience and the human free wills proceeds up until now in theological studies, therefore it is clear that a lot of other philosophical problems should be decided in parallel. At last, let us refer to P. Benacerraf's interesting reasoning in the article 'God, the Devil and Gödel' [13]. If the idealized versions of human beings are Turing machines, then they are not capable to execute Socrates' statement: "Learn itself." If the ideal person is a Turing machine, then he cannot know what kind of Turing machines he belongs to (according to Church-Turing's thesis, all Turing machines are equivalent).[5] Hence, there

[5] *Edit.*: More precisely, Turing's automated machines are equivalent approaches to computability. Notice that the conventional name of Turing machines actually refers, in Turing's words, to automatic machines, or a-machines. He also proposed other models of computation: c-machines (choice machines) and u-machines (unorganized machines). Turing argued for the claim (Turing's thesis) that whenever there is an effective method for obtaining the values of a mathematical function, the function can be computed by a Turing a-machine. At the same time, Church formulated the following thesis: a function of positive integers is effectively calculable only if it is recursive. If attention is restricted to functions of positive integers then Church's thesis and Turing's thesis are equivalent. It is important to distinguish between the Turing-Church thesis and the different proposition that whatever can be calculated by a machine can be calculated by a Turing machine. The two propositions are sometimes confused. Gandy termed the second proposition 'Thesis M': whatever can be calculated by a machine is Turing-machine-computable (see *Gandy, R. Church's Thesis and Principles for Mechanisms, [in:] Barwise, J., Keisler, H. J., Kunen, K. (eds). The Kleene Symposium. Amsterdam: North-Holland, 1980*).

is a classical problem about limits of human knowledge and, certainly, about limits of logic.

There is a strong demarcation line between reasonings of artificial intelligence and reasonings of human intelligence. Most likely, it is impossible to overcome this line, but the development of logic, which basic function is an *approximating* of various methods of human reasonings, takes place on the infinite path of overcoming this line. A limiting case of approximating (and while the most effective and fruitful) just also is the formalized deductive method implemented in computer programs.

There exist, certainly, other methods of approximating which also develop: hypothetic-deductive method, induction and abduction, formalizing of probable reasonings (see [42]). Recently non-monotonic logics develop, too. Non-monotonic reasonings, differently from classical, intuitionistic, classical-modal, etc., allow to operate adequately with the incomplete and changed information. The international school-seminar on non-monotonic reasonings[6] proceeds since 1994. Notice the big survey [19] and monographs [5] and [15]. Let us also pay attention to different argumentation theories (notice only the work [43] as the most approximate to the formal-logical modelling).

However, the future belongs to a computerization of logic and to its applications in computer science. We yet do not know completely what it is possible to wait from new computers for: quantum, neural, etc. Pay attention to rather remarkable fact. In 1960 the Nobel winner E. P. Wigner wrote the article about the difficultly explained efficiency of mathematics in natural sciences, following Galilei's words that "The book of the nature is written in the language of mathematics." Something similar corresponds to the attitude of logic to computer science. Now the concepts and methods of logic take one of the central places in computer science and it can even be called calculus of computer science. The article of the six American logicians [62], published in the beginning of the new century, is devoted just to this theme.

In D. Gabbay's preface to each volume of new *HPL* it is fairly noticed that the previous *HPL* became the Bible for the logical community. The basic intention of the new issue is that an exceptional value of logic in computer science, in the development of the formalized (computing) languages such as combinatory logic and λ-calculi and in artificial intelligence is shown in the most complete measure. Gabbay predicts that the day will be

[6] http://www.holds.medg.lcs.mit.edu/nm/

coming, when the scientist in the field of computer science will wake up with comprehension that his professional sort of activity belongs to the formal (symbolic) philosophy.

5. The researches financed by the Russian Foundation for Basic Research

The basic researches in the field of philosophical logic are fulfilled by the Russian scientists beyond the framework of the Russian Foundation for Basic Research (RFBR), which works only 17 years. Nevertheless, it is already possible to examine some tendencies.

First of all, there is a possibility of republishing works of well-known Russian logicians within the framework of appropriate research projects. For the first time the most important works are collected in the separate issues with the detailed introductions, extensive comments and the complete bibliography. It is (in the chronological order) the grant of the RFBR N 97-06-80360 (the chief A. A. Anisov). Here the comments have been prepared for V. A. Smirnov's monograph 'Formal inference and logical calculi'. In comments there is the comparative analysis of V. A. Smirnov's ideas and results in the field of the inference theory with works of representatives of other schools. The work is completed by the publishing grant of the RFBR N 99-06-87071 [134]. The comments were written by V. M. Popov, P. I. Bystrov, A. V. Smirnov and V. I. Shalak. V. A. Smirnov's articles, directly related to the inference theory, are also included in the issue. The works [102], [103], supported by grants N 97-06-80211 and N 00-06-80142 (the chief N. M. Nagornyj) are directed to the solution of the important problems of the comparative analysis of concrete results of one of the best-known present-day math-philosophical programs – 'Markov's constructivism' – with other conceptual programs such as 'Cantor's set-theoretic program', 'Brouwer's intuitionism' and 'Hilbert's proof theory'. Researches on the given project were some kind of summarizing to preparation of he two-volume issuing of A. A. Markov's 'Selected Works'. The publication of the two-volume book [92] is supported by the grant of the RFBR N 00-01-14195. M. N. Nagornyj was an editor, the introduction and comments also belong to him. The preparation of a commented publication of A. G. Dragalin's works on logic and philosophy of mathematics is supported by the grant of the RFBR N 00-06-80122 (the chief E. G. Dragalina-Chernaja), too. The two-volume issuing [31] and [82] is supported by the grant of the RFBR N 01-06-87068. G. E. Mints was the editor-in-chief and he also prepared comments, N. N. Nepejvoda wrote the introduction. In the second volume, A. N. Kolmogorov and A. G. Dragalin's textbook on logic is reprinted.

The other portion of grants of the RFBR supported researches in the field of non-classical logics. It was an international project INTASS–RFBR 95-8365 (the chief A. S. Karpenko). The project was aimed to solve problems of application of paraconsistent logic to philosophy, artificial intelligence and computability. Only the Russian participants published more than 40 articles. Consider some works published in the international journals. In [73] it is shown that a combination of two three-valued isomorphs of classical logic, which are contained in Bochvar's three-valued logic of senselessness, entails a paraconsistent logic (Sette's logic) and its dual 'weak intuitionistic logic'. In [113] it is established that the implicative logic with inverse negation which is defined in the pure implicative language, is paraconsistent. In [153] we can find the category approach to paraconsistent logics. Participants of the project have taken part in the 1st international congress on paraconsistent logics (Gent, 1997) [106] and in the international conference, devoted 50th years of publishing S. Jaśkowski's paper, in which the system paraconsistent logic is considered for the first time (Toruń, 1998) [139]. (Though actually the first system of similar logic has been considered by A. N. Kolmogorov in 1925 [81].) The special issue of one of the first electronic scientific journals in Russia 'Logical Studies' (A. S. Karpenko is the editor-in-chief) [87] is devoted to the problematics of paraconsistent logics, too.

The project of the RFBR N 00-06-80037 (the chief A. V. Chagrov) is directed on solution of a fundamental problem of the modern logical science connected to finding-out of ratio between non-classical logical systems, oriented on the description of information processes, and the comparative analysis of their expressive means. The old algorithmic problem of the matrixness and finite approximability of the normal modal logics, set by matrix logics, is solved in [22]. In [119] the problem of the complicated description of modal logics and their superintuitionistic fragments is solved by the limitation of the number of used variables. In [23] one considered problems of including the basic propositional logic and its extensions to modal logics, in particular, including formal propositional logic into Gödel-Löb's modal logic of demonstrability. The new relational semantics for extensions of the basic logic was constructed during this research. In [23] one built the propositional logic **LAP** with modified semantics of generalized, according to E. K. Vojshvillo, states and one defines the operations, including the classical propositional logic into **LAP**. In [72] it is shown that Kleene's regular operations are inexpressible in Yuriev's three-valued logic $\mathbf{Y}_3$, intended for formalizing the formal neuron, and this logic is essentially non-monotonic.

The works [10], [155] and [84] are supported by the grant of the RFBR N 97-06-80191 (the chief S. O. Kuznecov). The problem of simplification of formulation of DSM-method of automatic generation of hypotheses, based on formalizing rules of J. S. Mill's inductive logic, has been solved. The language of the stratified logical relational programs was used for this purpose. Another approach assumes to use the language of the naive set theory. One more result: the general effective method of axiomatization of algebra classes, corresponded to J-definable J-compact logics (many-valued logics), is described; the completeness theorem of appropriate calculi concerning these algebra classes is proved.

The project of the RFBR N 98-06-80177 (the chief A. S. Karpenko) is devoted to the special class of many-valued logics, namely to Łukasiewicz's finite-valued logics $Ł_n$, arisen from the problematics of logical fatalism. Within this project the corollaries of V. K. Finn's theorems about the number-theoretical nature of logics $Ł_n$ are proved: n is a prime number if and only if the set of functions of logic $Ł_n$ is a precomplete class (see [16]; later, this theorem twice rediscovered abroad). The obtained results have been summarized in the monograph [71], its publication was supported by the grant of the RFBR N 00-06-87014. The solution of the basic problem consists in the following: it is given a characterization of various classes of natural numbers (prime numbers, degrees of prime numbers, odd numbers, even numbers) by means of logical matrices. One of corollaries was a discovery of the law of generation of classes of the prime numbers, supplied with an appropriate computer program.

The interesting researches, published in [105], [104], were supported by the grant of the RFBR N 98-06-90205 (the chief N. N. Nepejvoda). In the first of them it is shown that the applied theories, based on superintuitionistic logics with Carnap's constructive rule, are classical, i.e. we can infer there the law of excluded middle. In the second work one researches the phenomenon of the non-formalizable, for the first time drawn attention after Gödel's theorem of incompleteness of formal theories.

Finally, notice the project supported by the RFBR N 00-06-80149 (the chief V. A. Bazhanov), in that the evolution of university logic in Russia in the period from the 19th to the middle of the 20th centuries was considered on the basis of the analysis of the different sources. The publication of the book [10] and the detailed work about I. E. Orlov [11], who is the founder of relevant logic, became a result of this research.

6. Conclusion

Let us pay attention to the obvious tendency that at the last time more professional math-logicians (mathematicians) began to study philosophical logic. The technical means of non-classical logic is more and more improved and becomes complicated. It is clear, as the future of logic, including philosophical logic in its modern understanding, is connected to computer science. In a word, we can see a mathematization and algebraization of philosophical logic. The half of the projects supported by the Russian Foundation for Basic Research and regarded by us belongs just to math-logicians.

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Andrew Schumann

A NOVEL TENDENCY IN PHILOSOPHICAL LOGIC

In this paper we consider perspectives of application of coinductive and corecursive methods of non-well-founded mathematics to philosophical logic. So, it is shown that the problem of analysis can be solved by using greatest fixed points. Means of well-founded mathematics are enough only for an explication of the trivial analysis. We claim that the nontrivial analysis should be explicated by means of non-well-funded mathematics. Further, we build a non-well-founded propositional logic with syntax and semantics whose objects are defined by coinduction as streams. We also survey perspectives of relationship between non-well-founded logics and unconventional computing.

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1. Non-well-foundedness and the main problem of analytical philosophy

A non-well-founded (non-WF) set theory belongs to axiomatic set theories that violate the rule of WF-ness and, as an example, allow sets to contain themselves: $X \in X$. In non-WF set theories, the foundation axiom of Zermelo-Fraenkel set theory is replaced by axioms implying its negation. The theory of non-WF sets has been explicitly applied in diverse fields such as logical modelling non-terminating computational processes and behavior of interactive systems in computer science (process algebra, coalgebra, logical programming based on coinduction and corecursion), linguistics and natural language semantics (situation theory), logic (analysis of semantic paradoxes).

Non-WF sets have been also implicitly used in non-standard (more precisely, *non-Archimedean*) analysis like infinitesimal and p-adic analysis [8], [29], [30], [49], [55], [58], [80]. Main advantages of non-WF sets consist in that we get an extension of standard sets such that the way of setting ma-

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thematical objects changes and we have a more general approach to computation without classical induction and recursion. The other extensions just as fuzzy or interval sets are defined by non-Boolean valuations of their membership relations, but the way of setting mathematical objects and the basic computing principles keep fulfilling. Non-WF sets suppose a unique way of extending conventional sets, changing the nature of setting the set hierarchy. We can define fuzzy, interval, continuous, and probabilistic systems on the base of non-WF sets. Therefore we can claim that non-WF sets are better for formalizing computing processes in natural systems than other extensions of conventional sets.

The *axiom of foundation* asserts that the membership relation $\in$ is WF (there is no descending sequence for $\in$), i.e. that any nonempty collection Y of sets has a member $y \in Y$ which is disjoint from Y. We can deny this axiom in order to postulate a set that has an infinite descending $\in$-chain, i.e. that is not WF. The particular case of such a set is one of the form $X = \{X\}$ with the circular membership relation. The set theory with the anti-foundation axiom (with denying the axiom of foundation) is considered in [4], [13]. Replacing the axiom of foundation in classical set theory with an alternative is not a new idea. For instance, in 1917 Miramanoff formulated the fundamental distinction between WF sets and hypersets.

The interest in non-WF phenomena is mainly motivated by some developments in computer sciences. Indeed, in this area, many objects and phenomena do have non-WF features: self-applicative programs, self-reference, graph circularity, looping processes, transition systems, paradoxes in natural languages, etc. Some others like strings, streams, and formal series are potentially infinite, and can only be approximated by partial and progressive knowledge. Also, it is natural to use universes containing adequate non-WF sets as frameworks to give semantics for these objects or phenomena. Moreover, it is often not relevant to use the classical principles of definition and reasoning by induction or recursion to define and reason about these objects. Therefore they assume some new metamathematical (logical) properties to be used.

So, instead of recursion, one applies *corecursion* as a type of operation that is dual to recursion. Corecursion is typically used to generate infinite data structures. The rule for primitive corecursion on codata is the dual to that for primitive recursion on data. Instead of descending on the argument, we ascend on the result. Notice that corecursion creates potentially infinite codata, whereas ordinary recursion analyzes necessarily finite data.

Induction and recursion are firmly entrenched as fundamentals for proving properties of inductively defined objects, e.g. of finite or enumerable objects. Discrete mathematics and computer science abound with such objects, and mathematical induction is certainly one of the most important tools. However, we cannot use the principle of induction for non-WF objects. Instead of this principle, the notion of *coinduction* appears as the dual to induction. Unfortunately, coinduction is still not fully established in the collective mathematical consciousness. A contributing factor is that coinduction is often presented in a relatively restricted form. Coinduction is often considered synonymous with bisimulation and is used to establish equality or other relations on infinite data objects such as streams [86] or recursive types [36]. But the sphere of applications of coinductive methods permanently became wider. For coinductive representation of real numbers see [16], [27]. Also, [56] shows possibilities of using the metric coinduction principle in the context of infinite streams as an alternative to traditional methods involving bisimulation (it is exemplified there by novel proof methods in theories of Markov chains and Markov decision processes). For applications of coinduction in logic see [61], [63], [67], [70], [77], [82], [83].

The difference between induction and coinduction may be well defined as follows. Firstly, let an operation $\Phi : \mathcal{P}(A) \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the powerset of A, be defined as monotone iff $X \subseteq Y$ implies $\Phi(X) \subseteq \Phi(Y)$ for $X, Y \subseteq A$. Any monotone operation Φ has the least and the greatest fixed point, X_Φ and X^Φ respectively, that is, $\Phi(X_\Phi) = X_\Phi$, $\Phi(X^\Phi) = X^\Phi$, and for any other fixed point $Y \subseteq A$ of Φ (i.e. $\Phi(Y) = Y$) we have $X_\Phi \subseteq Y \subseteq X^\Phi$. The sets X_Φ and X^Φ can be defined by $X_\Phi := \bigcap\{Y : Y \subseteq A, \Phi(Y) \subseteq Y\}$, $X^\Phi := \bigcup\{Y : Y \subseteq A, Y \subseteq \Phi(Y)\}$. It is easy to see that the monotonicity of Φ implies the required properties of X_Φ and X^Φ.

On the one hand, by definition of X_Φ, we have for any set $Y \subseteq A$ that $\Phi(Y) \subseteq Y$ implies $X_\Phi \subseteq Y$. This principle is called *induction*. On the other hand, by definition of X^Φ, we have for any set $Y \subseteq A$ that $Y \subseteq \Phi(Y)$ implies $Y \subseteq X^\Phi$. This principle is called *coinduction*.

The non-WF set can be exemplified as the following stream of seasons that unfolds without end but with a cyclic pattern to their nature: $seasons = \langle spring, \langle summer, \langle fall, \langle winter, seasons \rangle\rangle\rangle\rangle$. Over the past twenty years or so, ever more intrinsically circular phenomena have come to the attention of researchers in the areas of artificial intelligence and computer science. For instance, the broad fields of artificial intelligence and computer science have given urgency to the need to formulate models of self-referential structures (see [12]).

Denying the foundation axiom in number systems implies setting the *non-Archimedean ordering structure*. Remind that Archimedes' axiom affirms the existence of an integer multiple of the smaller of two numbers which exceeds the greater: for any positive real or rational number y, there exists a positive integer n such that $y = 1/n$ or $ny = 1$. The informal sense of Archimedes' axiom is that anything can be measured by a ruler. Refusing the Archimedean axiom entails the existence of infinitely large numbers.

A connection between denying the axiom of foundation and denying the Archimedean axiom may be shown as follows. In the beginning we demonstrate an informal meaning of the axiom of foundation. Imagine that all initial objects of mathematics (e.g. numbers) are ways and the operations over those initial objects are motions on them. Then this axiom says that there exist finite ways; in this case, we use the induction principle: it is possible to achieve an aim at the shortest distance between points. The negation of the axiom of foundation causes that all ways are infinite. Then we cannot apply the *induction principle*: there are no shortest distances. Therefore one uses there the *coinduction principle*: it is possible to achieve an aim at the largest distance between points. Taking into account the existence of infinitely large numbers in non-Archimedean mathematics (e.g. in p-adic analysis or in analysis of infinitesimals), we can state that initial objects of non-Archimedean mathematics are objects obtained implicitly by denying the axiom of foundation. Non-Archimedean numbers may be represented only as infinite ways. These objects are non-WF.

The non-Archimedean version of non-WF mathematics (i.e. mathematics without the axiom of foundation) is a new branch of modern mathematics. These unconventional mathematics have found the wide application in the p-adic case of non-WF physics [51], [52], [106], [107], in the p-adic case of non-WF probability theory [53]–[54], and in the non-Archimedean case of non-WF logic [92]–[98].

In this section we will consider the possibilities of application of non-WF ideas to analytical philosophy. Ones of the first works, devoted to the similar topic, were written by Barwise [11], [12]. In those works the ideas of non-WF-ness are used for an explanation of semantic paradoxes, mainly the Liar proposition. We shall try to show that in the semantic analysis it is impossible to avoid non-WF phenomena.

The reasoning is an initial concept of any (informal or symbolic) logic. A speech which is characterized by the simultaneous realization of the following conditions is called **reasoning**: (1) it is attributive – something is affirmed in relation to something (an attributive speech is called also an *analysis*; if B is affirmed in relation to A, then A is called an *analyzed expression*, B an *analyzing expression*); (2) it is informative – it explicitly shows the recipient the content, in other words, it refers him to real or invented objects and speaks about them something nontrivial; such a speech is not wholly reducible to the statement "$A \approx A$" (it is read as follows: A is analytically equal to A) and has the form "$A \approx B$"[1]; (3) it is inferable – it explicitly or implicitly includes certain deductions which substantiate its content; such a speech can be reduced to the expression "$A \approx B$, because…"; (4) it is convincing – it can convince an interlocutor of a truth-validity of any point of view; it is possible to reduce such a speech to the expression "somebody thinks that $A \approx B$ and it is possible to agree with it."

Thus, an informative, inferable and convincing analysis is called a reasoning. It is readily seen that we have: (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Indeed, every convincing speech is inferable, every inferable speech is informative, and every informative speech is attributive, but no vice versa, because there exists an attributive speech which is not informative, an informative speech, which is not inferable, and an inferable speech, which is not convincing.

Logic is a science which studies various modes of modelling reasonings.

It is possible to distinguish three levels in a reasoning: (1) syntactical, i.e. relations between signs used in a reasoning; (2) semantical, i.e. relations between denotations, truth-values, senses, etc. of signs; (3) pragmatical, i.e. relations between agents of a language competence, namely those who depending on a concrete situation are capable to ascribe semantical objects to well-formed combinations of signs. The class of all syntactical relations of an arbitrary language L (for example, English) is called syntax of L. The class of all semantical relations of a language L is called semantics of L. The class of all pragmatical relations of a language L is called pragmatics of L. A semiotic system of L is formed as a triple of syntax, semantics, and pragmatics of L.

Semantics of any language consists of rules of the following classes: (1) rules according to which real or invented object are ascribed to well-formed combinations of signs; (2) rules according to which state of affairs are ascribed to well-formed combinations of signs; the latter are called relations of semantical superposition.

A *semantical superposition* is a correspondence relation that maps states of affairs onto finite combinations of words. Such a combination is called a proposition. For example, the state of affairs 'Socrates goes for a walk' is assigned to the combination of five words "Socrates goes for a walk". A semantical superposition is formed due to special functions. For example, the proposition "Socrates is a man" is formed due to the function "… is… ".

For the successful modelling of reasonings it is necessary to have an algorithm of constructing a semantical superposition. Notice that a convincing

[1] We have $A = B \supset A \approx B$, but no vice versa, where $=$ is a sign for the equality.

attributive speech can satisfy the principle of induction: (1) the analysis "$A \approx A$" is convincingly inferable (i.e. this analysis is obvious without deductions); (2) if the analysis "$A \approx B$" is convincingly inferable by means of any argument C, then it is convincingly inferable by means of any other number of arguments among which C occurs; (3) every analysis is convincingly inferable if and only if it satisfies the two previous conditions. The inductive treatment of a convincing substantiation shows that the analysis in logic should contain the least type of semantical connection. It means that in the logical analysis, the natural logical function should assume a WF semantical superposition and it should satisfy the principle of induction. Thus, if A is a semantical superposition, then B is a semantical superposition if and only if $B \subseteq A$.

Conventional logical functions (negation, conjunction, disjunction, implication, etc.) are a variety of WF semantical superposition. For example, the implication is understood as the least type of semantical connection in a conditional proposition. Recall that the implication satisfies the following condition: $A \supset B$ is false if and only if A is true and B is false, and $A \supset B$ is true in all other cases.

Rules of WF semantical superpositions for appropriate propositions allow us to set semantical superpositions by finite algorithms. Using the example of implication we can show that indeed, the implication is a WF semantical superposition (i.e. it assumes the least fixed point, the least type of semantical connection). Firstly, from the false antecedent everything follows: $\{\text{false}\} \supset A$, i.e. any A, expressing any state of affairs. Secondly, we infer the true from any proposition: $A \supset \{\text{true}\}$. Consequently, the expression "If $2 \times 2 = 4$, then the moon is spherical" is a true implication, though any causal relationship between the antecedent and the consequent is not seen. The similar absurdities implied from the condition of WF semantical superposition for a conditional proposition are called in logic the paradoxes of implication.

Traditionally, the relations which assume a WF semantical superposition are regarded as logical relations. Let $\mathfrak{R}_L$ be the set of all logical relations and $\mathfrak{R}_S$ be the set of all semantical relations, i.e. the relations supposing any semantical superposition (WF and as well as non-WF). It is known that logical relations form the least nonempty subset of the set of semantical relations. This property is formulated as follows: $\mathfrak{R}_L \subset \mathfrak{R}_S$, and if $\mathfrak{R}_X \subset \mathfrak{R}_S$ and $\mathfrak{R}_X \subset \mathfrak{R}_L$, then $\mathfrak{R}_X = \mathfrak{R}_L$. In other words, for every $R_S \in \mathfrak{R}_S$ it is possible to find an appropriate $R_L \in \mathfrak{R}_L$ such that R_L is the least semantical connection in the semantical relation R_S.

Such an understanding of logical relations was proposed in the logical positivism (its representatives are Carnap, Frege, Hilbert, Quine, Russell, Tarski, and many others). We will call this understanding the **hypothesis of WF-ness of logical relations**.

The analytical philosophy (as a special direction of logical-and-philosophical researches) studies conditions of constructing attributive speeches, i.e. the expressions of the form "$A \approx B$". Logic studies not any, but only convincing attributive speeches. Therefore logic has the more restricted subject than analytical philosophy. However, algorithms of building the analysis "$A \approx B$" are exclusively studied within the framework of logic.

The logical positivism is the best known tradition of analytical philosophy. Within the framework of this tradition, the problem of analysis is solved only by means of logical methods, therefore the semantic analysis here is completely reduced to WF semantical superpositions. The main problem of the logical positivism is formulated as follows: what the least subset of the given set of terms, sufficient for analyzing all the remaining terms of this set? Thus, in the logical positivism, the correct analysis is reduced to the identity, because, according to logical positivists, every analysis is a WF semantical superposition.

The research of the problem of analysis should be carried out in frameworks of a system of postulates. For example, in the system of nonlogical postulates, in which the term 'triangle' is defined, we can conclude: "X is an equilateral triangle if and only if X is an equiangular triangle". Therefore "$A \approx B$" can be treated as "$A = B$" only in frameworks of a system L in which terms A and B with the corresponding semantical relations are well defined.

Let X be an analyzed expression, Y be an analyzing expression. If the analysis is correct, then, according to logical positivists, X is identical to Y; but in this case an opinion expressed by the expression $X = Y$ is identical to an opinion expressed by the expression $X = X$. In other words, if the analysis, containing a certain identity, is correct, then it is trivial. This refers to the paradox of analysis: the expression $X = Y$ cannot simultaneously be true and nontrivial [60], [75].

Let us consider two equations: $4 = IV$ and $4 = \sqrt[3]{64}$. We can always state that the equation $4 = IV$ is trivial, whereas the equation $4 = \sqrt[3]{64}$ is informative, as various numbers occur both on the right-side and on the left-side. To avoid the paradox of analysis, logical positivists distinguish two modes of identity: ontological and semantical. The expression "X and Y are identical concepts (or properties)" reflects an *ontological mode of identity*. A *semantical mode of identity* is formulated as follows: "X and Y are synonymous expressions". So, the equation $4 = \sqrt[3]{64}$ contains an ontological mode of identity, and the equation $4 = IV$ a semantical mode of identity.

Nevertheless, the paradox of analysis again arises in a semantic statement if we accept the following principle:

(i) *Let S_1 be a statement containing expression X and S_2 be a statement obtaining from S_1 if X is replaced by a synonymous expression Y; then S_1 is synonymous with S_2.*

This principle directly entails the paradox if we consider S_1 as a statement expressing the analysis, X as an analyzable expression, and Y as an analyzing expression. For example, while the word 'father' is synonymous with the expression 'male parent', the phrase 'the father is a male parent' means (i.e. is synonymous with) that 'father is father'. In the formal notation:

(ii) $X \in S_1$ *and* $Y \in S_1$, *therefore* $(X = Y) \in S_1$ *and* $(X = X) \in S_2$.

Let us consider another example. Let the analysis S_1 be represented by the expression "the attribute of being a brother = the attribute of being a male sibling". It allows us to paradoxically infer:

The statement "the attribute of being a brother = the attribute of being a male sibling" is identical to the statement "the attribute of being a brother = the attribute of being a brother".

If we regard S_2 as statement "somebody knows that brother = brother", then we infer that "somebody knows that brother = male sibling". It is obvious that this deduction cannot be logically correct. So, from my knowledge that $4 = 4$ does not follow that I know all the true equations of arithmetics concerning the number 4. This version of the paradox of analysis is called the epistemic paradox of analysis. It is formulated as follows:

(iii) *If $p = q$, $(p = p) = (p = q)$ and everybody knows that $p = p$, then everybody knows that $p = q$.*

In Church's opinion, we can deny paradox (ii), in particular paradox (iii), if we accept Frege's difference between sense (Sinn, connotation) and meaning (Bedeutung, denotation). The names 'father' and 'male parent', designating the same concept, have, nevertheless, a different sense, e.g. in the equation $2 + 2 = 4$ the expressions both on the left-side and on the right-side designate the same number (they have a common denotation), but they have a different sense. Clearly, the substitution rule formulated in (i) holds if the synonymy relation is understood as identification by sense.

Another way of denying paradoxes (ii) and (iii) was proposed by R. Carnap [23], [24]. He noticed that statements "Scott is identical to Scott" and "Scott is identical to the author of Waverly" are not identical, because the first is a tautology, and the second is a factual statement (assuming that the name 'Scott' is not an abbreviation of the expression 'the author of Waverly').

The most adequate way of denying the paradox of analysis consists in the difference between $X = Y$ (i.e. a nontrivial informative analysis), as a non-WF synonymous connection for X (*salva necessitate* is its traditional name), and $X = X$ (i.e. a trivial uninformative analysis), as a WF synonymous connection for X (*salva veritate* is its traditional name). Let i designate the members identical among themselves. The analysis $X = Y_i$ which takes place for some i is called a *non-WF synonymous connection for X*. This means that the identity is closed by using the greatest fixed point (coinduction). As a result, the nontrivial analysis "Scott is identical to the author of Waverly" should be considered as non-WF synonymous connection. The analysis $X = Y_i$ which takes place for all i is called a *WF synonymous connection for X*. This means that the identity is closed by using the least fixed point (induction). In this case the trivial analysis of the form "Scott is identical to Scott" should be described. It is obvious that the problem of analysis in relation to WF synonymous connection for X is solved by using conventional computing (e.g. using determined algorithms). We can show that the problem of analysis in relation to non-WF synonymous connection for X is solved by using unconventional computing (in particular, probabilistic algorithms).

Let C be an analyzable expression containing a certain theoretical term and the formula $Q_i \supset R_i$ be an analyzing expression of the form of conditional statement described in the language of i observations, namely in the language in which the theoretical term included in C is regarded. For instance, take as C the expression "At the moment t the electric light has a power equal l", and take as $Q_i \supset R_i$ the expression "At the observation i if at the moment t the given electric cable is connected to the ammeter, then the arrow of this ammeter moves at this moment to l from the initial position".

Let us assume that we have a WF synonymous connection for C in the analysis $C = Q_i \supset R_i$. This means that $Q_i \supset R_i$ logically follows from C for all i. Suppose further that $Q_i \supset R_i$ follows from C for some i, i.e. the expression $Q_i \wedge \neg R_i$ entails $\neg C$ for some i and $Q_j \wedge \neg R_j$ entails C for some $j \neq i$. But it cannot be correct for a WF synonymous connection for C. Suppose now that in the analysis $C = Q_i \supset R_i$, we should use a non-WF synonymous connection for C. Indeed, it is impossible to show that $C = Q_i \supset R_i$ for every observation i, because we should carry out infinitely many observations. In the given situation it is better to compare C with a degree of its falsification or confirmation. In other words, the analysis $C = Q_i \supset R_i$ should be considered not in all possible worlds (not for all i), but in some ones (for some i). Thus, the problem of the analysis $C = Q_i \supset R_i$ can be solved here by using probabilistic algorithms.

Let us also show that the problem of analysis is not solved by means of determined algorithms or other WF methods of logical analysis in the natural language. Let C be an analyzing expression, $Q_i \supset R_i$ be an analyzed expression, and the latter is pronounced by an agent of the language competence i with the following sense: "If the expression C is correct, then C is convincing for me." Assume that in the analysis $C = Q_i \supset R_i$, we have a WF synonymous connection for C. It means that $Q_i \supset R_i$ should be inferred from C for all i. Assume also that if the expression "C is correct and C is convincing for me" for some i, then we have $\neg C$. But it is not obvious for a WF synonymous connection for C. Thus, we have the condition of non-WF connection for C in the analysis $C = Q_i \supset R_i$. Interviewing all agents i to make clear, whether the expression C convinces them, is not a realizable task. As we see, the analysis $C = Q_i \supset R_i$ is necessary for considering not in all probable worlds, but at concrete i. Therefore the problem of the analysis $C = Q_i \supset R_i$ is always contextually solved in the natural language, in relation to the concrete native speaker.

So, paradoxes of analysis are bright witnesses that using WF methods it is not possible to explicate all the semantical relations originating in the natural language. Within the framework of logical positivism, the reduction of all semantical connections to WF was proposed, and the initial WF semantical superpositions were called logical functions (negation, conjunction, disjunction, implication). However, as we were convinced, there are the semantical relations which are not expressed by means of WF methods, namely the nontrivial analysis "$A \approx B$." Therefore we can assume that there are the natural logical functions producing the greatest type of semantical connection (non-WF logical relations). It means that each this logical function assumes a non-WF semantical superposition. In this case, if A is a semantical superposition and $A \subseteq B$, then B also is a semantical superposition.

Hence, we can put forward the **hypothesis of non-WF-ness of logical relations**. So, the task of construction of logical calculi not on the basis of WF semantical superpositions, but on the basis of non-WF ones seems to be very promising nowadays.

2. Non-well-founded logical predications

One of the most useful non-WF mathematical object is a stream – a recursive data-type of the form $s = \langle a, s' \rangle$, where s' is another stream. The notion of *stream calculus* was introduced by Escardó and Pavlović [72] as a means to do symbolic computation using the coinduction principle instead of the induction one. Let A be any set. We define the set A^ω of all streams over A as $A^\omega = \{\sigma \colon \{0, 1, 2, \ldots\} \to A\}$. For a stream σ, we call $\sigma(0)$ the initial value of σ. We define the *derivative* $\sigma(0)$ of a stream σ, for all $n \geq 0$, by $\sigma'(n) = \sigma(n + 1)$. For any $n \geq 0$, $\sigma(n)$ is called the n-th element of σ. It can also be expressed in terms of higher-order stream derivatives, defined, for all $k \geq 0$, by $\sigma^{(0)} = \sigma$; $\sigma^{(k+1)} = (\sigma^{(k)})'$. In this case the n-th element of a stream σ is given by $\sigma(n) = \sigma^{(n)}(0)$. Also, the stream is understood as an infinite sequence of derivatives. It will be denoted by an infinite sequence of values or by an infinite tuple: $\sigma = \sigma(0) :: \sigma(1) :: \sigma(2) :: \ldots :: \sigma(n-1) :: \sigma^{(n)}$, $\sigma = \langle \sigma(0), \sigma(1), \sigma(2), \ldots \rangle$.

A *bisimulation* on A^ω is a relation $R \subseteq A^\omega \times A^\omega$ such that, for all σ and τ in A^ω, if $\langle \sigma, \tau \rangle \in R$ then (i) $\sigma(0) = \tau(0)$ (*initial value*) and (ii) $\langle \sigma', \tau' \rangle \in R$ (*differential equation*).

If there exists a bisimulation relation R with $\langle \sigma, \tau \rangle \in R$ then we write $\sigma \sim \tau$ and say that σ and τ are *bisimilar*. In other words, the *bisimilarity* relation $\sim$ is the union of all bisimulations: $\sim := \bigcup\{R \subseteq A^\omega \times A^\omega \colon R$ is a bisimulation relation$\}$. Therewith, this relation $\sim$ is the greatest bisimulation. In addition, the bisimilarity relation is an equivalence relation.

Theorem 1 (Coinduction)

For all $\sigma, \tau \in A^\omega$, if there exists a bisimulation relation $R \subseteq A^\omega \times A^\omega$ with $\langle \sigma, \tau \rangle \in R$, then $\sigma = \tau$. In other words, $\sigma \sim \tau \Rightarrow \sigma = \tau$. $\square$

This proof principle is called *coinduction*. It is a systematic way of proving the statement using bisimularity: instead of proving only the single identity $\sigma = \tau$, one computes the greatest bisimulation relation R that contains the pair $\langle \sigma, \tau \rangle$. By coinduction, it follows that $\sigma = \tau$ for all pairs $\langle \sigma, \tau \rangle \in R$.

Now consider a *non-WF propositional logic* $\mathfrak{L}^\omega$, whose syntax and semantics are non-WF, i.e. they are defined by coinduction and their objects are streams. The syntax of $\mathfrak{L}^\omega$ is as follows:

Variables: $x ::= p \mid q \mid r \;\ldots$

Constants: $c ::= \top \mid \bot$

Formulas: $\varphi, \psi ::= x \mid c \mid \neg\psi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \supset \psi$

These definitions are coinductive. For instance, a variable x is of the form of a stream $x = x(0) :: x(1) :: x(2) :: \ldots :: x(n-1) :: x^{(n)}$, where $x(i) \in \{p, q, r, \ldots\}$ for each $i \in \omega$; a constant c is of the form of a stream $c = c(0) :: c(1) :: c(2) :: \ldots :: c(n-1) :: c^{(n)}$, where $c(i) \in \{\top, \bot\}$ for each $i \in \omega$; a formula $\varphi \wedge \psi$ has the differential equation $(\varphi \wedge \psi)' = \varphi' \wedge \psi'$ and its initial value is $(\varphi \wedge \psi)(0) = \varphi(0) \wedge \psi(0)$, etc.

Consider now semantics of $\mathfrak{L}^{\omega}$.

Truth-valuation of variables: $v(x) ::= v(p) \mid v(q) \mid v(r) \ \ldots$

Truth-valuation of formulas: $v(\varphi), v(\psi) ::= v(x) \mid v(\neg\psi) \mid v(\varphi \vee \psi) \mid v(\varphi \wedge \psi) \mid v(\varphi \supset \psi)$

These definitions are coinductive too. For example, truth-valuations of formulas are defined as follows:

- the differential equation of $v(\neg\psi)$ is $(v(\neg\psi))' = \top' - v(\psi)'$ and its initial value is $(v(\neg\psi))(0) = \top(0) - v(\psi)(0)$,
- the differential equation of $v(\varphi \vee \psi)$ is $(v(\varphi \vee \psi))' = \sup(v(\varphi)', v(\psi)')$ and its initial value is $(v(\varphi \vee \psi))(0) = \sup(v(\varphi)(0), v(\psi)(0))$,
- the differential equation of $v(\varphi \wedge \psi)$ is $(v(\varphi \wedge \psi))' = \inf(v(\varphi)', v(\psi)')$ and its initial value is $(v(\varphi \wedge \psi))(0) = \inf(v(\varphi)(0), v(\psi)(0))$,
- the differential equation of $v(\varphi \supset \psi)$ is $(v(\varphi \supset \psi))' = \top' - \sup(v(\varphi)', v(\psi)') + v(\psi)'$ and its initial value is $(v(\varphi \supset \psi))(0) = \top(0) - \sup(v(\varphi)(0), v(\psi)(0)) + v(\psi)(0)$.

We used very simple non-WF syntax and non-WF semantics. For non-WF syntax and non-WF semantics in functional programming see [61]. For coinductive definitions and proofs in big-step semantics, using both finite and infinite evaluations, see [63] and the works of Xavier Leroy[2]. The natural semantics of logic $\mathfrak{L}^{\omega}$ are given in [92]–[98] and these semantics assume a *non-Archimedean ordering structure*.

In paper [71], it is shown that Church's higher-order logic (HOL) is perfectly adequate for formalizing both inductive and coinductive definitions, e.g. a theory of recursive and corecursive definitions can be mechanized using Isabelle. Recall that least fixed points express inductive data types such as strict lists; greatest fixed points express coinductive data types, such as lazy lists. WF recursion expresses recursive functions over inductive data types; corecursion expresses functions that yield elements of coinductive data types.

Notice that coinduction has been well established for reasoning in concurrency theory [62]. Abramsky's lazy lambda calculus [1] has made coin-

[2] pauillac.inria.fr/$\sim$xleroy

duction equally important in the theory of functional programming. Later, Milner and Tofte motivated coinduction through a simple proof about types in a functional language [63].

Logical positivists suggested that proof should be WF, e.g. it should be inductively constructed. However, it is shown in [17], [19], [20], and [101] that *cyclic* (non-WF) *proof* provides a promising alternative to traditional inductive proof, modelled on streams (Fermat's infinite descent). WF proof systems are limited by the following principle: if in some case of a proof some inductive definition is unfolded infinitely often, then that case may be disregarded. Essentially, this principle is sound because each inductive definition has a least-fixed point interpretation which can be constructed as the union of a chain of approximations, indexed by ordinals; unfolding a definition infinitely often can thus be seen as inducing an infinite descending chain of these ordinals, which contradicts their WF-ness. In cyclic proof systems, the capacity for unfolding a definition infinitely often is built in to the system by allowing proofs to be non-WF, i.e. to contain infinite paths.

A simple example of *non-WF proof system* is given by Alex Simpson. This system is sound and complete on Borel sets and it captures inclusion between Borel sets of topological spaces. Recall that the Borel sets $\mathcal{B}(X)$ over a topological space X is the smallest σ-algebra containing the open sets $\mathcal{O}(X)$ (i.e. there is the closure of $\mathcal{O}(X)$ under complements and countable unions).

Let us consider a partially ordered set with: (1) finite infima (including top element $\top$), (2) countable suprema (including least element $\bot$), (3) and satisfying the distributive law for countable suprema: $x \wedge \bigvee_i y_i = \bigvee_i x \wedge y_i$. It is called a σ-frame. Further, let F be a σ-frame, and $B \subseteq F$ a base (i.e. every element of F arises as a countable supremum of elements of B). We define formulas for Borels by taking elements $b \in B$ as propositional constants and closing under negation, and countable conjunctions and disjunctions:

$$\psi ::= b \mid \neg\psi \mid \bigvee_i \psi_i \mid \bigwedge_i \psi_i$$

The proof system contains usual sequent proof rules on left and right for each connective, e.g.

$$\frac{\Gamma, \psi_i \vdash \Delta}{\Gamma \vdash \Delta} \bigwedge_i \psi_i \in \Gamma,$$

$$\frac{\{\Gamma \vdash \psi_i, \Delta\}_i}{\Gamma \vdash \Delta} \bigwedge_i \psi_i \in \Delta.$$

This system also includes atomic cuts:

$$\frac{\Gamma, b \vdash \Delta \quad \Gamma \vdash b, \Delta}{\Gamma \vdash \Delta}.$$

A basic entailment, written $C \Rightarrow D$, is given by a finite $C \subseteq B$ and countable $D \subseteq B$ such that: $\bigwedge C \leq \bigvee D$ in F. An infinite branch $(\Gamma_i \vdash \Delta_i)$ in a rule tree is justified if there exist $C \subseteq \bigcup_i \Gamma_i$ and $D \subseteq \bigcup_i \Delta_i$ such that $C \Rightarrow D$. A rule tree is a proof if every infinite branch is justified.

Evidently, non-WF logic with non-WF proof system can have unexpected metalogical properties that are not studied yet. The conventional computing is set in the frameworks of WF logics and WF proof systems. We can assume that setting computation using non-WF logics and non-WF proof systems can be provided by new horizons.

3. Unconventional computing

3.1. New computing media

It is a common misconception that computers have been changing in a qualitative sense for the last 50 years. All of the modern computers' basic procedures and architectures are still based on the principles formulated back in the 1930s and 1940s by Alan Turing and John von Neumann. But is it indispensable? Or at least effective? The answers to these questions is the subject of what is called unconventional computing – a relatively young yet many-sided branch of computer science.

Several conceptual question arise on the matter of practical realization of those basic principles. For example, a Turing machine is by definition unrealizable, because it involves an endless tape, whereas every computer's memory is finite. Unconventional models take a different approach to the axiomatization of computing: they try to make physical realizability an indigenous part of the axioms.

Despite having been a topic of discussion since at least as early as the 1970s, the terminology in this field is not quite established yet. Unconventional computing models can be divided into two groups:

1. Mechanical models: billiard ball computers, Domino computers etc., see [14], [37].
2. Biologically-inspired models: neural networks, genetic algorithms, cellular automata etc., see [5], [21], [46], [81].

Some of the models are purely theoretical, like the billiard ball models, and often they are also more important theoretically than being realized in practice. Others, like quantum computers, are variously close to practical implementations.

In the large sense of the word, computation (which is a much broader concept than calculation) is any mapping of input information into an output state. Hence every real-life system does some sort of computation; at the least, it computes its own future state. In other words, all that is needed to make a computer is an excitable medium and a way to channel the propagation of activity in it, i.e. the aim is to design (and, hopefully, construct) systems that do a predictable (or intentionally unpredictable, or probabilistic) computation.

There is also a popular concept of natural computing; most of what is called natural computing can be placed among mechanical or biological models, though there are some approaches that lie somewhere between, like chemical computing.

Here we come to one of the fundamental statements in computation theory, known as the Church-Turing thesis. It claims that any computation that is physically possible (notice that physically possible computations also include human thinking) can also be performed by a *conventional* computer. It is not a provable theorem, but rather an assertion connecting human intuition and the strict concept of algorithm. If this thesis is right, then the unconventional models can only find some applications in making ordinary algorithms more effective. But if it is not, then things get much more tricky: there appear to be problems that are principally unsolvable via conventional computing. Continuing in the same way, we can claim that in the last case we need to use logical systems of the new type as metalanguage, e.g. more expressible non-WF logics.

One of the objectionable points of unconventional computing models is the fact that they rely on many phenomena that are far from being completely described and explained by science, such as quantum physics or brain functioning. So these models inevitably simplify the actual state of affairs, and at some point go beyond the line where they start to bear more and more resemblance to conventional computing.

Human mind and ongoing theoretical research goes far beyond technical possibilities in what concerns unconventional computing – but that's not a reason for scepticism; let us recall that two-valued algebra was invented by George Boole more than a century before it found its application in modern computers. Adherents of mechanical models point at what is sometimes called the semantic gap the difference between visible logic and actual physical principles of computation. Unconventional computing tries to shorten this gap by introducing straightforward analogies between computational systems and the tasks solved by them.

3.1.1. The billiard ball model

It is one of the most ingenious models of unconventional computing [37]. Balls travelling on two-dimensional grid with a constant speed are considered. The grid is somewhat unusual: it is obtained from the ordinary uniform grid by a 45-degree turn; thus every grid point has 4 neighbors with equal distances to each of them, and this spacing is taken as the unit of distance. Time is discrete ($t = 0, 1, 2, \ldots$). Balls have the radius $\frac{1}{\sqrt{2}}$ and thus collide elastically (due to this radius two neighboring grid points cannot be simultaneously occupied by balls, and if two balls occupy two points exactly $\sqrt{2}$ apart, then they touch at exactly one point, because $\frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}$). Certain borders are set on the grid in order to realize conventional logic elements.

The billiard ball model is clearly reversible. But what can it be used for? It is an important universal logic model. It can be shown [37] that every logical circuit can be simulated by means of only one element called the Fredkin gate, which has 3 inputs c, p and q and 3 outputs $x = c$, $y = cp + \bar{c}q$, $z = \bar{c}p + cq$. The Fredkin gate can be built on the basis of a yet simpler element called the S-gate, which has 2 inputs c and x and 3 outputs c, cx and $\bar{c}x$ (the Fredkin gate is composed of two S-gates and two inverse S-gates). So, if we simulate the S-gate, thus we simulate all possible logical circuits. And indeed, the S-gate can be simulated via billiard balls and (in a rather intricate but harmonious manner) by cellular automata [64].

Note that the billiard ball model is a particular case of cellular automata. Using this model, we come to a very important conclusion in the logics of cellular automata: every logical circuit can be simulated by a reversible 2-dimensional 2-state cellular automaton.

3.1.2. Cellular automata

They constitute a young yet prolific field of research, first investigated by John von Neumann and Stanislaw Ulam in the 1940s. It arises at the boundary between several mathematical (like combinatorics or computability theory) and non-mathematical (like microbiology) branches of science, see [5], [21], [64], [111].

Cellular automata are used for modelling synchronous and uniform processes over large arrays, more precisely over infinite d-dimensional arrays of cells. At each iteration in the discrete time, each cell is updated according to a unique local transition function and the states of the neighboring cells.

Thus, in cellular automata there exist objects that may be interpreted as passive data (the neighboring cells in the initial configuration) and objects that may be interpreted as computing devices. In other words, here computation and construction are just two possible modes of activity. The dynamics is given by an explicit local transition rule by which at every step each cell determines its new state from the current state of its neighbors.

Notice that cellular automata are universal, because they can simulate any Turing machine.

3.1.3. Conservation and reversibility

Also, the main originality of unconventional computing is that just as conventional models of computation make a distinction between the structural part of a computer, which is fixed, and the data on which the computer operates, which are variable, so unconventional models assume that both structural parts and computing data are variable ones. Therefore the essential point of conventional computation is that the physics is segregated once and for all within the logic primitives. Once we have the formal specifications of these primitives and perhaps some design constraints, we can forget about the physics, i.e. about the structural part of a computer. However, the structural part has a permanent evolution in billiard ball model and cellular automata.

Notice that the reversibility of physical processes at a microscopic level makes a distinction between physics and logic in the computation. For example, in the AND-gate three of the four possible input configurations, namely $\langle 0,0\rangle$, $\langle 0,1\rangle$, and $\langle 1,1\rangle$, take the same result 0. As we see, the AND-gate is not reversible (e.g., the NOT-element is reversible).

Therefore to combine physics and logic in the computation, one proposed the idea of conservation which consists in the following principles (see [14], [37], [64])

1. each event has as many output signals as input ones;
2. the number of tokens of each kind is invariant.
3. each event establishes a one-to-one correspondence between the collective state of its input signals and that of its output signals.

3.2. Non-well-founded computing

3.2.1. Interactive computing

Interactive computing, developed in [34], [41]–[44], [109], is based on coinductive methods. Although such a computation model is abstract in the same measure as Turing machines, it assumes a combination of physics and logic in the computation, as well as that in cellular automata or in other unconventional computing models.

In conventional computing, the computation is performed in a closed-box fashion, transforming a finite input, determined by the start of the computation, to a finite output, available at the end of the computation, in a finite amount of time. Therefore physical implementation does not play role in such a computation that may be considered as process in a black box.

Consider now a simple example of interactive computing provided by Peter Wegner to show the role of interactive factors in the computation. Let us consider an automatic car whose task is to drive us across town from point W (work) to point H (home); we shall refer to it as the WH problem. The output for this problem should be a time-series plot of signals to the car's controls that enable it to perform this task autonomously. At issue is what form the inputs should take. We have two possible approaches to solve the WH problem: conventional and interctive/unconventional. Firstly, the car can be equipped with a map of the city, i.e. in the algorithmic scenario using conventional methods, where all inputs are provided a priori of computation. In a static world, such a map is in principle obtainable but ours is a dynamic environment. Secondly, the WH problem can be solved interactively. In this scenario, the inputs, or percepts, consist of a stream of images received in the car, as it is driving from W to H.

Thus, we see the main difference between conventional and unconventional computing. In the first case the computation is off-line (it takes place before the driving begins), and in the second on-line (it takes place as the car drives). This property allows Peter Wegner to claim that interactive/unconventional computation falls outside the bounds of the Church Turing thesis and it is shown to be a more powerful computational paradigm by allowing us to solve computational tasks that cannot be solved algorithmically [41], [109].

Recall that the Church-Turing thesis was initially formulated as follows: the intuitive notion of effective computability for functions and algorithms is formally expressed by Turing machines (Turing) or the lambda calculus (Church). This thesis equated WF logic, lambda calculus, Turing machines, and algorithmic computing as equivalent mechanisms of problem solving. Later, it was reinterpreted as a uniform complete mechanism for solving all computational problems. However, the simple example of the WH problem confirms that Turing machines are inappropriate as a universal foundation for computational problem-solving, because they are too weak to express interaction of object-oriented and distributed systems. Therefore Dina Goldin and Peter Wegner proposed interaction machines as a stronger model that better captures computational behavior for finite interactive computing agents [34], [41]–[44], [109]. Probably, for these machines, the following extension of the Church-Turing thesis should hold: the intuitive notion of sequential interaction is formally modelled by non-WF methods (coinduction, corecursion, etc.). For example, the least fixed point of the equation $S = A \times S$ is the empty set (i.e. it is so from the standpoint of conventional computing), while the greatest fixed point is the set of all streams over S (i.e. it is so from the standpoint of non-WF computing).

Notice, the statement that Turing machines completely express the intuitive notion of computing is a common misinterpretation of the Church-Turing thesis. For instance, Turing asserted in [104] that Turing machines could not provide a complete model for all forms of computation, but only for algorithms. Therefore he defined choice machines as an alternative model of computation, which added interactive choice as a form of computation, and later, he also defined unorganized machines as another alternative that modelled the brain.

Induction and recursion determine enumerable collections of finite structures, while coinduction and corecursion determine non-enumerable collections of infinite structures. As a result, WF logic cannot model interactive/unconventional computing. For example, sound and complete (first-order) logics have a recursively enumerable set of theorems and can formalize only semantic domains with a countable number of distinct properties. Therefore, in particular, the means of WF logics are not sufficient for a syntactical expressibility of all properties of arithmetic over the integers (Gödel's two incompleteness theorems). Thus, Gödel's reasoning may be extended to show that coinductive and corecursive systems are likewise incomplete because they have too many properties to be expressible as theorems of WF logics. However, the question how non-WF logics can syntactically express interactive/unconventional computing is still open.

Dina Goldin and Peter Wegner affirm that the gap between least- and greatest-fixed-point semantics is also the gap between operational (algorithm) and denotational (observation) semantics. It is also the same as the gap between deduction and abduction. It seems to be a restriction, because greatest-fixed-point semantics may be used in deductive non-WF logics, in particular in cyclic proof systems.

3.2.2. Coalgebras and hidden algebras

Instead of algebraic methods, their dual is applied to interactive systems defined by coinductive rules. This dual is called a theory of coalgebras. *Co-*

algebras, developed in [50], [73], [78], [88], [102], consider a notion of observational indistinguishability as bisimulation, a characterization of abstract behaviors as elements of final coalgebras and coinduction as a definition/proof principle for system behavior. *Hidden algebra*, introduced in [38] and further developed in [39], [40], combines algebraic and coalgebraic techniques in order to provide a semantic foundation for the object paradigm. Recall that the object paradigm is described as having: 1. objects with local state and operations that modify or observe them; 2. classes that classify objects through an inheritance hierarchy; and 3. concurrent distributed execution. The theory of hidden algebras is an extension of the theory of many sorted algebras that uses both constructor and destructor operations and a loose behavioral semantics over a fixed data universe for the states of objects.

Hidden algebras were introduced to give algebraic semantics for the object paradigm. One distinctive feature is a split of sorts into visible and hidden, where visible sorts are for data and hidden sorts are for objects. *Hidden logic* is the generic name for various logics closely related to hidden algebra, giving sound rules for behavioral reasoning that are easily automated [84], [85].

Let us remember that algebras and its associated inductive techniques have been successfully used for the specification of data types. Data types can be presented as **F**-algebras using *constructor operations* going into the type, i.e. tuples $\langle A, \alpha\rangle$, where A is an object and $\alpha : \mathbf{F}A \to A$ is a morphism in some category C, with $\mathbf{F} : \mathrm{C} \to \mathrm{C}$. Among **F**-algebras, initial ones $\iota : \mathbf{F}I \to I$ (least fixed points of **F**) are most relevant, their elements denote closed programs. Initial algebras come equipped with an induction principle stating that no proper subalgebras exist for initial algebras. This principle constitutes the main technique used in algebraic specifications for both definitions and proofs: defining a function on the initial algebra by induction amounts to defining its values on all the constructors; and proving that two functions on the initial algebra coincide amounts to showing that they agree on all the constructors.

The theory of coalgebras is viewed as a dualization of the theory of algebras. Object systems are presented as **G**-coalgebras using *destructor operations* going out of the object types, i.e. tuples $\langle C, \beta\rangle$, where C is an object and $\beta : C \to \mathbf{G}C$ is a morphism in some category C, with $\mathbf{G} : C \to C$. Final **G**-coalgebras $\zeta : Z \to \mathbf{G}Z$ (greatest fixed points of **G**) are in this case relevant, they incorporate all **G**-behaviors. The unique coalgebra homomorphism from a coalgebra to the final one maps object states to their behavior. A bisimulation between two coalgebras is a relation on their carriers, carrying itself coalgebraic structure. Bisimulations relate states that exhibit the same behavior. Final coalgebras come equipped with a coinduction principle stating that no proper bisimulations exist between a final coalgebra and itself; that is, two elements of a final coalgebra having the same behavior coincide.

While universal algebras [28] were applied to (WF) logics due to Lindenbaum and Tarski's well-known construction [76], coalgebras begin to be used in (non-WF) logics too [57], [67], [70], [82], [83].

4. Conclusion

A novel computational paradigm concerning a computation in non-WF systems is a burgeoning research area with much potential. The methodological frameworks of the future researches could be non-WF logical calculi, non-Archimedean mathematics, coalgebras, and their application to unconventional computing. Within these frameworks, non-WF computing could be regarded as a new model of non-Church-Turing computation. Advanced techniques for non-WF computing will include:

- Analysis of expressive powers of non-WF formal arithmetic.
- Setting probabilistic algorithms of non-WF computation.
- Applying these algorithms to unconventional computing media.

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HERBRAND THEOREMS: THE CLASSICAL AND INTUITIONISTIC CASES

A unified approach is applied for the construction of sequent forms of the famous Herbrand theorem for first-order classical and intuitionistic logics without equality. The forms do not explore skolemization, have wording on deducibility, and as usual, provide a reduction of deducibility in the first-order logics to deducibility in their propositional fragments. They use the original notions of admissibility, compatibility, a Herbrand extension, and a Herbrand universe being constructed from constants, special variables, and functional symbols occurring in the signature of a formula under investigation. The ideas utilized in the research may be applied for the construction and theoretical investigations of various computer-oriented calculi for efficient logical inference search without skolemization in both classical and intuitionistic logics and provide some new technique for further development of methods for automated reasoning in non-classical logics.

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1. Introduction

Herbrand's paper [1] contains a theorem called now the Herbrand theorem. This theorem permits to reduce the question of the deducibility (validity) of a formula F of first-order classical logic to the question of the deducibility (validity) of a quantifier-free ("propositional") formula F', so that the deducibility of F' can be established by means of using only propositional "calculations". When making the reduction of F to F', a certain set of terms (so-called Herbrand universe) is constructed. Different ways of construction of Herbrand universe(s) lead to different forms of the Herbrand theorem. In particular, three forms are given in [1]: A, B, and C. The Herbrand universes for B and C are defined as minimal sets of terms containing constants and functional symbols occurring in the Skolem functional form of F, and the

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unique difference between B and C consists in the ways of the skolemization of F[1].

The form A does not need the skolemization of F, and the Herbrand universe for it uses only constants, functional symbols, and certain quantifier variables from F. But its application requires checking a large number of quantifier sequences constructed from "schemes" (in the terminology of'[1]) in order to find at least one sequence satisfying a certain condition that guarantees the validity of the formula F.

Since intuitionistic logic does not keep the skolemization transformations in general, it is impossible to obtain the forms B and C for intuitionistic logic. Therefore, for intuitionistic logic, there can be made an attempt to construct a Herbrand theorem similar to the form A only, i.e. when preliminary skolemization is not obligatory and reduction of first-order investigations to propositional "calculations" is performed. Besides, it is desired to give forms of Herbrand's theorem for classical and intuitionistic cases providing the clear-cut distinction between them.

This research gives a possible decision of the problem under consideration: *the unified forms of Herbrand's theorem are formulated for classical and intuitionistic logics in a sequent form*[2]. It does not explore skolemization, have wordings on deducibility, and develops the approach suggested in [5, 6, 7] for classical logic and modified in a certain way for a tableau treatment of intuitionistic logic in [8], which permits to achieve the objective just reminded on the base of the original notions of admissibility and compatibility. Note that a similar style of inference search (not requiring skolemization and not giving Herbrand theorems at once) was exploited by the author of the paper and his coauthors in a number of sequent calculi for classical logic (see, for example, [9, 10]) and in the tableau method for intuitionistic logic from [11] in order to optimize an item-by-item examination arising when quantifier rules applications satisfying Gentzen's admissibility – i.e. to the eigenvariable condition [12] – are made[3].

Additionally note that our approach based on the notions of admissibility and compatibility shares some ideas with the papers [16] and [17]

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1 For example, see [2] and [3] for some details relating to such a type of Herbrand's theorem.

2 The announcement of the main results of the research was made in the slightly different form at the Kurt Gödel Centenary Symposium, Vienna, Austria, 2006 [4].

3 S. Kanger introduced his definition of admissibility [13] which has an advantage over Gentzen's one; its modified forms were used in a number of papers concerning inference search in classical logic (see, for example, [14]) and in intuitionistic logic (see, for example, [15]).

being exploited in the original way for the construction of various computer-oriented methods for classical and non-classical logics such as matrix characterization methods [17], different modifications of the connection method (see, for example, [16], [18], [19], [20]), and ordinal sequent and tableau methods (see, for example, [21], [22]). But all these papers do not contain any direct instructions how to construct both classical and intuitionistic forms (not requiring skolemization) of Herbrand's theorem.

2. Preliminaries

We use standard terminology of first-order sequent logic without equality. The basic signature Sig_0 of the first-order language consists of a (possibly empty) set of functional symbols (including constants), a (non-empty) set of predicate symbols, and logical connectives: the quantifier symbols for the universal character $\forall$ and for existential character $\exists$ as well as the propositional symbols for the implication ($\supset$), disjunction ($\vee$), conjunction ($\wedge$), and negation ($\neg$). And $\forall x$ and $\exists x$ are called quantifiers; they are considered as a single whole. A countable set of variables is denoted by Var.

Additionally, we extend the signature Sig_0 in the following way: for any natural number (*index*) k ($k = 1, 2, \ldots$) and for any symbol $\odot$ from Sig_0, we add the *indexed symbol* ${}^k\odot$ to Sig_0 and denote the constructed extension by Sig. For example, ${}^1\vee$, ${}^3\supset$, and ${}^5\forall$ are symbols of the extended signature. These upper-left indices are used for distinguishing different copies of the same formula.

For technical purpose only, we consider that Var consists of two disjoint countable sets: Var_0 and Var'_0 ($Var = Var_0 \cup Var'_0$), where the following condition is satisfies: for any $v \in Var_0$ and any natural number (*index*) k ($k = 1, 2, \ldots$), Var'_0 contains the *indexed variable* ${}^kv$.

The notions of terms, atomic formulas, literals, formulas, free and bound variables (over both $Sig_0 \cup Var$ and $Sig \cup Var$) are defined in the usual way [23] and assumed to be known for the reader.

Sequents also are defined in the usual manner, except that their succedents and antecedents are considered as multisets (cf. [23]).

Any syntactical object over $Sig_0 \cup Var$ is called an *original* one. As usual, we assume that no two quantifiers in any formula or in any sequent have a common variable, which can be achieved by renaming bound variables.

Without loss of generality, an *initial sequent* (i.e. a sequent being investigated on deducibility) always is considered to have the form $\rightarrow F$, where F is a *closed original* formula.

Any expression constructed over $Sig \cup Var$ may be viewed as an indexed one constructed by the renaming of a certain original object (by means of adding certain upper-left indices if needed). The extension of all the necessary semantic notions to indexed terms, indexed formulas, and indexed sequents is obvious: to do this, it is enough to consider all their upper-left indices in indexed terms, formulas, and sequents to be missed.

When it is important, we write the *"indexed formula"*, *"indexed sequent"*, etc. in order to underline the fact that an appropriate formula, sequent, etc. is constructed over $Sig \cup Var$ and may contain indexed symbols.

Sometimes, the original formula $\underline{\exists y \neg \exists x P(x, f(y)) \supset \neg \forall y' \exists x' P(x', y')}$ denoted by F^* will be used in a number of examples clarifying introduced notions and obtained results (P is a predicate symbol and f is a functional symbol).

If the *principal connective* of a (indexed) formula F is $\odot$ (i.e. F has the form $F' \odot F''$ or $\odot F'$, where $\odot$ is $\supset$, $\vee$, $\wedge$, $\neg$, $\forall$, or $\exists$), then F is called *$\odot$-formula.*

As in [2], we say that an occurrence of a subformula F in a formula G is

- *positive* if F is G;
- *positive* (*negative*) if G is of the form: $G_1 \wedge G_2$, $G_2 \wedge G_1$, $G_1 \vee G_2$, $G_2 \vee G_1$, $G_2 \supset G_1$, $\forall x G_1$, or $\exists x G_1$ and F is positive (negative) in G_1;
- *negative* (*positive*) if G is of the form $G_1 \supset G_2$ or $\neg G_1$ and F is *positive (negative)* in G_1.

Further, a formula F has a *positive (negative)* occurrence in a sequent $\Gamma \rightarrow \Delta$ if F has a positive occurrence in a formula from Δ (from Γ) or if F has a negative occurrence in a formula from Γ (from Δ). Moreover, if F has the form $\forall x F'$ ($\exists x F'$) and F has a positive (negative) occurrence in a formula G or in a sequent S, then $\forall x$ ($\exists x$) is called a *positive quantifier* in G or in S, respectively; $\exists x$ ($\forall x$) is called a *negative quantifier* in G or in S, if $\exists x F'$ ($\forall x F'$) has a positive (negative) occurrence in G or in S, respectively.

The variable of a positive quantifier occurring in a formula G or in a sequent S is called a *parameter* in G or in S, respectively; the variable of a negative quantifier occurring in a formula G or in a sequent S is called a *dummy* in G or in S, respectively.

Remark. The terms "parameters" and "dummies" are taken from [13], where they are used in the analogous sense.

For F^*, we have: x and y' are dummies and x' and y are parameters.

The way of the extension of the notions of dummies and parameters to sequents and (multi)sets of formulas or of sequents is obvious.

Since the property "to be a dummy" ("to be a parameter") is invariant w.r.t. logical rules applications in sequent calculi, any parameter (dummy)

x in a formula (in a sequent, in a (multi)set of formulas or of sequents) is convenient to be written as $\overline{x}$ ($\underline{x}$).

For a formula F (for a sequent S), $\mu(F)$ ($\mu(S)$) denotes the result of the elimination of all the quantifiers from F (from S).

If F (S) is a formula (a sequent) and x is its parameter or dummy then x considered to be a parameter or a dummy in $\mu(F)$ (in $\mu(S)$).

Convention. For any expression Ex over $Sig \cup Var$ and any index k, the notation ${}^kEx$ denotes the following expression: we delete all the upper-left indices in logical connectives, parameters, and dummies of Ex and add k as a upper-left index to all the symbols of the result of the deletion except for constant, for dummies, for predicate, and for functional symbols.

Therefore, we have for F^*: ${}^1F^*$ is ${}^1\exists^1\overline{y}^1\neg^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y}))^1 \supset {}^1\neg^1\forall^1 y'$ ${}^1\exists^1 x'\ P({}^1\overline{x}', {}^1\underline{y}')$ and $\mu({}^1F^*)$ is ${}^1\neg P({}^1\underline{x}, f({}^1\overline{y}))^1 \supset {}^1\neg\ P({}^1\overline{x}', {}^1\underline{y}')$.

We use slight modifications of the cut-free calculi LK and LJ [12] for technical purposes only. In this connection, we suppose a reader to know all the notions relating to deducibility in Gentzen (sequent) calculi. Draw you attention to the fact that all the *inference trees* in calculi under consideration are understood in the usual sense and grow "from top to bottom" by applying inference rules to an input sequent and afterwards to its "heirs", and so on. Additionally remind that any inference tree having only leaves with axioms is called a *proof tree*.

3. Admissibility and compatibility

Let F be a formula. By (i, F), we denote the i-th occurrence of its subformula if F is read from left to right. We write $(i, F) \sqsubseteq_F (j, F)$ if and only if (j, F) is a subformula of (i, F). Obviously, the relation $\sqsubseteq_F$ is partial ordered.

If (i, F) is the occurrence of a $\odot$-formula, where $\odot$ is a logical connective (a propositional connective or a quantifier), we also refer to this occurrence as to ${}_i\odot$-occurrence in F.

If a formula F has ${}_i\odot$- and ${}_j\odot'$-occurrences of its subformulas ($i \neq j$) and ${}_i\odot$-occurrence $\sqsubseteq_F$ ${}_j\odot$-occurrence, then ${}_j\odot'$ is said to be in the *scope* of ${}_i\odot$; this fact is denoted by ${}_i\odot \lessdot_F {}_j\odot'$. If $\odot$ ($\odot'$) is $\forall x$ or $\exists x$, we always write ${}_ix \lessdot_F {}_j\odot'$ (${}_i\odot \lessdot_F {}_jx$); and when $\odot'$ ($\odot$) is $\forall y$ or $\exists y$, we write ${}_ix \prec_F {}_jy$ (${}_iy \prec_F {}_jx$) underlining the fact that $\lessdot_F$ is restricted only in the case of the consideration of quantifiers variables.

Obviously, for any formula F, $\prec_F \subset \lessdot_F$ and the relations $\prec_F$ and $\lessdot_F$ are irreflexive and transitive.

We also *extend the (transitive and irreflexive) relations* $\prec_F$ *and* $\lessdot_F$

determined by an original formula F to the case of indexed symbols in the following way: for any natural numbers i and j we have ${}^{i}x \prec_F {}^{j}y$ and ${}^{i}\odot \lessdot_F {}^{j}\odot'$ if and only if $x \prec_F y$ and $\odot \lessdot_F \odot'$ respectively, where x, y, $\odot$, and $\odot'$ occur in F.

Moreover, any occurrence ${}_{i}\odot$ of a symbol $\odot$ in a formula F is treated as a new symbol. Therefore, ${}_{i}\odot$ and ${}_{j}\odot$ $(i \neq j)$ are different symbols denoting the same logical "operation" $\odot$. That is why ${}_{i}\odot$- and ${}_{j}\odot$-occurrences in F can be considered as different subformulas of F $(i \neq j)$, if needed.

The *extensions of* $\prec_F$ and *of* $\lessdot_F$ to the case of an original and indexed sequent S ($\prec_S$ and $\lessdot_S$) are obvious. The same relates to the case of a set Ξ of original and indexed formulas or of sequents ($\prec_\Xi$ and $\lessdot_\Xi$).

All the above-given extensions do not lead to confusion since we already begin all our investigations with an original formula or with an original sequent.

For the formula F^*, we have the following numbering of its logical connectives, predicate symbols and variables:

${}_1\exists\ {}_1\overline{y}\ {}_2\neg\ {}_3\exists\ {}_3\underline{x}\ {}_4P({}_3\underline{x}, f({}_1\overline{y}))\ {}_5\supset\ {}_6\neg\ {}_7\forall\ {}_7\underline{y}'\ {}_8\exists\ {}_8\overline{x}'\ {}_9P({}_8\overline{x}', {}_7\underline{y}')$.

Therefore, ${}_5\supset$ is the minimal element of $\lessdot_{F^*}$, ${}_1\overline{y} \prec_{F^*} {}_3\underline{x}$, ${}_2\neg \lessdot_{F^*} {}_3\underline{x}$; the variables ${}_3\underline{x}$ and ${}_8\overline{x}'$ are not comparable, the same concerns ${}_3\underline{x}$ and ${}_7\neg$ as well.

For the formula ${}_1\exists\ {}_1{}^3\overline{y}\ {}_2\neg\ {}_3\exists\ {}_3\underline{x}\ {}_4P({}_3\underline{x}, f({}_1{}^3\overline{y}))\ {}_5\supset {}_6\neg\ {}_7\forall\ {}_7\underline{y}'\, {}_8P({}^1\overline{x}', {}_7\underline{y}')$, being the result of the elimination of ${}_8\exists\ {}_8\overline{x}'$ in F^* and subsequent indexing of $\overline{y}$ and $\overline{x}'$, we have in particular that ${}_1{}^3\overline{y} \prec_{F^*} {}_3\underline{x}$ and the variables ${}_3\underline{x}$ and ${}^1\overline{x}'$ are not comparable.

A *substitution*, σ, is a finite mapping from variables to terms denoted by $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, where variables $x_1, \ldots, x_n$ are pairwise different and x_i is distinguished from t_i for all $i = 1 \ldots n$. For an expression Ex and a substitution σ, the result of the application of σ to the expression of Ex is understood in the usual sense; it is denoted by $Ex \cdot \sigma$.

For any set Ξ of expressions, $\Xi \cdot \sigma$ denotes the set obtained by the application of σ to every expression in Ξ. If Ξ is a set of (at least two) expressions and $\Xi \cdot \sigma$ is a singleton, then σ is called a *unifier* of Ξ. The notion of a *most general simultaneous unifier (mgsu)* of a set of expressions also is understood in the usual sense.

For any formula F (for any sequent S, for any set Ξ of formulas or sequents), each substitution σ induces a (possibly empty) relation $\ll_{F,\sigma}$ ($\ll_{S,\sigma}, \ll_{\Xi,\sigma}$) as follows: $y \ll_{F,\sigma} x$ ($y \ll_{S,\sigma} x$, $y \ll_{\Xi,\sigma} x$) if and only if there exists $x \mapsto t \in \sigma$ such that x is a dummy in F (S, Ξ), the term t contains y, and y is a parameter in F (S, Ξ). Obviously, $\ll_{F,\sigma}$ ($\ll_{S,\sigma}$, $\ll_{\Xi,\sigma}$) is an irreflexive relation (i.e. a relation that does not have any pair $\langle z, z\rangle$).

For example, consider the substitution $\sigma^* = \{\underline{x} \mapsto \overline{x}', \underline{y}' \mapsto f(\overline{y})\}$, where $\underline{x}$ and $\underline{y}'$ are dummies in F^* and $\overline{x}'$ and $\overline{y}$ are parameters in F^*. Then $\overline{x}' \ll_{F,\sigma} \underline{x}$ and $\overline{y} \ll_{F,\sigma} \underline{y}'$.

For a substitution σ and for an original formula F (for an original sequent S, for a set Ξ of expressions), $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$, $\lhd_{\Xi,\sigma}$) denotes the transitive closure of $\prec_F \cup \ll_{\{F\},\sigma}$ (of $\prec_S \cup \ll_{S,\sigma}$, of $\prec_\Xi \cup \ll_{\Xi,\sigma}$). At the same time, $\blacktriangleleft_{F,\sigma}$ ($\blacktriangleleft_{S,\sigma}$, $\blacktriangleleft_{\Xi,\sigma}$) denotes the transitive closure of $\lessdot_F \cup \ll_{\{F\},\sigma}$ (of $\lessdot_S \cup \ll_{S,\sigma}$, of $\lessdot_\Xi \cup \ll_{\Xi,\sigma}$).

We extend the notions of $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$, $\lhd_{\Xi,\sigma}$) and $\blacktriangleleft_{F,\sigma}$ ($\blacktriangleleft_{S,\sigma}$, $\blacktriangleleft_{\Xi,\sigma}$) to corresponding indexed units in the same way that was used when defining $\prec_F$ ($\prec_S$, $\prec_\Xi$) and $\lessdot_F$ ($\lessdot =_S$, $\lessdot_\Xi$).

A substitution σ is *admissible* (cf. [17]) for a formula F (for a sequent S, for a set Ξ of expressions) if and only if for every $x \mapsto t \in \sigma$, x is a dummy in F (in S, in Ξ), and $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$, $\lhd_{\Xi,\sigma}$) is an irreflexive relation.

For the above-given formula F^* and substitution σ^*, we have: ${}_1\overline{y} \lhd_{F^*,\sigma^*} {}_7\underline{y}' \lhd_{F^*,\sigma^*} {}_8\overline{x}' \lhd_{F^*,\sigma^*} {}_3\underline{x}$. Thus, σ^* is admissible substitution for F^*.

If $\sigma' = \{{}_7\underline{y}' \mapsto {}_8\overline{x}'\}$, then ${}_8\overline{x}' \ll_{\sigma'} {}_7\underline{y}'$. Since ${}_7\underline{y}' \prec {}_8\overline{x}'$, ${}_8\overline{x}' \lhd_{F^*,\sigma'} {}_8\overline{x}'$. Therefore, σ' is a reflexive relation and it is not admissible for F^*.

Obviously, $\lhd_{F,\sigma} \subseteq \blacktriangleleft_{F,\sigma}$ ($\lhd_{S,\sigma} \subseteq \blacktriangleleft_{S,\sigma}$, $\lhd_{\Xi,\sigma} \subseteq \blacktriangleleft_{\Xi,\sigma}$). Therefore, the following facts hold on the base of the definitions.

Proposition 1

The relation $\blacktriangleleft_{F,\sigma}$ as well as $\blacktriangleleft_{S,\sigma}$ and $\blacktriangleleft_{\Xi,\sigma}$ are irreflexive (antisymmetric) if and only if $\lhd_{F,\sigma}$ as well as and $\lhd_{S,\sigma}$ and $\lhd_{\Xi,\sigma}$ are irreflexive (antisymmetric) relations. Moreover, the irreflexivity of $\blacktriangleleft_{F,\sigma}$ (of $\blacktriangleleft_{S,\sigma}$, of $\blacktriangleleft_{\Xi,\sigma}$) implies the antisymmetry of $\blacktriangleleft_{F,\sigma}$ (of $\blacktriangleleft_{S,\sigma}$, of $\blacktriangleleft_{\Xi,\sigma}$) and vise versa.

This proposition permits the investigation of the irreflexivity (or the antisymmetry) of $\lhd$ to replace by the investigation of the irreflexivity (or the antisymmetry) of $\blacktriangleleft$ and vice versa.

Let F be a formula and ${}_{j_1}\odot_1, \ldots, {}_{j_r}\odot_r$ a sequence of all its logical connectives occurrences being maybe indexed. Let Tr_F be an inference tree for the initial sequent $\rightarrow F$ such that if $\alpha_{Tr_F}({}_{j_1 i}\odot_1)$ denotes an inference rule application eliminating the occurrence ${}_{j_1}\odot_1$ in F then Tr can be constructed in accordance with the order determined by the sequence $\alpha_{Tr_F}({}_{j_1}\odot_1), \ldots, \alpha_{Tr_F}({}_{j_r}\odot_r)$. In this case, ${}_{j_1}\odot_1, \ldots, {}_{j_r}\odot_r$ is called a *proper sequence* for Tr_F. (It is obvious that there may exist a connectives occurrences sequence for a formula F such that the sequence is not proper for any Tr_F. Besides, it must be clear that there may exist more than one proper sequence for an inference tree Tr_F in the case of the existence of one for F.)

Let F be a formula and Tr_F an inference tree for the initial sequent $\rightarrow F$. The tree Tr_F is called *compatible* with a substitution σ if and only if there exists a proper sequence ${}_{j_1}\odot_1, \ldots, {}_{j_r}\odot_r$ for Tr_F such that for any natural numbers m and n, the property $m < n$ implies that the ordered pair $\langle {}_{j_n}\odot_n, {}_{j_m}\odot_m \rangle$ does not belong to $\blacktriangleleft_{F,\sigma}$.

The results of the next section demonstrate the importance of the notion of compatibility for the intuitionistic case, while it is redundant for classical one as a whole and must be "transformed" into the notion of admissibility. There are several examples clarifying the role of compatibility in the next section.

4. Herbrand theorems

This section contains the main results of the paper, which condense the investigations presented in [7, 11, 8] in a unified form. Their proofs are omitted here; they contains in the subsequent sections. Besides, here we restrict ourselves only by considering *original* syntactical units for clearness of all the necessary constructions and theorems. Additionally note that without loss of generality, we are interested in establishing the deducibility of a sequent of the form $\rightarrow F$, where F is a closed formula.

Let F be a formula and $F_1, \ldots, F_n$ its variants. If $F_1, \ldots, F_n$ does not have any bound variables in pairs, then $F_1 \wedge \ldots \wedge F_n$ ($F_1 \vee \ldots \vee F_n$) is called a *variant $\wedge$-duplication* (a *variant $\vee$-duplication*).

Herbrand extension. Let G be a formula, F its subformula, and H a variant $\wedge$-duplication ($\vee$-duplication) of F not having common variables with G. Then the result of the replacement of F by H in G is called a *one-step Herbrand extension of* G. Further, the result $HE(G)$ of a finite sequence of one-step extensions consequently applied to G, then to a one-step Herbrand extension of G, and so on is called a *Herbrand extension* of G. If $HE(G)$ is generated by means of only $\wedge$-extensions, H is called an *intuitionistic Herbrand extension* of G.

Herbrand quasi-universe. Let F be a formula. Then $HQ(F)$ denotes the following minimal set of terms (called a *Herbrand quasi-universe*): (i) every constant and every parameter occurring in F belong to $HQ(F)$ (if there is no constant in F then the special constant $c_0 \in HQ(F)$); (ii) if f is a k-ary functional symbol and terms $t_1, \ldots, t_k \in HQ(F)$ then $f(t_1, \ldots, t_k) \in HQ(F)$.

In other words, $HQ(F)$ can be considered as a minimal set of terms constructing from constants and parameters occurring in F with the help of functional symbols of F with arities more that 0.

So, pLK and pLJ denote the propositional parts of LK and LJ, respectively, which means that pLK and pLJ do not contain quantifier rules, as well as $(Con \rightarrow)$ and $(\rightarrow Con)$ (see the next section) when antecedents and succedents of sequents are identified with multisets.

Theorem 1 (Sequent form of Herbrand's theorem for classical logic)

For a formula F, the sequent $\rightarrow F$ is deducible in the calculus LK (in any sequent calculus coextensive with LK) if and only if there are an Herbrand extension $HE(F)$ and a substitution σ of terms from the Herbrand quasi-universe $HQ(F)$ for all the dummies of $HE(F)$ such that

(i) there exists a proof tree $Tr_{\mu(HE(F))\cdot\sigma}$ for $\rightarrow \mu(HE(F))\cdot\sigma$ in pLK and

(ii) σ is an admissible substitution for $HE(F)$.

For intuitionistic logic, Theorem 1 transforms to the following form.

Theorem 2 (Sequent form of Herbrand's theorem for intuitionistic logic)

For a formula F, the sequent $\rightarrow F$ is deducible in the calculus LJ (in any sequent calculus coextensive with LJ) if and only if there are an intuitionistic Herbrand extension $HE(F)$ and a substitution σ of terms from the Herbrand quasi-universe $HQ(HE(F))$ for all the dummies of $HE(F)$ such that

(i) there exists a proof tree $Tr_{\mu(HE(F))\cdot\sigma}$ for $\rightarrow \mu(HE(F))\cdot\sigma$ in pLJ,

(ii) σ is an admissible substitution for $HE(F)$, and

(iii) $Tr_{\mu(HE(F))\cdot\sigma}$ is compatible with σ.

Draw your attention to the fact that Theorems 1 and 2 are distinguished by only the existence of (iii) in Theorem 2 (and by the calculi LK and LJ). The requirement (iii) is essential for intuitionistic logic. It is easy to check this fact with the help of the following simple examples.

Example 1. Let we have the sequent S: $\rightarrow F$, where F is $\neg\forall x P(x) \supset \exists y \neg P(y)$ ($\rightarrow F$ is deducible in LK and is not deducible in LJ). Obviously, for any intuitionistic Herbrand extension $HE(F)$, $\mu(HE(F))$ has the form $\neg(P(\overline{x}_{1,1}) \wedge \ldots \wedge P(\overline{x}_{1,p_1})) \wedge \ldots \wedge \neg(P(\overline{x}_{r,1}) \wedge \ldots \wedge P(\overline{x}_{r,p_r})) \supset \neg P(y)$ and Herbrand quasi-universe for it is equal to $\{c_0, \overline{x}_{1,1}, \ldots, \overline{x}_{r,p_r}\}$.

For $\rightarrow \mu(HE(F))$, any substitution $\sigma_{i,j} = \{y \mapsto \overline{x}_{i,j}\}$, where i and j are any natural numbers not exceeding r and p_r respectively, leads to a possibility to construct a proof tree $Tr_{i,j}$ for the selected extension $\mu(HE(F))$. (Obviously, the substitution $\{y \mapsto c_0\}$ does not have such a property.) It is easy to check the admissibility of $\sigma_{i,j}$ for $HE(F)$ and the absence of compatibility of $Tr_{i,j}$ with $\sigma_{i,j}$ regardless of the selection of $Tr_{i,j}$ and $\sigma_{i,j}$.

As a result, we have the deducibility of S in LK by Theorem 1 and the non-deducibility of S in LJ by Theorem 2. (When constructing any proof tree for S in LJ, any relation $\blacktriangleleft_{HE(F),\sigma_{i,j}}$ requires the application of the rule eliminating the first negation of F on the second step of deducing the proof tree, which is impossible to do in LJ for S.)

Example 2. If we slightly modify Example 1, taking $\rightarrow \exists x \neg P(x) \supset \neg \forall y P(y)$ as S, we have for S: $HQ(S) = \{c_0, \overline{x}\}$ and the substitution $\{y \mapsto \overline{x}\}$ is admissible for S. In this case, any proof tree for $\rightarrow \neg P(\overline{x}) \supset \neg P(\overline{x})$ is compatible with $\{y \mapsto \overline{x}\}$. Thus, S is deducible in LJ (and, of course, in LK).

Example 3. If we take $\rightarrow \forall y \exists x P'(y, x) \supset \exists y_1 \forall x_1 P'(x_1, y_1)$ as a sequent S, we have: $HQ(S) = \{x, x_1\}$ and for the substitution $\sigma = \{y \mapsto x_1, y_1 \mapsto x\}$, the sequent $\rightarrow P'(x_1, x) \supset P'(x_1, x)$ is deducible in pLK. Unfortunately, σ is not admissible for S and we cannot say anything about the deducibility of S even in LK. But it is easy to show that the construction of any Herbrand extension of $\forall y \exists x P'(y, x) \supset \exists y_1 \forall x_1 P'(x_1, y_1)$ cannot lead to an admissible substitution for any Herbrand extension and any its proof tree. Therefore, S is not deducible neither in LK nor in LJ.

As you can see, the above-given examples demonstrate that in comparison with Theorem 1, the grave disadvantage of Theorem 2 consists in the existence of the condition (iii) requiring a certain form of a proof tree for a sequent $\mu(\rightarrow F) \cdot \sigma$ in pLJ (or in its any analogue coextensive with pLJ): it must be *compatible* with σ, which does not permit any order of propositional rules applications leading to Tr while, in the classical case, any proof tree Tr for a sequent $\mu(\rightarrow F) \cdot \sigma$ in pLK admits any order of propositional rules applications leading to Tr.

Finally, note that since LK and LJ are sound and complete calculi, the obtained results permit to reduce the investigation the semantic characterization of classical and/or intuitionistic validity (of first-order formulas) to propositional deducibility satisfying certain conditions. Additionally, pay your attention to the fact that Theorem 1 can easily be transformed into some of sequent forms of Herbrand's theorem given in [7] for classical logic.

The rest of the paper is devoted to proving the Herbrand theorems. Only the syntactical approach is used at that.

5. Calculi LG, LK', and LJ'

For proving the main results, it is convenient to transform the cut-free calculi LK and LJ from [12] into calculi LK' and LJ' over $Sig \cup Var$ that are

$$\frac{\Gamma, A^k \wedge B \rightarrow \Delta}{\Gamma, A, B \rightarrow \Delta}\ (\wedge \rightarrow) \qquad \frac{\Gamma \rightarrow A^k \wedge B, \Delta}{\Gamma \rightarrow A, \Delta \quad \Gamma \rightarrow B, \Delta}\ (\rightarrow \wedge)$$

$$\frac{\Gamma, A^k \vee B \rightarrow \Delta}{\Gamma, A \rightarrow \Delta \quad \Gamma, B \rightarrow \Delta}\ (\vee \rightarrow)$$

$$\frac{\Gamma \rightarrow A^k \vee B, \Delta}{\Gamma \rightarrow A, \Delta}\ (\rightarrow \vee_1) \qquad \frac{\Gamma \rightarrow A^k \vee B, \Delta}{\Gamma \rightarrow B, \Delta}\ (\rightarrow \vee_2)$$

$$\frac{\Gamma, A^k \supset B \rightarrow \Delta}{\Gamma \rightarrow A, \Delta \quad \Gamma, B \rightarrow \Delta}\ (\supset \rightarrow) \qquad \frac{\Gamma \rightarrow A^k \supset B, \Delta}{\Gamma, A \rightarrow B, \Delta}\ (\rightarrow \supset)$$

$$\frac{\Gamma, {}^k\neg A \rightarrow \Delta}{\Gamma \rightarrow A, \Delta}\ (\neg \rightarrow) \qquad \frac{\Gamma \rightarrow {}^k\neg A, \Delta}{\Gamma, A \rightarrow \Delta}\ (\rightarrow \neg)$$

$$\frac{\Gamma, {}^k\forall {}^k\underline{x} A({}^k\underline{x}) \rightarrow \Delta}{\Gamma, A(t_k) \rightarrow \Delta}\ (\forall \rightarrow) \qquad \frac{\Gamma \rightarrow {}^k\forall {}^k\overline{x} A({}^k\overline{x}), \Delta}{\Gamma \rightarrow A({}^k\overline{x}), \Delta}\ (\rightarrow \forall)$$

$$\frac{\Gamma, {}^k\exists {}^k\overline{x} A({}^k\overline{x}) \rightarrow \Delta}{\Gamma, A({}^k\overline{x}) \rightarrow \Delta}\ (\exists \rightarrow) \qquad \frac{\Gamma \rightarrow {}^k\exists {}^k\underline{x} A({}^k\underline{x}), \Delta}{\Gamma \rightarrow A(t_k), \Delta}\ (\rightarrow \exists)$$

$$\frac{\Gamma, A \rightarrow \Delta}{\Gamma, A, {}^l A \rightarrow \Delta}\ (Con \rightarrow) \qquad \frac{\Gamma \rightarrow A, \Delta}{\Gamma \rightarrow A, {}^l A, \Delta}\ (\rightarrow Con)$$

$$\frac{\Gamma, A \rightarrow A, \Delta}{}\ (Ax)$$

In (Ax), A is an atomic formula. In $(Con \rightarrow)$ and $(\rightarrow Con)$, l denotes a new index w.r.t. an inference tree constructed before applications of $(Con \rightarrow)$ and $(\rightarrow Con)$. The rules $(\rightarrow \forall)$ and $(\exists \rightarrow)$ do not introduce any confusion w.r.t. the parameter ${}^k\overline{x}$ due to the convention about bound variables in an initial sequent. The term t_k satisfies the eigenvariable condition for ${}^k\underline{x}$ in both $(\rightarrow \exists)$ and $(\forall \rightarrow)$; moreover it contains only constants and functional symbols occurring in an initial sequent. This is the unique restriction for converting LG into LK'. But in the case of LJ', we must additionally require that the succedent of any sequent does not contain more than one formula; therefore, the rule $(\rightarrow Con)$ is redundant as well as Δ is an empty multiset in (Ax) and in all the rules eliminating logical connectives in succedents.

Figure 1: Calculi LG

adopted for using multisets as a succedent and an antecedent of any sequent. This permits to use the usual contraction rules (denoted by $(\rightarrow Con)$ and $(Con \rightarrow)$ here) as the only structural rules of LK' and LJ' since we have no restrictions to the number of formulas in succedents and antecedents of axioms in the case of classical logic and in the case of intuitionistic logic, we require the succedent of any sequent to contain no more than one formula.

The calculi LK' and LJ' are convenient to be determined with the help of a special calculus LG: LK' and LJ' are constructed from it by putting certain restrictions on LG.

Fig. 1 contains the description of LG. Remind that *inference trees in LG are applied "from top to bottom"* in order to attempt to construct a proof tree beginning with a sequent under consideration and finishing by axioms.

Another *distinctive feature* of LG concerns its axioms. Since we consider that any initial sequent (i.e. the sequent being investigated on deducibility) has the form $\rightarrow F$, where F is a closed formula, the process of the construction of a proof tree for it begins with the sequent $\rightarrow {}^1F$ called a *starting sequent* for LG and for calculi described below.

For the calculus LG, we have the following obvious result.

Proposition 2

For a formula F, a starting sequent $\rightarrow^1 F$ is deducible in the calculus LG in a way satisfying the restrictions for LJ' (for LK') if and only if an initial sequent $\rightarrow F$ is deducible in the calculus LJ (in LK) (in any sequent calculus coextensive with LJ (LK)) in the usual sense.

The process of inferring in LG is not essentially distinguished from inference search in both LK and LJ. That is why there are no examples for it; moreover, LG uses Gentzen's notion of admissibility, i.e. it contains the quantifier rules with the eigenvariable condition that leads to the great inefficiency of logical search. The calculus LB from the next section contains a certain idea how we can improve the efficiency of logical inference search in sequent (and tableau) calculi. (Also see [7], where some discussion of different notions of admissibility is made.)

6. Proofs of main results

To prove our main results, in this paper we follow the ideas of obtaining the Herbrand theorems suggested in [7, 8]. That is why we simply formulate "key" propositions giving the schemes of their proofs. So, all considerations concern the case of indexed formulas in general.

6.1. Basic calculus LB

An *equation* is an unordered pair of terms s and t written as $s \approx t$. Assume L is an atomc formula of the form $R(t_1, \ldots, t_n)$ and M is an atomic formula of the form $R(s_1, \ldots, s_n)$ where R is a predicate symbol and

$t_1, \ldots, t_n, s_1, \ldots, s_n$ are terms. Then $\Sigma(L, M)$ denotes the set of equations $\{t_1 \approx s_1, \ldots, t_n \approx s_n\}$. In this case, L and M are said to be *equal modulo* $\Sigma(L, M)$ ($L \approx M$ *modulo* $\Sigma(L, M)$).

The calculus LB without the quantifier rules, as well as without $(Con \rightarrow)$ and $(\rightarrow Con)$ is denoted by pLB. (Note that de facto pLB is pLK when succedents and antecedents of sequents are considered as multisets.)

Proposition 3

For a closed formula F, the starting sequent $\rightarrow^1 F$ is deducible in the calculus LG satisfying the restrictions for LJ' (for LK') if and only if there exists an inference tree $Tr_{^1F}$ in LB such that below-given (1), (2), and (3) ((1) and (2)) hold:

(1) all the leaves of $Tr_{^1F}$ are quasi-axioms and there exists the mgsu σ of the sets of equations from all the quasi-axioms of Tr,
(2) σ is admissible for the set of all the sequences of $Tr_{^1F}$, and
(3) $Tr_{^1F}$ is compatible with σ.

A *proof* of Prop. 3 can be obtained on the base of the following arguments in a way analogous to the one used when proving Prop. 2 in [7] and adapted to the consideration of both the classical and intuitionistic cases (cf. [8]).

First of all note that for proving the proposition, the item (2) can not be taken into consideration on account of Prop. 1. It is present in the proposition since the examination of compatibility for the classical case becomes redundant as a whole; it is replaced by the verification of only admissibility as its part. The validity of this also is confirmed by the results of [7].

For proving the *sufficiency*, let $Tr_{^1F}$ be a proof tree for a starting sequent $\rightarrow {}^1F$ in the calculus LB, σ a substitution that unifies all the equations of leaves of $Tr_{^1F}$ compatible with σ. Without loss of generality, we can assume that terms of σ do not contain dummies (otherwise, they could be replaced by a constant, say, c_0) and that for any dummy from $Tr_{^1F}$, there exists the term t such that $x \mapsto t \in \sigma$.

Since $\blacktriangleleft_{\Xi,\sigma}$ is an antisymmetric relation (see Prop. 1), the following statement has place: $\blacktriangleleft_{\Xi,\sigma}$ can be completed to a *strict linear order relation* ${\blacktriangleleft^+}_{\Xi,\sigma}$ (that is ${\blacktriangleleft^+}_{\Xi,\sigma}$ is determined for any different pair of elements from $\blacktriangleleft_{\Xi,\sigma}$ and $\odot \blacktriangleleft_{\Xi,\sigma} \odot'$ implies $\odot \, {\blacktriangleleft^+}_{\Xi,\sigma} \, \odot'$).

Due to the statement, the principle connective of ${}^1F$ is the first element of the linear order ${\blacktriangleleft^+}_{\Xi,\sigma}$; moreover we can construct the tree $Tr_{^1F}$ applying rules of LB that subsequently eliminate the logical connectives in sequents

$$\frac{\Gamma, A^k{\wedge}B \to \Delta}{\Gamma, A, B \to \Delta}\ (\wedge \to) \qquad \frac{\Gamma \to A^k{\wedge}B, \Delta}{\Gamma \to A, \Delta \quad \Gamma \to B, \Delta}\ (\to \wedge)$$

$$\frac{\Gamma, A^k{\vee}B \to \Delta}{\Gamma, A \to \Delta \quad \Gamma, B \to \Delta}\ (\vee \to)$$

$$\frac{\Gamma \to A^k{\vee}B, \Delta}{\Gamma \to A, \Delta}\ (\to \vee_1) \qquad \frac{\Gamma \to A^k{\vee}B, \Delta}{\Gamma \to B, \Delta}\ (\to \vee_2)$$

$$\frac{\Gamma, A^k \supset B \to \Delta}{\Gamma \to A, \Delta \quad \Gamma, B \to \Delta}\ (\supset\to) \qquad \frac{\Gamma \to A^k \supset B, \Delta}{\Gamma, A \to B, \Delta}\ (\to\supset)$$

$$\frac{\Gamma, {}^k\neg A \to \Delta}{\Gamma \to A, \Delta}\ (\neg \to) \qquad \frac{\Gamma \to {}^k\neg A, \Delta}{\Gamma, A \to \Delta}\ (\to \neg)$$

$$\frac{\Gamma, {}^k\forall {}^k\underline{x} A({}^k\underline{x}) \to \Delta}{\Gamma, A({}^k\underline{x}) \to \Delta}\ (\forall \to) \qquad \frac{\Gamma \to {}^k\forall {}^k\overline{x} A({}^k\overline{x}), \Delta}{\Gamma \to A({}^k\overline{x}), \Delta}\ (\to \forall)$$

$$\frac{\Gamma, {}^k\exists {}^k\overline{x} A({}^k\overline{x}) \to \Delta}{\Gamma, A({}^k\overline{x}) \to \Delta}\ (\exists \to) \qquad \frac{\Gamma \to {}^k\exists {}^k\underline{x} A({}^k\underline{x}), \Delta}{\Gamma \to A({}^k\underline{x}), \Delta}\ (\to \exists)$$

$$\frac{\Gamma, A \to \Delta}{\Gamma, A, {}^lA \to \Delta}\ (Con \to) \qquad \frac{\Gamma \to A, \Delta}{\Gamma \to A, {}^lA, \Delta}\ (\to Con)$$

$$\frac{\Gamma, A \to A', \Delta}{}\ (QuAx)$$

In $(Con \to)$ and $(\to Con)$, l denotes a new index w.r.t. an inference tree constructed before applications of $(Con \to)$ and $(\to Con)$. In a quasi-axiom $(QuAx)$, $A \approx A'$ *modulo* $\Sigma(A, A')$, where A and A' are atomic formulas. In compliance with the above-given convention about parameters and dummies, ${}^k\overline{x}$ always is considered as a parameter in the rules $(\to \forall)$ and $(\exists \to)$; ${}^k\underline{x}$ always is considered as a dummy in the rules $(\to \exists)$ and $(\forall \to)$. There are no other restrictions in the classical case. But in the intuitionistic case, the succedent of any sequent does not contain more than one formula and, therefore, the rule $(\to Con)$ is redundant as well as Δ is an empty multiset in $(QuAx)$ and in all the rules eliminating logical connectives in succedents.

Figure 2: Calculi LB

under consideration in accordance with strict linear order $\blacktriangleleft_{\Xi,\sigma}$; such a possibility is guaranteed by the statement and by that any rule of any sequent calculus under consideration always eliminates the principle connective of a formula in a sequent which the rule is applied to).

Since the calculi LG and LB are distinguished by the quantifier rules only, it is easy to transform Tr_{1F} to a tree Tr'_{1F} for the starting sequent $\to {}^1F$ by replacing applications of the quantifier rules $(\forall \to)$ and $(\to \exists)$ of

LB by appropriate applications of quantifier rules $(\forall \rightarrow)$ and $(\rightarrow \exists)$ of LG in accordance with the following instructions: in $Tr_{{}^1F}$, every rule application

$$\frac{\Gamma, {}^k\forall^k \underline{x} A({}^k\underline{x}) \rightarrow \Delta}{\Gamma, A({}^k\underline{x}) \rightarrow \Delta}\ (\forall \rightarrow) \qquad \left(\frac{\Gamma \rightarrow {}^k\exists^k \underline{x} A({}^k\underline{x}), \Delta}{\Gamma \rightarrow A({}^k\underline{x}), \Delta}\ (\rightarrow \exists)\right)$$

is replaced by the rule application

$$\frac{\Gamma, {}^k\forall^k \underline{x} A({}^k\underline{x}) \rightarrow \Delta}{\Gamma, A(t_k) \rightarrow \Delta}\ (\forall \rightarrow) \qquad \left(\frac{\Gamma \rightarrow {}^k\exists^k \underline{x} A({}^k\underline{x}), \Delta}{\Gamma \rightarrow A(t_k), \Delta}\ (\rightarrow \exists)\right)$$

where ${}^kx \mapsto t_k \in \sigma$. Note that in two last rules, t_k satisfies the eigenvariable condition for ${}^kx$ in $Tr'_{{}^1F}$ because of the compatibility of $Tr_{{}^1F}$ with σ (and, as a result, the admissibility of σ for the set of all the sequents in $Tr_{{}^1F}$). Thus, $Tr'_{{}^1F}$ is a proof tree or $\rightarrow^1 F$ in LG and the sufficiency is proved.

To prove the *necessity*, we must perform the "converse" transformation. This means that in a proof tree $Tr'_{{}^1F}$ for a starting sequent $\rightarrow {}^1F$, every application of the rule $(\forall \rightarrow)$ $((\rightarrow \exists))$:

$$\frac{\Gamma, {}^k\forall^k \underline{x} A({}^k\underline{x}) \rightarrow \Delta}{\Gamma, A(t_k) \rightarrow \Delta}\ (\forall \rightarrow) \qquad \left(\frac{\Gamma \rightarrow {}^k\exists^k \underline{x} A({}^k\underline{x}), \Delta}{\Gamma \rightarrow A(t_k), \Delta}\ (\rightarrow \exists)\right)$$

is replaced by the rule application:

$$\frac{\Gamma, {}^k\forall^k \underline{x} A({}^k\underline{x}) \rightarrow \Delta}{\Gamma, A({}^k\underline{x}) \rightarrow \Delta}\ (\forall \rightarrow) \qquad \left(\frac{\Gamma \rightarrow {}^k\exists^k \underline{x} A({}^k\underline{x}), \Delta}{\Gamma \rightarrow A({}^k\underline{x}), \Delta}\ (\rightarrow \exists)\right)$$

"preserving" all the other rules application without any modification, which leads to an inference tree $Tr_{{}^1F}$ in LB containing only leaves with quasi-axioms.

If σ' is determined as the set containing all the ${}^kx \mapsto t_k$ from $Tr'_{{}^1F}$ and only them, then obviously, σ' is a unifier of the sets of equations from all the quasi-axioms of $Tr_{{}^1F}$. By the main property of unifiers, we conclude that there is a mgsu σ of the sets of equations from all the quasi-axioms of $Tr_{{}^1F}$. Since $Tr'_{{}^1F}$ is a proof tree in LG and the eigenvariable condition is satisfied for every ${}^kx \mapsto t_k \in \sigma'$, the substitution σ is admissible for the set of all the sequents in $Tr_{{}^1F}$ and, therefore, $Tr_{{}^1F}$ is compatible with σ by Prop. 1. This *finishes* the proof of Prop. 3.

Example 4. Let us establish the deducibility of the stating sequent $\rightarrow {}^1F^*$ in LG (denoted by S^*) by constructing a tree $Tr_{{}^1F^*}$ satisfying Prop. 3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\rightarrow {}^1\exists^1\overline{y}\,{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y}))\ {}^1\supset{}^1\neg{}^1\forall^1\underline{y}'\,{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \quad (\rightarrow {}^1F^*)}{{}^1\exists^1\overline{y}\,{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y})) \rightarrow {}^1\neg{}^1\forall^1\underline{y}'\,{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \quad (\text{by } (\rightarrow\supset))}}{{}^1\exists^1\overline{y}\,{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y})), {}^1\forall^1\underline{y}'\,{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \rightarrow \quad (\text{by } (\rightarrow \neg))}}{{}^1\exists^1\overline{y}\,{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y})), {}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \rightarrow \quad (\text{by } (\forall \rightarrow)}}{{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y})), {}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \rightarrow \quad (\text{by } (\exists \rightarrow)}}{{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \rightarrow {}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y})) \quad (\text{by } (\neg \rightarrow)}}{{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \rightarrow P({}^1\underline{x}, f({}^1\overline{y})) \quad (\text{by } (\rightarrow \exists))}}{P({}^1\overline{x}',{}^1\underline{y}') \rightarrow P({}^1\underline{x}, f({}^1\overline{y})) \quad ((QuAx), \text{ by } (\exists \rightarrow))}$$

Draw your attention to the fact that the order of quantifier rule applications in LB is immaterial and has no influence on the final result on deducibility.

The constructed tree contains the only one list being a quasi-axiom. For its left atomic formula A and right atomic formula A', have $A \approx A'$ *modulo* $\Sigma(A, A')$, where $\Sigma(A, A') = \{{}^1\overline{x}' \approx {}^1\underline{x}, {}^1\underline{y}' \approx f({}^1\overline{y})\}$

The substitution $\sigma^* = \{{}^1\underline{x} \mapsto {}^1\overline{x}', {}^1\underline{y}' \mapsto f({}^1\overline{y})\}$ is the unique mgsu of $\Sigma(A, A')$. In section 3 it was proven that σ^* is an admissible substitution for F^* and, as a result, for the set of all the sequents of $Tr_{{}^1F^*}$. Obviously, $Tr_{{}^1F^*}$ is compatible with σ^*. By Prop. 3, the starting sequent S^* is deducible in LG satisfying the restrictions for LJ' (and, of course, for LK'). Therefore, the initial sequent $\rightarrow F^*$ is deducible in both LJ and LK.

Example 5. If we consider the formula ${}^1F^{**}$: ${}^1\neg\ {}^1\forall^1\overline{y}\ {}^1\exists^1\underline{x}\ P({}^1\underline{x}, f({}^1\overline{y}))$ ${}^1\supset\ {}^1\exists^1\underline{y}'\ {}^1\neg\ {}^1\exists^1\overline{x}'\ P({}^1\overline{x}',{}^1\underline{y}')$ instead of ${}^1F^*$, it is easy to construct an inference tree $Tr_{{}^1F^{**}}$ for the starting sequent $\rightarrow {}^1F^{**}$ such that (1) and (2) from Prop. 2 hold. But it is impossible for the item (3) to take place for both $Tr_{{}^1F^{**}}$ and any other inference tree for $\rightarrow {}^1F^{**}$ independent of a generated mgsu at that. By Prop. 3 and Prop. 2, we have that the initial sequent $\rightarrow F^{**}$ is deducible in LK but not deducible in LJ.

Example 6. If we consider the formula ${}^1F^{***}$: ${}^1\forall^1\overline{y}\ {}^1\neg{}^1\forall^1\underline{x}\ P({}^1\underline{x}, f({}^1\overline{y}))$ ${}^1\supset\ {}^1\neg{}^1\forall^1\underline{y}'\ {}^1\exists^1\overline{x}'\ P({}^1\overline{x}',{}^1\underline{y}')$, we can prove that for any inference tree $Tr_{{}^1F^{***}}$ for the starting sequent $\rightarrow {}^1F^{***}$ it is impossible to construct a mgsu admissible for all the sequents of $Tr_{{}^1F^{***}}$. Therefore, by Prop. 3 and Prop. 2, the initial sequent $\rightarrow F^{***}$ cannot be deduced even in LK.

6.2. Convolution of inference trees

This section explores the idea suggested in [7] and modified for the "convolution" ("reduction") of a inference tree in LB into a certain sequent

$$\frac{\Gamma, \iota(A) \wedge \iota(B) \rightarrow \Delta}{\Gamma, \iota(A), \iota(B) \rightarrow \Delta}\ (\uparrow \wedge \rightarrow) \qquad \frac{\Gamma \rightarrow \iota(A) \wedge \iota(B), \Delta}{\Gamma \rightarrow \iota(A), \Delta \quad \Gamma \rightarrow \iota(B), \Delta}\ (\uparrow \rightarrow \wedge)$$

$$\frac{\Gamma, \iota(A) \vee \iota(B) \rightarrow \Delta}{\Gamma, \iota(A) \rightarrow \Delta \quad \Gamma, \iota(B) \rightarrow \Delta}\ (\uparrow \vee \rightarrow)$$

$$\frac{\Gamma \rightarrow \iota(A) \vee \iota(B), \Delta}{\Gamma \rightarrow \iota(A), \Delta}\ (\uparrow \rightarrow \vee_1) \qquad \frac{\Gamma \rightarrow \iota(A) \vee \iota(B), \Delta}{\Gamma \rightarrow \iota(B), \Delta}\ (\uparrow \rightarrow \vee_2)$$

$$\frac{\Gamma, \iota(A) \supset \iota(B) \rightarrow \Delta}{\Gamma \rightarrow \iota(A), \Delta \quad \Gamma, \iota(B) \rightarrow \Delta}\ (\uparrow \supset \rightarrow) \qquad \frac{\Gamma \rightarrow \iota(A) \supset \iota(B), \Delta}{\Gamma, \iota(A) \rightarrow \iota(B), \Delta}\ (\uparrow \rightarrow \supset)$$

$$\frac{\Gamma, \neg\iota(A) \rightarrow \Delta}{\Gamma \rightarrow \iota(A), \Delta}\ (\uparrow \neg \rightarrow) \qquad \frac{\Gamma \rightarrow \neg\iota(A), \Delta}{\Gamma, \iota(A) \rightarrow \Delta}\ (\uparrow \rightarrow \neg)$$

$$\frac{\Gamma, \forall^k \underline{x}\iota(A({}^k\underline{x})) \rightarrow \Delta}{\Gamma, \iota(A({}^k\underline{x})) \rightarrow \Delta}\ (\uparrow \forall \rightarrow) \qquad \frac{\Gamma \rightarrow \forall^k \overline{x}\iota(A({}^k\overline{x})), \Delta}{\Gamma \rightarrow \iota(A({}^k\overline{x}))\Delta}\ (\uparrow \rightarrow \forall)$$

$$\frac{\Gamma, \exists^k \overline{x}\iota(A({}^k\overline{x})) \rightarrow \Delta}{\Gamma, \iota(A({}^k\overline{x})) \rightarrow \Delta}\ (\uparrow \exists \rightarrow) \qquad \frac{\Gamma \rightarrow \exists^k \underline{x}\iota(A({}^k\underline{x})), \Delta}{\Gamma \rightarrow \iota(A({}^k\underline{x})), \Delta}\ (\uparrow \rightarrow \exists)$$

$$\frac{\Gamma, \iota(A) \wedge {}^k\iota(A) \rightarrow \Delta}{\Gamma, \iota(A), {}^k\iota(A) \rightarrow \Delta}\ (\uparrow Con \rightarrow) \qquad \frac{\Gamma \rightarrow \iota(A) \vee {}^k\iota(A), \Delta}{\Gamma \rightarrow \iota(A), {}^k\iota(A), \Delta}\ (\uparrow \rightarrow Con)$$

There are no restrictions in the classical case. But in the intuitionistic case, the succedent of any sequent does not contain more than one formula and, therefore, the rule $(\rightarrow Con)$ is redundant as well as Δ is an empty multiset in $(QuAx)$ and in all the rules "restored" logical connectives in succedents when reading rules "from bottom to up".

Figure 3: Convolution Calculus for LB

that can be modified after this into a sequent deduced by applying the only (propositional) rules of pLB. This gives us a possibility to obtain the main results of the paper – the "syntactical" forms of Herbrand theorems.

Let Tr be an inference tree for a starting sequent ${}^1S_0$ of the form $\rightarrow^1 F$ in the calculus LB, where F is an original closed formula. To every sequent S in Tr, we assign the sequent $\iota_{Tr}(S)$ or simply $\iota(S)$ (an analogue of the expression called the *spur* of S in [7]), as follows:

– If S is a leaf of Tr, then $\iota(S)$ is the result of deleting all the upper-left indices in logical connectives occurring in S.

– If S is not a leaf node and spurs are assigned to all its successors in an inference rule of LB, we assign $\iota(S)$ to S in accordance with the rules of the convolution calculus. Note that if a rule R of the calculus LB is applied to the sequent S in Tr ("from top to bottom"), the spur is assigned to S as prescribed by the rule R of the convolution calculus applied "bottom up".

Further, $\iota(Tr)$ denotes the result of the replacement of every sequent S from Tr by $\iota(S)$.

The following properties of proof trees can easily be proved by induction on the number of rules applications.

Proposition 4

Let Tr be an inference tree for a stating sequent ${}^1S_0$ in the calculus LB. Suppose all the leaves of Tr are quasi-axioms and there exists the mgsu σ of all the sets of equations from the quasi-axioms of Tr. If σ' denotes the result of the deletion all the upper-left indices in σ then the following properties hold w.r.t. $\iota(Tr)$, $\iota_{Tr}({}^1S_0)$, σ, and σ':

1) $\iota(Tr)$ is an inference tree for the initial (original) sequent $\iota_{Tr}({}^1S_0)$ in LB;

2) All the leaves of $\iota(Tr)$ are quasi-axioms containing only equations unified by the substitution σ';

3) $\mu(\iota(Tr)) \cdot \sigma'$ can be considered as an inference tree in the calculus pLB for the initial sequent $\mu(\iota_{Tr}({}^1S_0)) \cdot \sigma'$;

4) σ is admissible for the set of all the sequents of Tr if and only if σ' is admissible for $\iota_{Tr}({}^1S_0)$;

5) $\iota(Tr)$ is compatible with σ' if and only if Tr is compatible with σ.

Example 7. For $\rightarrow^1 F^*$ and σ^* from the Example 4, we have: $\iota_{Tr}(\rightarrow^1 F^*)$ is $\exists y \neg \exists x P(x, f(y)) \supset \neg \forall y' \exists x' P(x', y')$, $\mu(\iota_{Tr}(\rightarrow^1 F^*)$ is $\neg P(\underline{x}, f(\overline{y})) \supset \neg P(\overline{x}', \underline{y}')$, and $\sigma^{*\prime} = \{\underline{x} \mapsto \overline{x}', \underline{y}' \mapsto f(\overline{y})\}$.

For $Tr_{{}^1F^*}$, $\iota(Tr_{\rightarrow Tr_{{}^1F^*}})$ presents the tree

$$\dfrac{\rightarrow \exists \overline{y} \neg \exists \underline{x} P(\underline{x}, f(\overline{y})) \supset \neg \forall \underline{y}' \exists \overline{x}' P(\overline{x}', \underline{y}')}{} \quad (\rightarrow F^*)$$

$$\dfrac{\exists \overline{y} \neg \exists \underline{x} P(\underline{x}, f(\overline{y})) \rightarrow \neg \forall \underline{y}' \exists \overline{x}' P(\overline{x}', \underline{y}')}{} \quad (\text{by } (\rightarrow \supset))$$

$$\dfrac{\exists \overline{y}^1 \neg \exists \underline{x} P(\underline{x}, f(\overline{y})), \forall \underline{y}' \exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow}{} \quad (\text{by } (\rightarrow \neg))$$

$$\dfrac{\exists \overline{y} \neg \exists \underline{x} P(\underline{x}, f(\overline{y})), \exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow}{} \quad (\text{by } (\forall \rightarrow)$$

$$\dfrac{\neg \exists \underline{x} P(\underline{x}, f(\overline{y})), \exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow}{} \quad (\text{by } (\exists \rightarrow)$$

$$\dfrac{\exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow \exists \underline{x} P(\underline{x}, f(\overline{y}))}{} \quad (\text{by } (\neg \rightarrow)$$

$$\dfrac{\exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow P(\underline{x}, f(\overline{y}))}{} \quad (\text{by } (\rightarrow \exists))$$

$$P(\overline{x}', \underline{y}') \rightarrow P(\underline{x}, f(\overline{y})) \quad ((QuAx), \text{ by } (\exists \rightarrow))$$

The tree $\iota(Tr_{\to Tr_{1_{F^*}}})$ can be transformed into the tree $\mu(\iota(Tr_{\to Tr_{1_{F^*}}}))$:

$$\frac{\dfrac{\dfrac{\to \neg P(\underline{x}, f(\overline{y})) \supset \neg P(\overline{x}', \underline{y}') \quad (\to \iota F^*)}{\neg P(\underline{x}, f(\overline{y})) \to \neg P(\overline{x}', \underline{y}') \quad (\text{by } (\to\supset))}}{{}^1\neg P(\underline{x}, f(\overline{y})), P(\overline{x}', \underline{y}') \to \quad (\text{by } (\to \neg))}}{P(\overline{x}', \underline{y}') \to P(\underline{x}, f(\overline{y})) \quad ((QuAx), \text{ by } (\neg \to))}$$

It is easy to check that items (1)–(5) from Prop. 4 take place w.r.t. $\iota_{Tr}(\to^1 F^*)$, $\mu(\iota(Tr_{\to Tr_{1_{F^*}}}))$, σ^*, and $\sigma^{*\prime}$.

6.3. Proving Herbrand theorems

"Summarizing" the results of section 6.1 and 6.2, we obtain the "key statement" leading to the Herbrand theorems.

Proposition 5 ("Key statement")

A sequent ${}^1S_0$ of the form $\to^1 F$ is deducible in the calculus LG, where F is a closed original formula, if and only if there are an inference tree Tr for ${}^1S_0$ and a substitution σ of terms from the Herbrand quasi-universe $HQ(\iota_{Tr}({}^1S_0))$ for all the dummies in $\iota_{Tr}({}^1S_0)$ such that

(i) $\mu(\iota(Tr)) \cdot \sigma$ is a proof tree in the calculus pLB for the initial sequent $\mu(\iota_{Tr}({}^1S_0)) \cdot \sigma$;

(ii) σ is an admissible substitution for $HE(F)$;

(iii) the tree $\mu(\iota(Tr)) \cdot \sigma$ is a compatible with σ.

Indeed, Prop. 3 and 4 provide the truth of items (i), (ii), and (iii) since $\mu(\iota_{Tr}({}^1S_0))$ coincides with $HE(F)$. The requirement about taking terms from $HQ(\iota_{Tr}(S))$ is provided by the subformula property of LB (relating to terms of the quantifier rules) and the mgsu properties.

Example 8. Let ${}^1S_0$ denote the sequent $\to {}^1F^*$ and σ the substitution σ^* from the Example 7. If $\mu(\iota(Tr))$ is the tree $\mu(\iota(Tr_{\to Tr_{1_{F^*}}}))$ taking from the Example 7 too, then all the items of Prop. 5 hold. Thus, the sequent is deducible in LG.

Now, it is easy to convert the "key statement" to the Herbrand theorems.

First of all note that the construction of sequent $\iota_{Tr}({}^1S_0)$ for a sequent ${}^1S_0$ of the form $\to^1 F$, in accordance with the convolution, calculus produces a certain Herbrand extension $HE(F)$ being the succedent of $\iota_{Tr}({}^1S_0)$.

Moreover, when constructing a tree $\iota(Tr)$ for $\iota_{Tr}(^1S_0)$ by means of only rules of the calculus LJ', we obtain that $\iota_{Tr}(^1S_0)$ represents itself an intuitionistic Herbrand extension. Therefore, Theorem 2 holds in accordance with Prop. 2, 3, and 5.

The proof of Theorem 1 can be obtained in the same way, if we take into account the fact that checking compatibility is unnecessary for classical logic, since any step of the reduction of any formula to its prenex normal form have no influence on the deducibility of the formula in LK'. (Obviously, $\iota_{Tr}(^1S_0)$ represents itself a “full” Herbrand extension, which cannot obligatory be intuitionistic.)

If we compare the results of this paper with the results of some of papers being reminded in the introduction, we observe that in contrast to [17] and to papers based on it, Prop. 1 asserts that for checking admissibility in both the classical and the intuitionistic cases, we can restrict ourselves only by consideration of quantifier structures of a formula (of a sequent) investigating on deducibility, i.e. by examining $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$) only but not $\blacktriangleleft_{F,\sigma}$ ($\blacktriangleleft_{S,\sigma}$).

7. Conclusion

The paper presents author's results on Herbrand theorems for the sequent form of first-order classical and intuitionistic logics. The sequent formalism under consideration permitted to present the unified approach to wording and proving the Herbrand forms suggested here. Besides, it also gave a possibility to achieve enough general considerations: many famous Herbrand theorem forms for classical logic can be produced as its applications. Additionally note that obtained proofs have a transparent character and are connected with deducibility only.

The approach suggested in the paper is based on the development of the special technique of inference search in sequent calculi that has relatively high efficiency in comparison with the traditional methods based on Gentzen's or Kanger's notions of admissibility of substitutions. It can be improved in the direction of the optimization of rules applications, for example, in ways similar to that was explored for constructing the sequent calculi in [7] for classical logic and in [11, 8] for intuitionistic logic.

Another positive feature of the approach is that it may give a possibility for redescribing matrix characterization methods ([17]) and different modifications of the connection method ([16], [18], [19]) in its notions and notations. It may also be applied for constructing efficient methods of inference search in modal and other logics.

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AN INFINITE SEQUENCE OF PROPOSITIONAL CALCULI

It is described an infinite sequence of propositional calculi where the next calculus has more compound form of deduction theorem.

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1. Introduction

J. Łukasiewicz was the first logician who introduced a 3-valued propositional calculus [1] in order to consider a three values: falsehood, truth, and admissibility. This calculus has a sole binary bundle $\supset$. A. Church proposed to prove [2] that the deduction theorem of Łukasiewicz's calculus said: if $G, A \vdash B$ then $G \vdash A \supset (A \supset B)$. This fact is generalized in this paper.

2. Syntax

It is defined a next infinite sequence of propositional calculi Cn, $n \geq 1$. The propositional calculus Cn has next symbols: $\supset$ (binary bundle), (,) (two round brackets), $f_1, f_2, \ldots, f_n$ (constants) and an infinite list of variables p, q, r, s, p_i, q_i, r_i, s_i $(i \geq 1)$.

Formula (in the sense of a well-formed formula) of the propositional calculus Cn is defined inductively.

1. Every constant is a formula.

2. Every variable is a formula.

3. If A and B are formulas then $(A \supset B)$ is a formula too.

Let $A \overset{n}{\supset} B$ be a formula $A \supset (A \supset \ldots (A \supset B) \ldots)$ that contains n occurrences of the formula A and one occurrence of the formula B, $n \geq 1$; for example, $p \overset{3}{\supset} q$ is $p \supset (p \supset (p \supset q))$.

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Now for every n ($n \geq 1$) we introduce next formulas

$$A \overset{n}{\supset} \left(B \overset{n}{\supset} A\right), \tag{1n}$$

$$\left(A \overset{n}{\supset} \left(B \overset{n}{\supset} C\right)\right) \overset{n}{\supset} \left(\left(A \overset{n}{\supset} B\right) \overset{n}{\supset} \left(A \overset{n}{\supset} C\right)\right), \tag{2n}$$

$$\left(\left(A \overset{n}{\supset} f_n\right) \overset{n}{\supset} f_n\right) \overset{n}{\supset} A, \tag{3n}$$

where A, B and C are arbitrary formulas of the propositional calculus Cn.

Axiom schemes of the propositional calculus Cn ($n \geq 1$) are the formulas (1n), (2n), (3n).

A formula D is called an axiom according to the given axiom scheme if the D is received from this axiom scheme by substitution for A, B, C concrete formulas; for example, $p \supset (q \supset p)$ is an axiom of the calculus $C1$ according to the axiom scheme (11).

The calculus Cn has one rule of inference: B is derived from $A \overset{n}{\supset} B$ and A (this rule is called modus ponens).

A sequence

$$F_1, F_2, \ldots, F_m \tag{4}$$

of formulas of the calculus Cn is called a proof of this calculus if every F_i is an axiom according to some axiom scheme or it is derived from F_j and F_k by modus ponens where $j < i$ and $k < i$ ($m \geq 1$). The formula F_m from the sequence (4) is called a theorem of calculus Cn and it is written $\vdash_{Cn} F_m$.

Theorem 1

$\vdash_{Cn} A \overset{n}{\supset} A$ where the A is any formula of the calculus Cn, $n \geq 1$.

Proof. Let A be any formula of the calculus Cn, $n \geq 1$. An adequate sequence consists of 5 formulas F_i.

The F_1 is $A \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)$; it is the axiom scheme (1n).

The F_2 is $A \overset{n}{\supset} \left(\left(A \overset{n}{\supset} A\right) \overset{n}{\supset} A\right)$; it is the axiom scheme (1n).

The F_3 is $\left(A \overset{n}{\supset} \left(\left(A \overset{n}{\supset} A\right) \overset{n}{\supset} A\right)\right) \overset{n}{\supset} \left(\left(A \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)\right) \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)\right)$; it is the axiom scheme (2n).

The F_4 is $\left(A \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)\right) \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)$; it is derived from the F_3 and the F_2 by modus ponens.

The F_5 is $A \overset{n}{\supset} A$; it is derived from the F_4 and the F_1 by modus ponens.

Theorem 1 is proved.

If it is assumed that an occurrence of a formula F_i into the sequence (4) is yet founded by its membership to a formula list G then such sequence is called the definition of hypothesis proof and it is written $G \vdash_{Cn} F_m$.

Theorem 2 (Deduction theorem)
If $G, A \vdash_{Cn} B$ then $G \vdash_{Cn} A \overset{n}{\supset} B$, $n \geq 1$.

Proof. Proof scheme of this theorem coincides with the proof scheme of deduction theorem involved in [2] as the calculus Cn has analogous formulas which are used in Church's proof.

Theorem 2 is proved.

3. Semantics

Let $m \dot{-} k$ be $max(0, m-k)$ where m and k are arbitrary natural numbers. Elements of $\{0, 1, \ldots, n\}$ will be assigned to formulas of the calculi Cn if its variables are assigned values from this set, the constant f_i is assigned to value i and take value of $k \supset m$ equal to $m \dot{-} k$.

At first it is noticed that value of the formula $p \overset{n}{\supset} q$ when $p = k$ and $q = m$ is equal to $m \dot{-} nk$ (it is proven by mathematical induction).

Examples.

1. Find value of the formula $p \overset{n}{\supset} \left(q \overset{n}{\supset} p\right)$ when $p = k$ and $q = m$. We have

$$k \overset{n}{\supset} \left(m \overset{n}{\supset} k\right) = \left(m \overset{n}{\supset} k\right) \dot{-} nk = \left(k \dot{-} nm\right) \dot{-} nk = 0$$

for every natural numbers k, m.

2. Find value of the formula $\left(\left(p \overset{n}{\supset} f_n\right) \overset{n}{\supset} f_n\right) \overset{n}{\supset} p$ when $p = k$. We have

$$\left(k \overset{n}{\supset} n\right) \overset{n}{\supset} n \overset{n}{\supset} k = k \dot{-} n\,(n \dot{-} nk) = 0$$

for every natural numbers k.

3. It is easy to verify that value of the formula

$$\left(p \overset{n}{\supset} \left(q \overset{n}{\supset} r\right)\right) \overset{n}{\supset} \left(\left(p \overset{n}{\supset} q\right) \overset{n}{\supset} \left(p \overset{n}{\supset} r\right)\right)$$

is equal to zero when $p = k$, $q = m$, $r = t$ where k, m, t are arbitrary natural numbers.

A formula of the calculus Cn is called exceptional if its value is equal to zero when its variables are assigned arbitrary values.

Theorem 3

If $\vdash_{Cn} A$ then the formula A is exceptional, $n \geq 1$.

Proof. Every axiom obtaining on any axiom scheme $(1n)$, $(2n)$ and $(3n)$ is exceptional according to the examples 1, 2, 3. Further modus ponens holds the exception property. Theorem 3 is proved.

Theorem 4

$\vdash_{C1} A$ if and only if the formula A is exceptional.

Proof. Formulas of the calculus $C1$ will be converted in formulas of Church's calculus P_1 [2] if the constant f_1 is changed to the constant f of the calculus P_1 and round brackets are changed to square ones. Then tautologies of the calculus P_1 will be converted in exceptional formulas of the calculus $C1$ if in the calculus P_1 truth is represented by zero and falsehood is represented by one (the constant f of the calculus P_1 is represented by falsehood). Church's Theorems 150 and 152 say that a formula of the calculus P_1 is proven if and only if it is tautology. So for as the calculi $C1$ and P_1 have the same axioms and rules of inference then this completes the proof.

Theorem 5

For every calculus Cn $(n \geq 2)$ there is its exceptional formula which is not a theorem.

Proof. Every theorem of the calculus Cn has the form $A \overset{n}{\supset} B$ according to Theorem 3 and the forms of axiom schemes. So the formula $p \overset{n-1}{\supset} p$ of the calculus Cn where $n \geq 2$ is not its theorem but this formula is exceptional. Theorem 5 is proved.

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STUDIES IN LOGIC, GRAMMAR AND RHETORIC 14 (27) 2008

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5-VALUED NON-DETERMINISTIC SEMANTICS FOR THE BASIC PARACONSISTENT LOGIC *mCi*

One of the most important paraconsistent logics is the logic **mCi**, which is one of the two basic logics of formal inconsistency. In this paper we present a 5-valued characteristic nondeterministic matrix for **mCi**. This provides a quite non-trivial example for the utility and effectiveness of the use of non-deterministic many-valued semantics.

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1. Introduction

A *paraconsistent* logic is a logic which allows nontrivial inconsistent theories. There are several approaches to the problem of designing a useful paraconsistent logic (see e.g. [8, 13, 10, 9]). One of the oldest and best known is da Costa's approach ([14, 15]), which seeks to allow the use of classical logic whenever it is safe to do so, but behaves completely differently when contradictions are involved. da Costa's approach has led to the family of LFIs (Logics of Formal (In)consistency – see [12]). This family is based on two main ideas. The first is that propositions should be divided into two sorts: the "normal" (or consistent) propositions, and the "abnormal" (or inconsistent) ones. Classical logic can (and should) be applied freely to normal propositions, but not to abnormal ones. The second idea is to formally introduce this classification into the language. When this is done by employing a special (primitive or defined) unary connective $\circ$ (where the intuitive meaning of $\circ\varphi$ is : "φ is consistent") we get a special type of LFIs: the C-systems ([11]). The class of C-systems is the most important and useful subclass of the class of logics of formal (in)consistency.

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For a long time the class of C-systems has had one major shortcoming: it lacked a corresponding intuitive semantics, which would be easy to use and would provide real insight into these logics.[1] In [3] this was remedied by providing simple, modular *non-deterministic* semantics for almost all the propositional C-systems considered in the literature. This semantics is based on the use of non-deterministic matrices (Nmatrices). These are multi-valued structures (introduced in [5, 6]) where the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options. Although applicable to a much larger family of logics, the semantics of finite Nmatrices has all the advantages that the semantics of ordinary finite-valued semantics provides. In particular:

1. The semantics of finite Nmatrices is *effective* in the sense that for determining whether $T \vdash_{\mathcal{M}} \varphi$ (where $\mathcal{M}$ is an Nmatrix) it always suffices to check only *partial* valuations, defined only on subformulas of $T \cup \{\varphi\}$. It follows that a logic which has a finite characteristic Nmatrix is necessarily decidable.
2. A logic with a finite characteristic Nmatrix is finitary (i.e.: the compactness theorem obtains for it – see [6]).
3. There is a well-known uniform method ([16, 7]) for constructive cut-free calculus of n-sequents for any logic which has an n-valued characteristic matrix. This method can easily be extended to logics which have a finite characteristic Nmatrix (see [4]).

Now [3] has left one major gap: no semantics was provided in it for one of the most basic systems considered in [12]. This is Marco's system **mCi**, to which the whole of section 4 of [12] is devoted, and is the minimal C-system in which an appropriate *inconsistency* operator (dual to the consistency operator $\circ$) can be defined. The main goal of this paper is to complete the work started in [3] by closing this gap. Another goal is to give still another quite non-trivial example for the utility and effectiveness of the use of non-deterministic many-valued semantics. Both goals are achieved here by presenting a finite (in fact: 5-valued) characteristic Nmatrix for **mCi**.[2]

[1] The bivaluations semantics and the possible translations semantics described in [11, 12, 17] are not satisfactory from these points of view, since their effectiveness (in the sense explained below) is not aprioriy guaranteed, and so a corresponding proposition should be proved from scratch for any useful instance of these types of semantics.

[2] A possible-translation semantics for **mCi** has been provided in [17].

2. Preliminaries

2.1. The System mCi

Let $\mathcal{L}_{\text{cl}}^{+} = \{\wedge, \vee, \supset\}$, $\mathcal{L}_{\text{cl}} = \{\wedge, \vee, \supset, \neg\}$, and $\mathcal{L}_{\text{C}} = \{\wedge, \vee, \supset, \neg, \circ\}$. For $n \geq 0$, let $\neg^0\varphi = \varphi$, $\neg^{n+1}\varphi = \neg(\neg^n\varphi)$.

Definition 1

Let $\mathbf{HCL}^+$ be some Hilbert-type system which has Modus Ponens as the only inference rule, and is sound and strongly complete for the $\mathcal{L}_{\text{cl}}^{+}$-fragment of CPL (classical propositional logic).[3] The logic **mCi** is the logic in $\mathcal{L}_{\text{C}}$ obtained from $\mathbf{HCL}^+$ by adding the schemata:

(**t**) $\neg\varphi \vee \varphi$

(**p**) $\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi)$

(**i**) $\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)$

(**cc**) $\circ\neg^n\circ\varphi$ (for every $n \geq 0$[4]).

2.2. Non-deterministic Matrices

Our main semantic tool in what follows will be the following generalization of the concept of a matrix:

Definition 2

A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V}$ is a non-empty set of *truth values.*

(b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.

(c) For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} - \{\emptyset\}$.

We say that $\mathcal{M}$ is *(in)finite* if so is $\mathcal{V}$.

2. A *(legal) valuation* in an Nmatrix $\mathcal{M}$ is a function $v : \mathcal{L} \to \mathcal{V}$ (where we identify a language with its set of formulas) that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \ldots, \psi_n \in \mathcal{L}$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

---

[3] I.e.: for every sentence φ and theory $\mathbf{T}$ in $\mathcal{L}_{\text{cl}}^{+}$, $\mathbf{T} \vdash_{\mathbf{HCL}^+} \varphi$ iff $\mathbf{T} \vdash_{CPL} \varphi$.

[4] Actually, it suffices to take here $n \geq 1$, since $\circ\circ\varphi$ is a theorem of $\mathbf{HCL}^+ + \{(t), (p), (i)\}$.

3. A valuation v in an Nmatrix $\mathcal{M}$ is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a *model* in $\mathcal{M}$ of a set $\mathbf{T}$ of formulas (notation: $v \models^{\mathcal{M}} \mathbf{T}$) if it satisfies every formula in $\mathbf{T}$.
4. $\vdash_{\mathcal{M}}$, the consequence relation induced by the Nmatrix $\mathcal{M}$, is defined as follows: $T \vdash_{\mathcal{M}} \varphi$ if for every v such that $v \models^{\mathcal{M}} T$, also $v \models^{\mathcal{M}} \varphi$.
5. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is *sound* for an Nmatrix $\mathcal{M}$ (where $\mathcal{L}$ is the language of $\mathcal{M}$) if $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{M}}$. $\mathbf{L}$ is *complete* for $\mathcal{M}$ if $\vdash_{\mathbf{L}} \supseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is *characteristic* for $\mathbf{L}$ if $\mathbf{L}$ is both sound and complete for it (i.e.: if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{M}}$). $\mathcal{M}$ is *weakly-characteristic* for $\mathbf{L}$ if for every formula φ of $\mathcal{L}$, $\vdash_{\mathbf{L}} \varphi$ iff $\vdash_{\mathcal{M}} \varphi$.

3. An Nmatrix for mCi

In our semantics for **mCi** we shall employ five truth values: T, F, t, f, and I. Intuitively, I is the truth value of inconsistent propositions. T and F are the truth values of propositions which are *necessarily* consistent, while t and f are the truth values of propositions which are *contingently* consistent.

Definition 3

The Nmatrix $\mathcal{M}_{mCi} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V} = \{I, T, F, t, f\}$
- $\mathcal{D} = \{I, T, t\}$
- $\mathcal{O}$ is defined by:

$$a \tilde{\vee} b = \begin{cases} \{t, I\} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \{f\} & \text{if } a, b \notin \mathcal{D} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \{t, I\} & \text{if either } a \notin \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{f\} & \text{if } a \in \mathcal{D} \text{ and } b \notin \mathcal{D} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \{f\} & \text{if either } a \notin \mathcal{D} \text{ or } b \notin \mathcal{D} \\ \{t, I\} & \text{otherwise} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \{F\} & \text{if } a = T \\ \{T\} & \text{if } a = F \\ \{f\} & \text{if } a = t \\ \{t, I\} & \text{if } a \in \{f, I\} \end{cases}$$

$$\tilde{\circ} a = \begin{cases} \{F\} & \text{if } a = I \\ \{T\} & \text{if } a \neq I \end{cases}$$

These tables reflect the fact that the only sentences which are necessarily consistent according to **mCi** are sentences of the form $\neg^n \circ \varphi$.

4. The Soundness and Completeness Theorem

Theorem 1

mCi is sound for $\mathcal{M}_{mCi}$.

Proof. Obviously, it suffices to show that if v is a legal valuation in $\mathcal{M}_{mCi}$ then $v(\varphi) \in \mathcal{D}$ whenever φ is an axiom of **mCi**, and that v respects MP (in the sense that $v(\psi) \in \mathcal{D}$ whenever $v(\varphi) \in \mathcal{D}$ and $v(\varphi \supset \psi) \in \mathcal{D}$). This is straightforward for the axioms of $\mathbf{HCL}^+$ (and in fact follows from Theorem 1 of [3]). Now we show the validity in $\mathcal{M}_{mCi}$ of the special axioms of **mCi**:

(**t**) From the table for negation it follows that if $v(\varphi) \notin \mathcal{D}$ then $v(\neg\varphi) \in \mathcal{D}$. Hence the validity of axiom (**t**) follows from the table for $\vee$.

(**p**) From the table for negation it follows that if $v(\varphi) \neq I$ then either $v(\varphi) \notin \mathcal{D}$, or $v(\neg\varphi) \notin \mathcal{D}$. Therefore it follows from the tables for $\wedge$ and $\supset$ that $v(\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi) \in \mathcal{D}$ in this case. On the other hand, if $v(\varphi) = I$ then $v(\circ\varphi) = F$ and so again $v(\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi) \in \mathcal{D}$ by the table for $\supset$.

(**i**) The tables for $\neg$ and $\circ$ entail that if $v(\varphi) \neq I$ then $v(\neg\circ\varphi) = F$, and so $v(\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)) \in \mathcal{D}$ by the table for $\supset$. On the other hand, if $v(\varphi) = I$ then $v(\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)) \in \mathcal{D}$ by the tables for $\neg$, $\wedge$, and $\supset$.

(**cc**) By the truth tables for $\circ$, $v(\circ\varphi) \in \{T, F\}$. By the table for $\neg$, this fact entails that for every $n \geq 0$, $v(\neg^n \circ \varphi) \in \{T, F\}$. By the table for $\circ$ it follows therefore that $v(\circ \neg^n \circ\varphi) = T \in \mathcal{D}$.

We leave the proof that v respects MP for the reader. □

Theorem 2

mCi is complete for $\mathcal{M}_{mCi}$.

Proof. Assume that $\mathbf{T}$ is a theory and φ_0 a sentence such that $\mathbf{T} \nvdash_{mCi} \varphi_0$. We construct a model of $\mathbf{T}$ in $\mathcal{M}_{mCi}$ which is not a model of φ_0. For this extend $\mathbf{T}$ to a maximal theory $\mathbf{T}^*$ such that $\mathbf{T}^* \nvdash_{mCi} \varphi_0$ (and so $\varphi_0 \notin \mathbf{T}^*$). $\mathbf{T}^*$ has the following properties:

1. $\psi \notin \mathbf{T}^*$ iff $\psi \supset \varphi_0 \in \mathbf{T}^*$.
2. If $\psi \notin \mathbf{T}^*$ then $\psi \supset \varphi \in \mathbf{T}^*$ for every sentence φ of $\mathcal{L}_C$.
3. $\varphi \vee \psi \in \mathbf{T}^*$ iff either $\varphi \in \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.
4. $\varphi \wedge \psi \in \mathbf{T}^*$ iff both $\varphi \in \mathbf{T}^*$ and $\psi \in \mathbf{T}^*$.
5. $\varphi \supset \psi \in \mathbf{T}^*$ iff either $\varphi \notin \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.

6. For every sentence φ of $\mathcal{L}_C$, either $\varphi \in \mathbf{T}^*$ or $\neg\varphi \in \mathbf{T}^*$.
7. If both $\varphi \in \mathbf{T}^*$ and $\neg\varphi \in \mathbf{T}^*$ then $\circ\varphi \notin \mathbf{T}^*$.
8. If $\neg\circ\varphi \in \mathbf{T}^*$ then both $\varphi \in \mathbf{T}^*$ and $\neg\varphi \in \mathbf{T}^*$.
9. $\circ\neg^n\circ\varphi \in \mathbf{T}^*$ for every $n \geq 0$.

Property 1 follows from the deduction theorem (which is obviously valid for **mCi**) and the maximality of $\mathbf{T}^*$. Property 2 is proved first for $\psi = \varphi_0$ as follows: by 1, if $\varphi_0 \supset \varphi \notin \mathbf{T}^*$ then $(\varphi_0 \supset \varphi) \supset \varphi_0 \in \mathbf{T}^*$. Hence $\varphi_0 \in \mathbf{T}^*$ by the positive tautology $((\varphi_0 \supset \varphi) \supset \varphi_0) \supset \varphi_0$. A contradiction. Property 2 then follows for all $\psi \notin \mathbf{T}^*$ by 1 and the transitivity of implication. Properties 3–5 are easy corollaries of 1, 2, and the closure of $\mathbf{T}^*$ under positive classical inferences (for example: suppose $\varphi \vee \psi \in \mathbf{T}^*$, but neither $\varphi \in \mathbf{T}^*$, nor $\psi \in \mathbf{T}^*$. By property 1, $\varphi \supset \varphi_0 \in \mathbf{T}^*$ and $\psi \supset \varphi_0 \in \mathbf{T}^*$. Since φ_0 follows in positive classical logic from $\varphi \vee \psi$, $\varphi \supset \varphi_0$, and $\psi \supset \varphi_0$, we get $\varphi_0 \in \mathbf{T}^*$. A contradiction). Property 6 is immediate from Property 3 and Axiom **(t)**. Property 7 follows from Axiom (**p**), while Property 8 follows from Axiom (**i**). Finally, Property 9 follows from Axiom (**cc**).

Define now a valuation v in $\mathcal{M}_{mCi}$ as follows:

$$v(\psi) = \begin{cases} I & \text{if} \quad \psi \in \mathbf{T}^*, \neg\psi \in \mathbf{T}^* \\ F & \text{if} \quad \psi \notin \mathbf{T}^* \text{ and } \psi \text{ is of the form } \neg^n\circ\varphi \\ f & \text{if} \quad \psi \notin \mathbf{T}^* \text{ and } \psi \text{ is not of the form } \neg^n\circ\varphi \\ T & \text{if} \quad \neg\psi \notin \mathbf{T}^* \text{ and } \psi \text{ is of the form } \neg^n\circ\varphi \\ t & \text{if} \quad \neg\psi \notin \mathbf{T}^* \text{ and } \psi \text{ is not of the form } \neg^n\circ\varphi \end{cases}$$

From Property 6 of $\mathbf{T}^*$ it follows that v is well-defined. From the same property it easily follows also that $v(\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}^*$. We use this to prove that v is a legal valuation, i.e.: it respects the interpretations of the connectives in $\mathcal{M}_{mCi}$. That this is the case for the positive connectives easily follows from Properties 3–5 of $\mathbf{T}^*$, and the fact that by definition of v, $v(\psi_1 * \psi_2) \in \{I, t, f\}$ for every ψ_1, ψ_2, and $* \in \{\vee, \wedge, \supset\}$. We prove next the cases of $\neg$ and $\circ$:

- Suppose $v(\psi) = I$. Then by the definition of v, both ψ and $\neg\psi$ are in $\mathbf{T}^*$. Hence $\circ\psi \notin \mathbf{T}^*$ by Property 7. Since $\circ\psi$ is $\neg^0\circ\psi$, this means that $v(\circ\psi) = F$.
- Suppose $v(\psi) \neq I$. Then by the definition of v, either ψ or $\neg\psi$ is not in $\mathbf{T}^*$. Hence $\neg\circ\psi \notin \mathbf{T}^*$ by Property 8. Since $\circ\psi$ is $\neg^0\circ\psi$, this means that $v(\circ\psi) = T$.
- Suppose $v(\psi) = T$. Then ψ is of the form $\neg^n\circ\varphi$, and $\neg\psi \notin \mathbf{T}^*$. But then $\neg\psi$ is $\neg^{n+1}\circ\varphi$, and so the fact that $\neg\psi \notin \mathbf{T}^*$ entails that $v(\neg\psi) = F$.

- Suppose $v(\psi) = F$. Then the formula ψ is of the form $\neg^n \circ \varphi$, and $\psi \notin \mathbf{T}^*$. Therefore by Properties 9 and 6, $\circ\neg\psi \in \mathbf{T}^*$, and $\neg\psi \in \mathbf{T}^*$. It follows by Property 7 that $\neg\neg\psi \notin \mathbf{T}^*$. Since $\neg\psi$ is in this case $\neg^{n+1} \circ \varphi$, this entails that $v(\neg\psi) = T$.
- Suppose $v(\psi) = t$. Then $\neg\psi \notin \mathbf{T}^*$, and ψ is not of the form $\neg^n \circ \varphi$. Hence also $\neg\psi$ is not of the form $\neg^n \circ \varphi$, and since $\neg\psi \notin \mathbf{T}^*$, we have $v(\neg\psi) = f$.
- Suppose $v(\psi) = f$. Then $\psi \notin \mathbf{T}^*$, and so (by Property 6) $\neg\psi \in \mathbf{T}^*$. Hence $v(\neg\psi) \neq f$. Since in this case ψ and $\neg\psi$ are not of the form $\neg^n \circ \varphi$, $v(\neg\psi) \notin \{T, F\}$ as well. It follows that $v(\neg\psi) \in \{t, I\}$.
- Suppose $v(\psi) = I$. Then both ψ and $\neg\psi$ are in $\mathbf{T}^*$. Hence $\circ\psi \notin \mathbf{T}^*$ by Property 7, and so by Property 9 ψ is not of the form $\neg^n \circ \varphi$. This implies that also $\neg\psi$ is not of this form, and so $v(\neg\psi) \notin \{T, F\}$. Since $\neg\psi \in \mathbf{T}^*$, $v(\neg\psi) \neq f$ as well. Hence $v(\neg\psi) \in \{t, I\}$.

Since $v(\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}^*$, $v(\psi) \in \mathcal{D}$ for every $\psi \in \mathbf{T}$, while $v(\varphi_0) \notin \mathcal{D}$. Hence v is a model of $\mathbf{T}$ which is not a model of $v(\varphi_0)$. □

Together Theorems 1 and 2 provide the main result of this paper:

Theorem 3

$\mathcal{M}_{mCi}$ is a characteristic Nmatrix for **mCi**.

Corollary 1

mCi is decidable.

Example 1

$\vdash_{mCi} \neg\neg\circ\varphi \supset \circ\varphi$

Proof. Let v be a valuation in $\mathcal{M}_{mCi}$ and let φ be a sentence. Then $v(\circ\varphi) \in \{T, F\}$, and so $v(\neg\neg \circ \varphi) = v(\circ\varphi)$ by the table for $\neg$. It follows from the table for $\supset$ that $v(\neg\neg \circ \varphi \supset \circ\varphi) \in \{t, I\} \subseteq \mathcal{D}$. □

Example 2

$\circ p \supset \circ\neg p$ is not a theorem of **mCi**.

Proof. Define a (partial) valuation v by $v(p) = f$, $v(\neg p) = I$, $v(\circ p) = T$, $v(\circ\neg p) = F$, and $v(\circ p \supset \circ\neg p) = f$. Then v is a legal partial valuation, and by the effectivity of the semantics (see the introduction) it can be extended to a countermodel of $\circ p \supset \circ\neg p$. □

5. Extensions of mCi and Modularity

One of the most important advantages of the semantics of Nmatrices is its *modularity*. The idea is as follows. Let **L** be some basic logic, and suppose that $\mathcal{M}$ is a characteristic Nmatrix for **L**. Then to each natural axiom Ax that one might like to add to **L** there usually corresponds a condition that refinements of $\mathcal{M}$ should satisfy in order for Ax to be sound in them. A characteristic Nmatrices for natural extensions of **L** by a finite set of such axioms can then be produced in a modular way by refining $\mathcal{M}$ according to these conditions. Proving the soundness and completeness of such an extension of **L** for the corresponding resulting refinement of $\mathcal{M}$ usually involves only a straightforward adaptation of the proof of the soundness and completeness of **L** for $\mathcal{M}$. A lot of examples of this modularity have been given in [1, 2] and [3]. The methods of the latter can be applied to extensions of **mCi** in the most obvious way. Here are 3 examples:

- Let **(c)** be the scheme $\neg\neg\varphi \supset \varphi$. A characteristic Nmatrix for the extension of **mCi** by **(c)** is obtained from $\mathcal{M}_{mCi}$ by letting $\tilde{\neg} f = \{t\}$ (rather than $\tilde{\neg} f = \{t, I\}$).
- Let **(e)** be the scheme $\varphi \supset \neg\neg\varphi$. A characteristic Nmatrix for the extension of **mCi** by **(e)** is obtained from $\mathcal{M}_{mCi}$ by letting $\tilde{\neg} I = \{I\}$.
- A characteristic Nmatrix for the extension of **mCi** by both **(c)** and **(e)** is obtained from $\mathcal{M}_{mCi}$ by letting $\tilde{\neg} f = \{t\}$ and $\tilde{\neg} I = \{I\}$.

We leave the proofs of these claims for the reader.

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Andrei Khrennikov

TOWARD THEORY OF *P*-ADIC VALUED PROBABILITIES

We present a short review on generalization of probability theory in that probabilities take values in the fields of p-adic numbers, $\mathbf{Q}_p$. Such probabilities were introduced to serve p-adic theoretical physics. In some quantum physical models a wave function (which is a complex probability amplitude in ordinary QM) takes vales in $\mathbf{Q}_p$ (for some prime number p) or their quadratic extensions. Such a wave function can be interpreted probabilistically in the framework of p-adic probability theory. This theory was developed by using both the frequency approach (by generalizing von Mises) and the measure-theoretic approach (by generalizing Kolmogorov). In particular, some limit theorems were obtained. However, theory of limit theorems for p-adic valued probabilities is far from being completed. Another interesting domain of research is corresponding theory of complexity. We obtained some preliminary results in this direction. However, it is again far from to be completed. Recently p-adic models of classical statistical mechanics were considered and some preliminary results about invariant p-adic valued measures for dynamical systems were obtained.

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1. Introduction

This paper is a short review on generalization of probability theory with probabilities taking values in $\mathbf{Q}_p$. The main attention will be paid to limit theorems and p-adic theory of recursive statistical tests generalizing Martin-Löf's theory. We start with recollection of formalization of theory with real valued probabilities – taking values in the segment [0,1] of the real line.

Since the creation of the modern probabilistic axiomatics by A. N. Kolmogorov [1] in 1933, probability theory was merely reduced to the theory of normalized σ-additive measures taking values in the segment [0,1] of the field of real numbers $\mathbf{R}$. In particular, the main competitor of Kolmogorov's measure-theoretic approach, von Mises' frequency approach to proba-

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bility [2], [3], practically totally disappeared from the probabilistic arena. On one hand, this was a consequence of difficulties with von Mises' definition of randomness (via place selections), see e.g., [4]–[6].[1] On the other hand, von Mises' approach (as many others) could not compete with precisely and simply formulated Kolmogorov's theory.

We mentioned von Mises' approach not only, because its attraction for applications, but also because von Mises' model with frequency probabilities played the important role in the process of formulation of the conventional axiomatics of probability theory.

We would like to mention that Kolmogorov's (as well as von Mises') assumptions were also based on a fundamental, but hidden, assumption: *Limiting behavior of relative frequencies is considered with respect to one fixed topology on the field of rational numbers* $\mathbf{Q}$*, namely, the real topology.* In particular, the consideration of this asymptotic behavior implies that probabilities belong to the field real numbers $\mathbf{R}$.

However, it is possible to study asymptotic behavior of relative frequencies (which are always rational numbers) in other topologies on field of rational numbers $\mathbf{Q}$. In this way we derive another probability-like structure that recently appeared in theoretical physics. This is so called *p-adic probability.*

We recall that p-adic numbers are applied intensively in different domains of physics – quantum logic, string theory, cosmology, quantum mechanics, quantum foundations, see, e.g., [8]–[12], dynamical systems [13], [11], [14], biological and cognitive models [11], [14]–[16].

P-adic valued probabilities were introduced in [23]–[25], [7] to serve p-adic theoretical physics [8]–[12]. In some quantum physical models [10], [11] a wave function (which is a complex probability amplitude in ordinary QM) takes vales in $\mathbf{Q}_p$ (for some prime number p) or its quadratic extensions. Such a wave function can be interpreted probabilistically in the framework of p-adic probability theory. This theory was developed by using both the frequency approach (by generalizing von Mises) and the measure-theoretic approach (by generalizing Kolmogorov). In particular, some limit theorems were obtained. However, theory of limit theorems for p-adic valued probabilities is far from being completed. Another interesting domain of research is corresponding theory of complexity. We obtained some preliminary results in this direction. However, it is again far from being completed. Recently p-adic models of classical statistical mechanics were consi-

[1] However, see also [7], where von Mises' approach was simplified, generalized, and then fruitfully applied to theoretical physics.

dered and some preliminary results about invariant p-adic valued measures for dynamical systems were obtained.

2. p-adic Numbers

The field of real numbers $\mathbf{R}$ is constructed as the completion of the field of rational numbers $\mathbf{Q}$ with respect to the metric $p(x,y) = |x-y|$, where $|\cdot|$ is the usual valuation given by the absolute value. The fields of p-adic numbers $\mathbf{Q}_p$ are constructed in a corresponding way, but by using other valuations. For a prime number p the p-adic valuation $|\cdot|_p$ is defined in the following way. First we define it for natural numbers. Every natural number n can be represented as the product of prime numbers, $n = 2^{r_2} 3^{r_3} \dots p^{r_p} \dots$, and we define $|n|_p = p^{-r_p}$, writing $|0|_p = 0$ and $|-n|_p = |n|_p$. We then extend the definition of the p-adic valuation $|\cdot|_p$ to all rational numbers by setting $|n/m|_p = |n|_p/|m|_p$ for $m \neq 0$. The completion of $\mathbf{Q}$ with respect to the metric $\rho_p(x,y) = |x-y|_p$ is the locally compact field of p-adic numbers $\mathbf{Q}_p$.

The number fields $\mathbf{R}$ and $\mathbf{Q}_p$ are unique in a sense, since by Ostrovsky's theorem, see e.g., [26], $|\cdot|$ and $|\cdot|_p$ are the only possible valuations on $\mathbf{Q}$, but have quite distinctive properties. The field of real numbers $\mathbf{R}$ with its usual valuation satisfies $|n| = n \to \infty$ for valuations of natural numbers n and is said to be *Archimedian*. By a well known theorem of number theory [26] the only complete Archimedian fields are those of the real and the complex numbers. In contrast, the fields of p-adic numbers, which satisfy $|n|_p \leq 1$ for all $n \in N$, are examples of *non-Archimedian* fields.

Unlike the absolute value distance $|\cdot|$, the p-adic valuation satisfies the strong tringle inequality:

$$|x+y|_p \leq \max[|x|_p, |y|_p], x, y \in \mathbf{Q}_p.$$

Consequently the p-adic metric satisfies the strong triangle inequality $\rho_p(x,y) \leq \max[\rho_p(x,z), \rho_p(z,y)], x,y,z \in \mathbf{Q}_p$, which means that the metric ρ_p is an *ultrametric*, [26]. Write $U_r(a) = \{x \in \mathbf{Q}_p : |x-a|_p \leq r\}$, where $r = p^n$ and $n = 0, \pm 1, \pm 2, \dots$ These are the "closed" balls in $\mathbf{Q}_p$ while the sets $\mathbf{S}_r(a) = \{x \in \mathbf{Q}_p : |x-a|_p = r\}$ are the spheres in $\mathbf{Q}_p$ of such radii r. These sets (balls and spheres) have a somewhat strange topological structure from the viewpoint of our usual Euclidian intuition: they are both open and closed at the same time, and as such are called *clopen* sets. Finally, any p-adic ball $U_r(0)$ is an additive subgroup of $\mathbf{Q}_p$, while the ball $U_1(0)$ is also a ring, which is called the *ring of p-adic integers* and is denoted by $\mathbf{Z}_p$.

The p-adic *exponential function* $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The series converges in $\mathbf{Q}_p$ if $|x|_p \leq r_p$, where $r_p = 1/p, p \neq 2$ and $r_2 = 1/4$. p-adic trigonometric functions $\sin x$ and $\cos x$ are defined by the standard power series. These series have the same radius of convergence r_p as the exponential series.

3. p-adic Frequency Probability Model

As in the ordinary probability theory [2], [3], the first p-adic probability model was the frequency one, [23]–[25], [7], [9], [10]. This model was based on the simple remark that relative frequencies $\nu_N = \frac{n}{N}$ always belong to the field of rational numbers $\mathbf{Q}$. And $\mathbf{Q}$ can be considered as a (dense) subfield of $\mathbf{R}$ as well as $\mathbf{Q}_p$ (for each prime number p). Therefore behaviour of sequences $\{\nu_N\}$ of (rational) relative frequencies can be studied not only with respect to the real topology on Q, but also with respect to any p-adic topology on Q. Roughly speaking a p-adic probability (as real von Mises' probability) is defined as:

$$\mathbf{P}(\alpha) = \lim_{N} \nu_N(\alpha). \tag{1}$$

Here α is some label denoting a result of a statistical experiment. Denote the set of all such labels by the symbol Ω. In the simplest case $\Omega = \{0, 1\}$. Here $\nu_N(\alpha)$ is the relative frequency of realization of the label α in the first N trials. The $\mathbf{P}(\alpha)$ is the frequency probability of the label α.

The main p-adic lesson is that it is impossible to consider, as we did in the real case, limits of the relative frequencies ν_N when the $N \to \infty$. Here the point "∞" belongs, in fact, to the real compactification of the set of natural numbers. So $|N| \to \infty$, where $|\cdot|$ is the real absolute value. The set of natural numbers $\mathbf{N}$ is bounded in $\mathbf{Q}_p$ and it is densely embedded into the ring of p-adic integers $\mathbf{Z}_p$ (the unit ball of $\mathbf{Q}_p$). Therefore sequences $\{N_k\}_{k=1}^{\infty}$ of natural numbers can have various limits $m = \lim_{k\to\infty} N_k \in \mathbf{Z}_p$.

In the p-adic frequency probability theory we proceed in the following way to provide the rigorous mathematical meaning for the procedure (1), see [24], [25]. We fix a p-adic integer $m \in \mathbf{Z}_p$ and consider the class, L_m, of sequences of natural numbers $s = \{N_k\}$ such that $\lim_{k\to\infty} N_k = m$ in $\mathbf{Q}_p$.

Let us consider the fixed sequence of natural numbers $s \in L_m$. We define a p-adic s-probability

$$\mathbf{P}(\alpha) = \lim_{k\to\infty} \nu_{N_k}(\alpha), s = \{N_k\}.$$

This is the limit of relative frequencies with respect to the fixed sequence $s = \{N_k\}$ of natural numbers. For any subset A of the set of labels Ω, we define its s-probability as

$$\mathbf{P}(A) = \lim_{k\to\infty} \nu_{N_k}(A), s = \{N_k\},$$

where $\nu_{N_k}(A)$ is the relative frequency of realization of labels α belonging to the set A in the first N trials. As $\mathbf{Q}_p$ is an additive topological semigroup (as well as $\mathbf{R}$), we obtain that the p-adic probability is additive:

Theorem 3.1

$$\mathbf{P}(A_1 \cup A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2), A_1 \cap A_2 = \emptyset. \tag{2}$$

As $\mathbf{Q}_p$ is even an additive topological group (as well as $\mathbf{R}$), we get that

Theorem 3.2

$$\mathbf{P}(A_1 \setminus A_2) = \mathbf{P}(A_1) - \mathbf{P}(A_1 \cap A_2). \tag{3}$$

Trivially, for any sequence $s = \{N_k\}$, $\mathbf{P}(\Omega) = \lim_{k\to\infty}\nu_{N_k}(\Omega) = 1$, as $\nu_N(\Omega) = \frac{N}{N} = 1$ for any N. As $\mathbf{Q}_p$ is a multiplicative topological group (as well as $\mathbf{R}$), we get (see von Mises [2], [3] for the real case and [7] for the p-adic case) Bayes' formula for conditional probabilities:

Theorem 3.3

$$\mathbf{P}(A|B) = \lim_{k\to\infty} \frac{\nu_{N_k}(A\cap B)}{\nu_{N_k}(A)} = \frac{\mathbf{P}(A\cap B)}{\mathbf{P}(A)}, \quad \mathbf{P}(A) \neq 0. \tag{4}$$

As we know, frequency probability played the crucial role in conventional probability theory in determination of the range of values (namely, the segment [0,1]) of a probabilistic measure, see remarks on von Mises' theory in Kolmogorov's book [1]. Frequencies always lie between zero and one. Thus their limits (with respect to the real topology) belong to the same range.

In the p-adic case we can proceed in the same way. Let $r \equiv r_m = \frac{1}{|m|_p}$ (where $r = \infty$ for $m = 0$). We can easily get, see [23], [24], that for the p-adic frequency s-probability, $s \in L_m$, the values of $\mathbf{P}$ always belong to the p-adic ball $U_r(0) = \{x \in \mathbf{Q}_p : |x|_p \leq r\}$. In the p-adic probabilistic model such a ball $U_r(0)$ plays the role of the segment [0,1] in the real probabilistic model.

4. Measure-Theoretic Approach

As in the real case, the structure of an additive topological group of $\mathbf{Q}_p$ induces the main properties of probability that can be used for the axiomatization in the spirit of Kolmogorov, [1]. Let us fix $r = p^{\pm l}, l = 0, 1, \ldots,$ or $r = \infty$.

Axiomatics. *Let* Ω *be an arbitrary set (a sample space) and let* F *be a field of subsets of* Ω *(events). Finally, let* $\mathbf{P} : F \to U_r(0)$ *be an additive function (measure) such that* $\mathbf{P}(\Omega) = 1$*. Then the triple* $(\Omega, F, \mathbf{P})$ *is said to be a* p*-adic* r*-probabilistic space and* $\mathbf{P}$ p*-adic* r*-probability.*

Following to Kolmogorov we should find some technical mathematical restriction on $\mathbf{P}$ that would induce fruitful integration theory and give the possibility to define averages. Kolmogorov (by following Borel, Lebesque, Lusin, and Egorov) proposed to consider the σ-additivity of measures and the σ-structure of the field of events. Unfortunately, in the p-adic case the situation is not so simple as in the real one. One could not just copy Kolmogorov's approach and consider the condition of σ-additivity. There is, in fact, a No-Go theorem, see, e.g., [27]:

Theorem 4.1

All σ-additive p-adic valued measures defined on σ-fields are discrete.

Here the difficulty is not induced by the condition of σ-additivity, but by an attempt to extend a measure from the field F to the σ-field generated by F. Roughly speaking there exist σ-additive "continuous" $\mathbf{Q}_p$-valued measures, but they could not be extended from the field F to the σ-field generated by F. Therefore it is impossible to choose the σ-additivity as the basic integration condition in the p-adic probability theory.

The first important condition (that was already invented in the first theory of non-Archimedian integration of Monna and Springer [28]) is *boundedness*: $||A||_{\mathbf{P}} = \sup\{|\mathbf{P}(A)|_p : A \in F\} < \infty$.

Of course, if $\mathbf{P}$ is a p-adic r-probability with $r < \infty$, then this condition is fulfilled automatically. It is nontrivial only if the range of values of a p-adic probability is unbounded in $\mathbf{Q}_p$.[2] We pay attention to one important particular case in that the condition of boundedness alone implies

[2] In the frequency formalism this corresponds to considering of p-adic (frequency) s-probabilities for $s \in L_0$; e.g., $s = \{N_k = p^k\}$. In this case $m = \lim_{k \to \infty} p^k = 0$.

fruitful integration theory. Let Ω be a compact zero-dimensional topological space.[3] Then the integral

$$E\xi = \int_\Omega \xi(\omega)\mathbf{P}(d\omega)$$

is well defined for any continuous function $\xi : \Omega \to \mathbf{Q}_p$. For example, this theory works well for the following choice: Ω is the ring of q-adic integers Z_q, and $\mathbf{P}$ is a bounded p-adic r-probability, $r < \infty$. The integral is defined as the limit of Riemannian sums [28], [27].

But in general boundedness alone does not imply fruitful integration theory. We should consider another condition, namely *continuity* of $\mathbf{P}$. The most general continuity condition was proposed by A. van Rooij [27].[4]

Definition 4.1

A p-adic valued measure that is bounded, continuous, and normalized is called p-adic probability measure.

Everywhere below we consider p-adic probability spaces endowed with p-adic probability measures.

Let $(\Omega, F, \mathbf{P})$ be a p-adic probabilistic space. *Random variables* $\xi : \Omega \to \mathbf{Q}_p$ are defined as $\mathbf{P}$-integrable functions.

As the frequency p-adic probability theory induces, see [7], (as a Theorem) Bayes' formula for conditional probability, we can use (4) as the definition of conditional probability in the p-adic axiomatic approach (as it was done by Kolmogorov in the real case).

Example 4.1. (p-adic valued uniform distribution on the space of q-adic sequences). Let p and q be two prime numbers. We set $X_q = \{0, 1, \ldots, q-1\}, \Omega_q^n = \{x = (x_1, \ldots, x_n) : x_j \in X_q\}, \Omega_q^\star = \bigcup_n \Omega_q^n$ (the space of finite sequences), and

$$\Omega_q = \{\omega = (\omega_1, \ldots, \omega_n, \ldots) : \omega_j \in X_q\}$$

(the space of infinite sequences). For $x \in \Omega_q^n$, we set $l(x) = n$. For $x \in \Omega_q^\star, l(x) = n$, we define a cylinder U_x with the basis x by $U_x = \{\omega \in \Omega_q : \omega_1 = x_1, \ldots, \omega_n = x_n\}$. We denote by the symbol F_{cyl} the field of subsets of Ω_q generated by all cylinders. In fact, the F_{cyl} is the collection of all finite unions of cylinders.

[3] There exists a basis of neighborhoods that are open and closed at the same time.

[4] We remark that in many cases continuity coincides with σ-additivity.

First we define the uniform distribution on cylinders by setting $\mu(U_x) = 1/q^{l(x)}, x \in \Omega_q^\star$. Then we extend μ by additivity to the field F_{cyl}. Thus $\mu : F_{\text{cyl}} \to \mathbf{Q}$. The set of rational numbers can be considered as a subset of any $\mathbf{Q}_p$ as well as a subset of $\mathbf{R}$. Thus μ can be considered as a p-adic valued measure (for any prime number p) as well as the real valued measure. We use symbols $\mathbf{P}_p$ and $\mathbf{P}_\infty$ to denote these measures. The probability space for the uniform p-adic measure is defined as the triple

$$\mathcal{P} = (\Omega, F, \mathbf{P}), \text{ where } \Omega = \Omega_q, F = F_{\text{cyl}} \text{ and } \mathbf{P} = \mathbf{P}_p.$$

The $\mathbf{P}_p$ is called a *uniform p-adic probability distribution*.

The uniform p-adic probability distribution is a probabilistic measure iff $p \neq q$. The range of its values is a subset of the unit p-adic ball.

Remark 4.1. Values of $\mathbf{P}_p$ on cylinders coincide with values of the standard (real-valued) uniform probability distribution (Bernoulli measure) $\mathbf{P}_\infty$. Let us consider, the map $j_\infty(\omega) = \sum_{j=0}^\infty \frac{\omega_j}{2^{j+1}}$. The j_∞ maps the space Ω_q onto the segment [0,1] of the real line $\mathbf{R}$ (however, j_∞ is not one to one correspondence). The j_∞-image of the Bernoulli measure is the standard Lebesque measure on the segment [0,1] (the uniform probability distribution on the segment [0,1]).

Remark 4.2. The map $j_q : \Omega_q \to \mathbf{Z}_q, j_q(\omega) = \sum_{j=0}^\infty \omega_j q^j$, gives (one to one!) correspondence between the space of all q-adic sequences Ω_q and the ring of q-adic integers Z_q. The field F_{cyl} of cylindrical subsets of Ω_q coincides with the field $B(\mathbf{Z}_q)$ of all clopen (closed and open at the same time) subsets of $\mathbf{Z}_q$. If Ω_q is realized as $\mathbf{Z}_q$ and F_{cyl} as $B(Z_q)$, then μ_p is the p-adic valued Haar measure on $\mathbf{Z}_q$. The use of the topological structure of $\mathbf{Z}_q$ is very fruitful in the integration theory (for $p \neq q$). In fact, the space of integrable functions $f : \mathbf{Z}_q \to \mathbf{Q}_p$ coincides with the space of continuous functions (random variables) $C(\mathbf{Z}_q, \mathbf{Q}_p)$, see [28], [27], [26], [7].

5. *p*-adic Limit Theorems

5.1. *p*-adic Asymptotics of Bernoulli Probabilities

Everywhere in this section p is a prime number distinct from 2. We start with considering the classical Bernoulli scheme (in the conventional probabilistic framework) for random variables $\xi_j(\omega) = 0, 1$ with probabilities $1/2, j = 1, 2, \ldots$. First we consider a finite number n of random variables: $\xi_1(\omega), \ldots, \xi_n(\omega)$. A sample space corresponding to these random variables can be chosen as the space $\Omega_2^n = \{0, 1\}^n$. The probability of an event A is defined as

$$\mathbf{P}^{(n)} = \frac{|A|}{|\Omega_2^n|} = \frac{|A|}{2^n},$$

where the symbol $|B|$ denotes the number of elements in a set B. The typical problem of ordinary probability theory is to find the asymptotic behavior of the probabilities $\mathbf{P}^{(n)}(A), n \to \infty$. It was the starting point of the theory of limit theorems in conventional probability theory.

But the probabilities $\mathbf{P}^{(n)}(A)$ belong to the field of rational numbers $\mathbf{Q}$. We may study behavior of $\mathbf{P}^{(n)}(A)$, not only with respect to the usual real metric $\rho_\infty(x, y)$ on $\mathbf{Q}$, but also with respect to an arbitrary metric $\rho(x, y)$ on $\mathbf{Q}$. We have studied the case of the p-adic metric on $\mathbf{Q}$, see [29], [30]. We remark that $\mathbf{P}^{(n)}(A) = \sum_{x \in A} \mu(U_x)$, where μ is the uniform distribution on Ω_2. By realizing μ as the (real valued) probability distribution $\mathbf{P}_\infty$ we use the formalism of conventional probability theory. By realizing μ as the p-adic valued probability distribution $\mathbf{P}_p$ we use the formalism of p-adic probability theory.

What kinds of events A are naturally coupled to the p-adic metric? Of course, such events must depend on the prime number p. As usual, we consider the sums

$$S_n(\omega) = \sum_{k=1}^{n} \xi_n(\omega).$$

We are interested in the following question. Does p divide the sum $S_n(\omega)$ or not? Set $A(p, n) = \{\omega \in \Omega_2^n : p \text{ divides the sum } S_n(\omega)\}$. Then $\mathbf{P}^{(n)}(A(p, n)) = L(p, n)/2^n$, where $L(p, n)$ is the number of vectors $\omega \in \Omega_2^n$ such that p divides $|\omega| = \sum_{j=1}^{n} \omega_j$. As usual, denote by $\bar{A}$ the complement of a set A. Thus $\bar{A}(p, n)$ is the set of all $\omega \in \Omega_q^n$ such that p does not divide the sum $S_n(\omega)$. We shall see that the sets $A(p, n)$ and $\bar{A}(p, n)$ are asymptotically symmetric from the p-adic point of view:

$$\mathbf{P}^{(n)}(A(p, n)) \to \tfrac{1}{2} \quad \text{and} \quad \mathbf{P}^{(n)}(\bar{A}(p, n)) \to \tfrac{1}{2} \tag{5}$$

in the p-adic metric when $n \to 1$ in the same metric. Already in this simplest case we shall see that the behavior of sums $S_n(\omega)$ depends crucially on the choice of a sequence $s = \{N_k\}_{k=1}^\infty$ of natural numbers. A limit distribution of the sequence of random variables $S_n(\omega)$, when $n \to \infty$ in the ordinary sense, does not exist. We have to describe all limiting distributions for different sequences s converging in the p-adic topology.

Let $(\Omega, F, \mathbf{P})$ be a p-adic probabilistic space and $\xi_n : \Omega \to \mathbf{Q}_p (n = 1, 2, \ldots)$ be a sequence of equally distributed independent random variables,

$\xi_n = 0, 1$ with probability $1/2$.[5] We start with the following result that can be obtained through purely combinatorial considerations (behavior of binomial coefficients C_m^r in the p-adic topology).

Theorem 5.1

Let $m = 0, 1, \ldots, p^s - 1 (s = 1, 2, \ldots), r = 0, \ldots, m$, and $l \geq s$. Then

$$\lim_{n \to m} \mathbf{P}(\omega : S_n(\omega) \in U_{1/p^l}(r)) = \frac{C_m^r}{2^m}.$$

Formally this theorem can be reformulated as the following result for the convergence of probabilistic distributions: *The limiting distribution on $\mathbf{Q}_p$ of the sequence of the sums $S_n(\omega)$, where $n \to m$ in $\mathbf{Q}_p$, is the discrete measure $\kappa_{1/2,m} = 2^{-m} \sum_{r=0}^m C_m^r \delta_m$.*

We consider the event $A(p, n, r) = \{\omega : S_n(\omega) = pi + r\}$ for $r = 0, 1, \ldots, p - 1$. This event consists of all ω such that the residue of $S_n(\omega)$ mod p equals to r. Note that the set $A(p, n, r)$ coincides with the set $\{\omega : S_n(\omega) \in U_{1/p}(r)\}$.

Corollary 5.1

Let $n \to m$ in $\mathbf{Q}_p$, where $m = 0, 1, \ldots, p - 1$. Then the probabilities $\mathbf{P}^{(n)}(A(p, n, r))$ approach $C_m^r / 2^m$ for all residues $r = 0, \ldots, m$.

In particular, as $A(p, n) \equiv A(p, n, 0)$, we get (5). What happens in the case $m \geq p$? We have only the following particular result:

Theorem 5.2

Let $n \to p$ in $\mathbf{Q}_p$ and $r = 0, 1, 2, \ldots, p$. Then

$$\lim_{n \to p} \mathbf{P}(\omega : S_n(\omega) \in U_{1/p^l}(r)) = \frac{C_p^r}{2^p},$$

where $s \geq 2$ for $r = 0, p$ and $s \geq 1$ for $r = 1, \ldots, p - 1$.

5.2. Laws of Large Numbers

We now study the general case of dichotomous equally distributed independent random variables: $\xi_n(\omega) = 0, 1$ with probabilities q and $q' = 1 - q$, $q \in \mathbf{Z}_p$. We shall study the weak convergence of the probability distributions

[5] Here $1/2$ is considered as a p-adic number. In the conventional theory $1/2$ is considered as a real number.

$\mathbf{P}_{S_{N_k}}$ for the sums $S_{N_k}(\omega)$. We consider the space $C(\mathbf{Z}_p, \mathbf{Q}_p)$ of continuous functions $f : \mathbf{Z}_p \to \mathbf{Q}_p$. We will be interested in convergence of integrals

$$\int_{\mathbf{Z}_p} f(x) d\mathbf{P}_{S_{N_k}}(x) \to \int_{\mathbf{Z}_p} f(x) d\mathbf{P}_S(x), f \in C(\mathbf{Z}_p, \mathbf{Q}_p),$$

where $\mathbf{P}_S$ is the limiting probability distribution (depending on the sequence $s = \{N_k\}$). To find the limiting distribution $\mathbf{P}_S$, we use the method of characteristic functions. We have for characteristic functions

$$\phi_{N_k}(z, q, a) = \int_\Omega \exp\{z S_{N_k}(\omega)\} d\mathbf{P}(\omega) = (1 + q'(e^z - 1))^{N_k}.$$

Here z belong to a sufficiently small neighborhood of zero in the $\mathbf{Q}_p$; see [10] for detail about the p-adic method of characteristic functions. Let a be an arbitrary number from $\mathbf{Z}_p$. Let $s = \{N_k\}_{k=1}^\infty$ be a sequence of natural numbers converging to a in the $\mathbf{Q}_p$. Set $\phi(z, q, a) = (1 + q'(e^z - 1))^a$. This function is analytic for small z. It is easy to see that the sequence of characteristic functions $\{\phi_{N_k}(z, q, a)\}$ converges (uniformly on every ball of a sufficiently small radius) to the function $\phi(z, q, a)$. Unfortunately, we could not prove (or disprove) a p-adic analogue of Levy's theorem. Therefore in the general case the convergence of characteristic functions does not give us anything. However, we shall see that we have Levy's situation in the particular case under consideration: There exists a bounded probability measure distribution, denoted by $\kappa_{q,a}$, having the characteristic function $\phi(z, q, a)$ and, moreover, $\mathbf{P}_{S_{N_k}} \to \mathbf{P}_S = \kappa_{q,a}$, $N_k \to a$.

We start with the first part of the above statement. Here we shall use Mahlers integration theory on the ring of p-adic integers, see e.g., [26], [27], [9], [10]. We introduce a system of binomial polynomials: $C(x, k) = C_x^k = \frac{x(x-1)\ldots(x-k+1)}{k!}$ (that are considered as functions from $\mathbf{Z}_p$ to $\mathbf{Q}_p$). Every function $f \in C(\mathbf{Z}_p, \mathbf{Q}_p)$ is expanded into a series (a Mahler expansion, see [40]) $f(x) = \sum_{k=0}^\infty a_k C(x, k)$. It converges uniformly on $\mathbf{Z}_p$. If μ is a bounded measure on $\mathbf{Z}_p$, then

$$\int_{\mathbf{Z}_p} f(x) \mu(dx) = \sum a_k \int_{\mathbf{Z}_p} C(x, n) \mu(dx).$$

Therefore to define a p-adic valued measure on $\mathbf{Z}_p$ it suffices to define coefficients $\int_{\mathbf{Z}_p} C(x, n) \mu(dx)$. A measure is bounded iff these coefficients are bounded. Using the Mahler expansion of the function $\phi(z, q, a)$, we obtain

$$\lambda_m(q, a) = \int_{\mathbf{Z}_p} C(x, m) \kappa_{q,a}(dx) = (1 - q)^m C(a, m).$$

As $|C(a,m)|_p \leq 1$ for $a \in \mathbf{Z}_p$, we get that the distribution $\kappa_{q,a}$ (corresponding to $\phi(z,q,a)$) is bounded measure on $\mathbf{Z}_p$. Set $\lambda_{mn}(q,a) = \int_\Omega C(S_n(\omega),m)dP(\omega)$. We find

$$\lambda_{mN_k}(q,a) = (1-q)^m C_{N_k}^m.$$

Thus $\lambda_{mN_k}(q,a) \to \lambda_m(q,a), N_k \to a$. This implies the following limit theorem.

Theorem 5.3 (p-adic Law of Large Numbers)

The sequence of probability distributions $\{\mathbf{P}_{S_{N_k}}\}$ converges weakly to $\mathbf{P}_S = \kappa_{q,a}$, when $N_k \to a$ in $\mathbf{Q}_p$.

One might say that in the p-adic case there is no the law of large numbers in the ordinary meaning. We could not consider asymptotics for $n \to \infty$. Only properly selected subsequencies of natural numbers generate fruiful asymptotics of probabilities. However, even such a weakened law of large numbers may serve for applications. In physics sometimes one selects special sequences of observation times; similar subselections may appear in other statistical applications.

5.3. The Central Limit Theorem

Here we restrict our considerations to the case of symmetric random variables $\xi_n(\omega) = 0, 1$ with probabilities $1/2$. We study the p-adic asymptotic of the normalized sums

$$G_n(\omega) = \frac{S_n(\omega) - ES_n(\omega)}{\sqrt{DS_n(\omega)}}, \tag{6}$$

Here $ES_n = n/2, D\xi_n = E\xi^2 - (E\xi)^2 = 1/4$ and $DS_n = n/4$. Hence

$$G_n(\omega) = \frac{S_n(\omega) - n/2}{\sqrt{n}/2} = \sum_{j=1}^{n} \frac{2\xi_n}{\sqrt{n}} - \sqrt{n}.$$

By applying the method of characteristic functions we can find the characteristic function of the limiting distribution. Let us compute the characteristic function of random variables $G_n(\omega)$:

$$\psi_n(z) = (\cosh\{z/\sqrt{n}\})^n.$$

Set $\psi(z,a) = (\cosh\{z/\sqrt{a}\})^a, a \in \mathbf{Z}_p, a \neq 0$. This function belongs to the space of locally analytic functions. There exists the p-adic analytic generalized function, see [10] for detail, γ_a with the Borel-Laplace transform $\psi(z,a)$.

Unfortunately, we do not know so much about this distribution (an analogue of Gaussian distribution?). We only proved the following theorem:

Theorem 5.4

The γ_1 is the bounded measure on $\mathbf{Z}_p$.

Open Problems:
1). Boundedness of γ_a for $a \neq 1$.
2). Weak convergence of $\mathbf{P}_{G_n}$ to $\mathbf{P}_G = \gamma_a$ (at least for $a = 1$).

6. *p*-adic Valued Probabilities – Coupling with Statistics

In fact, Kolmogorov's probability theory has two (more or less independent) counterparts:
(a) axiomatics (a mathematical representation);
(i) interpretation (rules for application).

The first part is the measure-theoretic formalism. The second part is a mixture of frequency and ensemble interpretations: "... we may assume that to an event A which has the following characteristics: (a) one can be practically certain that if the complex of conditions $\sum$ is repeated a large number of times, N, then if n be the number of occurrences of event A, the ratio n/N will differ very slightly from $\mathbf{P}(A)$; (b) if $\mathbf{P}(A)$ is very small, one can be practically certain that when conditions $\sum$ are realized only once the event A would not occur at all", [1].

As we have already noticed, (a) and (i) are more or less independent. Therefore Kolmogorov's measure-theoretic formalism, (a), is used successfully, for example, in the subjective probability theory.

In practice we apply Kolmogorov's (conventional) interpretation, (i), in the following way. First of all we have to fix $0 < \epsilon < 1$, *significance level*. If the probability $\mathbf{P}(A)$ of some events A is less than ϵ, this event is considered as practically impossible. We now generalize the conventional interpretation of probability to the case of $\mathbf{Q}_p$-valued probabilities. First of all we have to fix some neighborhood of zero, V, *significance neighborhood*.

If the probability $\mathbf{P}(A)$ of some event A belongs to V, this event is considered as practically impossible.

Since the group $\mathbf{Q}_p$ is metrizable, then the situation is similar to the standard (real) probability. We choose $\epsilon > 0$ and consider the ball

$$V_\epsilon = \{x \in \mathbf{Q}_p : \rho_p(0, x) < \epsilon\}.$$

If $\rho_p(0, \mathbf{P}(A)) < \epsilon$, then the event A is considered as practically impossible.

Let us borrow some ideas from statistics. We are given a certain sample space Ω with an associated distribution $\mathbf{P}$. Given an element $\omega \in \Omega$, we want to test the hypothesis "ω belongs to some reasonable majority". A reasonable majority $\mathcal{M}$ can be described by presenting *critical regions* $\Omega^{(\epsilon)} (\in F)$ of the significance level $\epsilon, 0 < \epsilon < 1 : \mathbf{P}(\Omega^{(\epsilon)}) < \epsilon$. The complement $\bar{\Omega}^{(\epsilon)}$ of a critical region $\Omega^{(\epsilon)}$ is called $(1 - \epsilon)$ confidence interval. If $\omega \in \Omega^{(\epsilon)}$, then the hypothesis '$\omega$ belongs to majority $\mathcal{M}$' is rejected with the significance level ϵ. We can say that ω fails the test to belong to $\mathcal{M}$ at the level of critical region $\Omega^{(\epsilon)}$.

$\mathbf{Q}_p$-statistical machinery works in the same way. We consider significance levels V given by neighborhoods of zero in $\mathbf{Q}_p$. Thus we consider critical regions $\Omega^{(V)} (\in F)$:

$$\mathbf{P}(\Omega^{(V)}) \in V.$$

If $\omega \in \Omega^{(V)}$, then the hypothesis "ω belongs to majority $\mathcal{M}$" (represented by the statistical test $\{\Omega^{(V)}\}$) is rejected with the significance level V. Since $\mathbf{Q}_p$ is metrizable, then we can always choose $V = V_\epsilon, \epsilon > 0$.

Of course, the strict mathematical description of the above statistical considerations can be presented in the framework of Martin-Löf [6], [4], [7] statistical tests. We remark that such a p-adic framework was already developed in [7]. We emphasize some similarities and differences between real and p-adic theories:

In the p-adic case (as in the real case) it is possible to enumerate effectively all p-adic tests for randomness.

However, a universal p-adic test for randomness does not exist [7].

We now define $\mathbf{Q}_p$-random sequences, namely sequences

$$\omega = (\omega_1, \ldots, \omega_N, \ldots), \omega_j = 0, 1,$$

that are random with respect to a $\mathbf{Q}_p$-valued probability distribution in the same way as in the real Martin-Löf approach.

The general scheme of the application of $\mathbf{Q}_p$-valued probabilities is the same as in the ordinary case:

1) we find initial probabilities;
2) then we perform calculations by using calculus of $\mathbf{Q}_p$-valued probabilities;
3) finally, we apply the above interpretation to resulting probabilities.

The main question is "How can we find initial probabilities?" Here the situation is more or less similar to the situation in the ordinary probability theory. One of possibilities is to apply the frequency arguments (as R. von Mises). We have already discussed such an approach for p-adic probabilities. Another possibility is to use subjective approach to probability. I think that everybody agrees that there is nothing special in segment $[0, 1]$ as the set of labels for the measure of belief in the occurrence of some event. In the same way we can use, for example, the segment $[-1, 1]$ (signed probability) or the unit complex disk (complex probability) or the set of p-adic integers $\mathbf{Z}_p$ (p-adic probability). Since $\mathbf{Q}_p$ is a field we can apply the machinery of Bayesian probabilities and, finally, use our interpretation of probabilities to make a statistical decision. The third possibility is to use symmetry arguments, Laplacian approach. For example, by such arguments we can choose (in some situations) the uniform $\mathbf{Q}_p$-valued distribution.

Example 6.1. (A p-adic statistical test) Theorem 5.1. implies that, for each p-adic sphere $\mathbf{S}_{1/p^l}(r)$, where l, r, m were done in Theorem 3.1:

$$\lim_{k\to\infty} \mathbf{P}(\{\omega \in \Omega_2 : S_{N_k}(\omega) \in \mathbf{S}_{1/p^l}(r)\}) = 0,$$

for each sequence $s = \{N_k\}, N_k \to m, k \to \infty$. We can construct a statistical test on the basis of this limit theorem (as well as any other limit theorem). Let $s = \{N_k\}, N_k \to m$, be a fixed sequence of natural numbers. For any $\epsilon > 0$, there exists k_ϵ such that, for all $k \geq k_\epsilon$,

$$|\mathbf{P}(\{\omega \in \Omega_2 : S_{N_k}(\omega) \in \mathbf{S}_{1/p^l}(r)\})|_p < \epsilon.$$

We set $\Omega^{(\epsilon)} = \bigcup_{k \geq k_\epsilon} \{\omega \in \Omega_2 : S_{N_k}(\omega) \in \mathbf{S}_{1/p^l}(r)\}$. We remark that

$$|\mathbf{P}(\Omega^{(\epsilon)})|_p < \epsilon.$$

We now define reasonable majority of outcomes as sequences that do not belong to the sphere $\mathbf{S}_{\frac{1}{p^l}}(r)$, "nonspherical majority." Here the set $\Omega^{(\epsilon)}$ is the critical region on the significance level ϵ.

Suppose that a sequence ω belongs to the set $\Omega^{(\epsilon)}$. Then the hypothesis "ω belongs nonspherical majority" must be rejected with the significance level ϵ. In particular, such a sequence ω is not random with respect to the uniform p-adic distribution on Ω_2. If, for some sequence of 0 and 1, $\omega = (\omega_j)$ we have $\omega_1 + \ldots + \omega_{N_k} - r = \alpha \mod p^l, \alpha = 1, \ldots, p-1$, for all $k \geq k_\epsilon$, then it is rejected.

The simplest test is given by $m = 1, r = 0, N_k = 1 + p^k$ and $\omega_1 + \ldots + \omega_{N_k} = \alpha \mod p, \alpha = 1, \ldots, p-1$.

At the end we recall some recent applications of p-adic numbers: a) in cognitive science and neurophysiology [31], [32]; b) logical foundation of p-adic probability [33]; c) modeling disordered systems (spin glasses) [34]–[36]; d) p-adic cosmology and quantum physics [37].

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THE MAIN GENERALIZATION OF CONTINUOUS-VALUED LOGIC

The paper outlines basic results in the generalization of continuous-valued logic. The survey is based on Russian publications. We consider an order logic, which is a generalization of continuous-valued logic where the operations of maximum (disjunction) and minimum (conjunction) are substituted with the operation of selection of r-th order argument, corresponding to the values of arguments. We show that this new operation is expressed in a superposition of disjunctions and conjunctions of continuous-valued logic. Various classes of logical determinants are considered; they are thought as numerical characteristics of matrices, expressible in operations of continuous-valued logic. Namely, we investigate order determinants, which generalize order logical operation of several arguments in matrix form, and determinants with various constraints on subsets of matrix elements. Properties of all logical determinants are discussed, compared with properties of algebraic determinants; techniques of computation of logical determinants are supplied. We also investigate a predicate algebra of choice, which generalizes continuous-valued logic in case of simulation of discontinuous functions; a hybrid logic of continuous and discrete variables; a logic-arithmetic algebra, which includes, in addition to continuous-logic operations, four arithmetical operations; a complex algebra of logic, where supportive set C is a field of complex numbers. A description of each algebra includes basic laws, which are compared with the laws of conventional continuous-valued logic. Several generalizations of continuous-logic operations to operations over matrices, random and interval variables are discussed. Some applications of continuous-valued logics are shown.

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1. Introduction

Continuous-valued logic (CL) is a very rich mathematical structure, particulars of which include expressiveness, constructability and visibility. These features open a wide road to the huge varieties of kinetic potential applications in mathematics, engineering, economy, biology, sociology, and history. Much more areas of possible application are still uncovered; therefore we have to search for novel generalizations of CL.

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There is a large variety of generalizations of CL. Some of them, namely logic determinants, are oriented toward application of CL to systems of higher dimensions. Others aim to expand domains of logical functions by enriching continuous set with discrete set (hybrid logic) or by transition from real numbers to complex numbers (complex logic). There are also generalizations of CL derived from the modification of a structure of variables, they are related to matrix, random or interval variables. At last, we can generalize CL by including some non-logical operations, e.g. arithmetical operations. All these generalizations allow us to apply mathematical apparatus of CL to investigation of complex natural and engineering systems. Thus, in particular, hybrid logic may be applied to design and analysis of analog-digital devices, interval logic is appropriate for systems with uncertain parameters, logic-arithmetic algebra is perfect for systems with discontinuity. All these classes will be discussed in this paper.

2. Order logic and order logical determinants

Let us start with a function

$$f^r(a_1, \ldots, a_n) = a^r, \quad a^1 \leq \ldots \leq a^n, \quad r = \overline{1, n}, \tag{1}$$

which selects the r-th serial element a^r of a set $A = \{a_1, \ldots, a_n\}$, $a_i \in C$. The r-operation f^r generalizes operations of disjunction $\vee = \max$ and conjunction $\wedge = \min$ of CL; it is equivalent to disjunction when $r = n$, and it is a conjunction-like function when $r = 1$. The algebra $\{A; f^r, \ r = \overline{1, n}\}$ is called an algebra of serial logic. Every possible operation in the serial logic is build of the serial operations like f^r, $r = \overline{1, n}$, and their various superpositions. To define some function of the serial logics (as well as a functions of CL) we have to assign an argument a_i, whose value is accepted by the function, to every order $a_1, \ldots, a_n$ of arguments. It is quite easy to follow an analytical representation with the superposition of the operations $\vee$ and $\wedge$ of CL from the tabular form. The analytical representations of function of serial logic do not usually differ from those of CL. In addition to the common laws of CL the following triple of the laws whose representation is specific for the serial logic should be displayed:

Tautology: $$f^r(a, \ldots, a) = a \tag{2}$$

Commutativity: $$f^r(a_1, \ldots, a_n) = f^r(a_{i_1}, \ldots, a_{i_n}) \tag{3}$$

Distributivity: $$f^r(\varphi^{q_1}, \varphi^{q_2}, \ldots, \varphi^{q_p}) = \varphi^{q_r}, \ q_1 < q_2 < \ldots < q_p \tag{4}$$

The disjunctive expression of an arbitrary function of serial logic in the operations of CL is quite an obvious one

$$f^r(a_1,\ldots,a_n) = \bigvee_{i_1\neq\ldots\neq i_{n-r+1}} (a_{i_1}\wedge\ldots\wedge a_{i_{n-r+1}}), a_{i_k}\in\{a_1,\ldots,a_n\}. \quad (5)$$

If the set A from (1) is partially ordered as a quasi-matrix

$$A_n = \begin{Vmatrix} a_{11} & \cdots & a_{1m_1} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm_n} \end{Vmatrix} = \|a_{ij}\|, \quad a_{i1}\leq\ldots\leq a_{im_i}, \quad i=\overline{1,n}, \qquad (6)$$

then we have the generalization of serial r-operation (1) in form of the serial logical determinant (LD)

$$A_n^r \equiv |a_{ij}|_n^r = \begin{vmatrix} a_{11} & \cdots & a_{1m_1} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm_n} \end{vmatrix}^r = a^r, \quad r=\overline{1,m}, \quad M=\sum_{i=1}^n m_i. \qquad (7)$$

In the particular case of an unordered set A_n the quasi-matrices (6) are transformed into a column $A = \begin{Vmatrix} a_1 \\ \cdots \\ a_n \end{Vmatrix}$, and LD (7) becomes a usual serial r-operation (1). The algebra $\{A_n; A_n^r, \ \ r=\overline{1,m}\}$ is an algebra of the serial LD. Every possible operation in the algebra of serial LD are expressed via A_n^r, $r=\overline{1,m}$ and their superpositions. We can define LD and represent it analytically in the same way as we did with the functions of CL and serial logic; see (2). In the analytical representations, serial LD does not differ from the functions of CL and serial logic; it also obeys the laws of both logics. Also, LDs have a number of properties similar to the properties of algebraic determinants of square matrices. For example, (1) values of the LD A_n^r are non-decreasing functions of the rank r, (2) rearrangements of any two lines of the LD A_n^r do not change the values:

$$|a_{ij}+c|_n^r = |a_{ij}|_n^r + c, \ |a_{ij}\vee c|_n^r = |a_{ij}|_n^r \vee c, \ |a_{ij}\wedge c|_n^r = |a_{ij}|_n^r \wedge c, \ (8)$$

$$|a_{ij}\cdot c|_n^r = \begin{cases} c\,|a_{ij}|_n^r, & c>0 \\ c\,|a_{ij}|_n^{n-r+1}, & c<0. \end{cases} \qquad (9)$$

Any serial LD can be expressed by the operations of CL:

$$\begin{vmatrix} a_{11} & \cdots & a_{1m_1} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm_n} \end{vmatrix}^r = \bigvee_{\sum_{s=1}^n i_s = n+r-1} (a_{1i_1}^{m_1}\wedge\cdots\wedge a_{ni_n}^{m_n}) \qquad (10)$$

The entry $a_{ki_k}^{m_k}$ means that the element a_{ki_k} does not enter into conjunctions for which the conditions on $\sum i_s$ implies $i_k > m_k$. In addition to the DNF expressions of the serial r-order functions (5) and LD (10) there are exist dual CNFs. The serial LD can be decomposed on a smaller LD, e.g.:

$$|a_{ij}|_n^r = \bigvee_{i,j} a_{ij}\,|a_{ij}|_{n\setminus a_{ij}}^r\,. \tag{11}$$

In the equation (11), term $|a_{ij}|_{n\setminus a_{ij}}^r$ is LD, obtained from the LD $|a_{ij}|_n^r$ by the elimination of the element a_{ij} (by a logical addition of the element a_{ij}). There are also other decompositions of LD, in particular the decompositions minimal on complexity and so-called block decompositions. Their sequential use allows, alongside with the obvious formula (10), to open LD. The complexity of disclosure of serial LD by using (10) is polynomial on n: $N \approx n^r/(r-1)!$ (r is fixed); the complexity of sequential decomposition of LD onto blocks has $O(rM)$ bound. The calculating of serial LDs A_n^r by sequential ordering of their elements is even easier. An appropriate algorithm has a complexity $N = (n-1)r$; however, it requires to store the sub-products. A complexity of the approximation of a value of serial LD is significantly smaller because

$$\bigwedge_{i=1}^{n} a_{i]d_i[}^{m_i} \le |a_{ij}|_n^r \le \bigvee_{i=1}^{n} a_{i[l_i]},\ d_i = (n+r-1)m_i/M,\ l_i = rm_i/M, \tag{12}$$

where $]a[$ and $[a]$ are the rounds up to the nearest integer downwards and up.

Order logic and the order LD are used in analytical expressions of processes in high-dimensional systems, when the same low-dimensional processes are well elaborated analytically with the help of CL. Thus, e.g., in the example 2, part 1, of the paper we saw that the process on output of two-input automaton is expressed using operations of CL. This means that the process on the output of a multi-input automaton might be expressed analytically by the order LD. Actually, two LDs are enough: (i) LD, any i-th row of which represent moments when signal is changed, $0 \to 1$, on i-th input; (ii) LD, any row of which consists of the moments of signal changing, $1 \to 0$, on i-th input; the moments are arranged in ascending order.

For order logic, the order LD and their applications to a simulation of applied systems see [1–3, 6, 7]; numerous results of the approach can be found in the proceedings of the conferences on continuous-valued logic [9–15].

3. Logical determinants with the restrictions on a sum of elements

Let us consider a rectangular matrix with real elements

$$A_n = \left\| \begin{matrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right\| = \|a_{ij}\| \,. \tag{13}$$

We are interested in the various sums $\sum_q^{/} a_{ij}$, $q = 1, 2, \ldots$, of the matrix elements which include exactly one element from each column (we can include as many elements as necessary from each row) and the functions like those below:

$$\begin{aligned} A^{1\vee} &\equiv \overset{1}{\left| \begin{matrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right|^{\vee}} = \overset{1}{|a_{ij}|}^{\vee} = \bigvee_q \sum\nolimits_q^{/} a_{ij}, \\ A^{1\wedge} &\equiv \overset{1}{\left| \begin{matrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right|^{\wedge}} = \overset{1}{|a_{ij}|}^{\wedge} = \bigwedge_q \sum\nolimits_q^{/} a_{ij}, \end{aligned} \tag{14}$$

Let us consider a possibility of the elementary LD with limitations of the 1st sort for a matrix A. In the particular case of the single non-zero column of the matrix A, we see that LD $A^{1\vee}$ passes into the CL disjunction of the elements of this column; the LD $A^{1\wedge}$ is transformed correspondingly into their conjunction. The analytical representation of LD with the 1st sort restrictions is different, in general, of the functions of CL, both serial logic and serial LD, because the algebraic operation $+$ can be used in addition to the CL operations $\wedge$ and $\vee$. The expressions of LD with limitations of the 1st sort obey all laws of CL, and, moreover, have a number of properties similar to the properties of algebraic determinants. Thus, the rearrangement of any two lines (columns) of the LDs $A^{1\vee}$ or $A^{1\wedge}$ does not change its value. We also have

$$|ca_{ij}|^{\vee} = c\,|a_{ij}|^{\vee}, \quad |ca_{ij}|^{\wedge} = c\,|a_{ij}|^{\wedge}, \quad c > 0;$$

$$|ca_{ij}|^{\vee} = c\,|a_{ij}|^{\wedge}, \quad |ca_{ij}|^{\wedge} = c\,|a_{ij}|^{\vee}, \quad c < 0;$$

$$|a_{ij} + c|_{n\times m}^{\vee} = |a_{ij}|_{n\times m}^{\vee} + cm,$$

$$|a_{ij} + c|_{n\times m}^{\wedge} = |a_{ij}|_{n\times m}^{\wedge} + cm;$$

$$|c - a_{ij}|_{n\times m}^{\overset{\vee}{\wedge}} = cm - |a_{ij}|_{n\times m}^{\overset{\wedge}{\vee}} \,. \tag{15}$$

The LD $A^{1\vee}$ can be decomposed on smaller LDs, on column or the collections of columns

$$A^{1\vee} = \bigvee_{i=1}^{n} a_{ij} + A_{j}^{1\vee}, \quad A^{1\vee} = A^{1\vee}_{\{j_1,\dots,j_r\}} + A^{1\vee}_{j_1,\dots,j_r} \tag{16}$$

The same decomposition takes place for the LD $A^{1\wedge}$. In (16), $A^{1\vee}_{j_1,\dots,j_r}$ is LD, derived from $A^{1\vee}$ by the exclusion of $j_1,\dots,j_r$ columns (logical complement of $j_1,\dots,j_r$ columns), $A^{1\vee}_{\{j_1,\dots,j_r\}}$ is LD, formed by the columns $j_1,\dots,j_r$ (minor from these columns). The sequential decomposition LD, which is an agreement with (1), has the following obvious expression:

$$A^{1\vee} = \sum_{j=1}^{m} \bigvee_{k=1}^{n} a_{kj}, \qquad A^{1\wedge} = \sum_{j=1}^{m} \bigwedge_{k=1}^{n} a_{kj}, \tag{17}$$

The complexity of the decomposition is $N = (n-1)(m-1)$.

Now we operate with the sums of the elements of a matrix (13) like the one below

$$\sum\nolimits_{q}^{\prime\prime} a_{ij}, \quad q = 1, 2, \dots,$$

which includes exactly one element from each column and p_i elements from the ith row; $b_i \le p_i \le c_i$, $i = \overline{1,n}$, $\sum\limits_{i=1}^{n} p_i = m$. So, we have

$$A^{2\vee} \equiv \left|\begin{matrix} a_{11} & \overset{2}{\cdots} & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix}\right|^{\vee} = \left|\overset{2}{a_{ij}}\right|^{\vee}_{(b_i,c_i)} = \bigvee_{q} \sum\nolimits_{q}^{//} a_{ij},$$

$$A^{2\wedge} \equiv \left|\begin{matrix} a_{11} & \overset{2}{\cdots} & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix}\right|^{\wedge} = \left|\overset{2}{a_{ij}}\right|^{\wedge}_{(b_i,c_i)} = \bigwedge_{q} \sum\nolimits_{q}^{//} a_{ij} \tag{18}$$

Let us consider LD with limitations of the 2nd sort for a matrix A. Such LD only when $\sum\limits_{i=1}^{n} c_i \ge m$; there is a difference from LD (14). In the specific case of single non-zero column in the matrix A, the LD $A^{2\vee}$ is transformed into the CL disjunction, and the LD $A^{2\wedge}$ is transformed into the CL conjunction of elements of this column. In a common case, the analytical representation of LD (18) differs from the CL functions of both serial logic and serial LD; besides the CL operation $\vee$ and $\wedge$ include the operation $+$. The expressions of LD with the limitations of the 2nd sort obey all laws of CL but have also a number of specific properties reminding properties of the algebraic determinants. Thus, for example: (1) rearrangements of two lines together with their limitations do not change the value of LD; (2) lines, the area of which limitations is empty, can be eliminated without change of the value of LD. The LDs $A^{2\vee}$, $A^{2\wedge}$ can be decomposed on smaller LDs from any column

$$A^{2\vee} = \bigvee_{i=1}^{n} (a_{ij} + A_{ij}^{2\vee}), \quad A^{2\wedge} = \bigwedge_{i=1}^{n} (a_{ij} + A_{ij}^{2\wedge}). \tag{19}$$

In (19), $A_{ij}^{2\vee}$ is LD, obtained from the LD $A_{ij}^{2\vee}$ by the exclusion of jth column and shift of an interval $[b_i, c_i]$ of values of the parameter p_i along one position to the left (but not further than 0); this is called a logical adjunct (complement) of an element a_{ij} in the LD $A^{2\vee}$. The logical addition of element a_{ij} in the LD $A^{2\wedge}$ is similar to (19). We can also implement decomposition on the collection of columns

$$\begin{aligned} A^{2\vee} &= \bigvee_{k} (A^{2\vee}_{\{j_1,\ldots,j_r\},U_k} + A^{2\vee}_{j_1,\ldots,j_r,V_k}), \\ A^{2\wedge} &= \bigwedge_{k} (A^{2\wedge}_{\{j_1,\ldots,j_r\},U_k} + A^{2\wedge}_{j_1,\ldots,j_r,V_k}) \end{aligned} \tag{20}$$

In 1st formula of (20), $A^{2\vee}_{\{j_1,\ldots,j_r\},U_k}$ is LD, formed from the columns $j_1, \ldots, j_r$ of LD $A^{2\vee}$, with the restrictions on the number of elements in different lines by area U_k (minor from these columns). $A^{2\vee}_{j_1,\ldots,j_r,V_k}$ is LD, obtained from LD $A^{2\vee}$ by the exclusion of columns $j_1, \ldots, j_r$ and restrictions on numbers of elements in different lines by area V_k (logical complement of the specified minor in the LD $A^{2\vee}$). The sum of U_k and V_k for any k gives us the restriction on number of elements in lines of the whole LD $A^{2\vee}$. The definitions for the terms of the 2nd formula of (20) are similar. Subsequent decomposition of the LD $A^{2\vee}$, $A^{2\wedge}$, with usage of the formulas (19) and (20) produces simple LDs, which can be calculated directly. Complexity of the calculation of $n \times m$-size LD with the limitations of the 2nd sort is $N < 3n^{m-1}$.

Now we will consider every possible sum $\sum_q^{///} a_{ij}$, $q = 1, 2, \ldots$, of the elements of matrix A (13) of sizes $n \times n$, a sum includes equally one unit from each row and each column. Thus we obtain the following functions:

$$A^{3\vee} \equiv \begin{vmatrix} a_{11} & \overset{3}{\cdots} & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}^{\vee} = \left| \overset{3}{a_{ij}} \right|^{\vee} = \bigvee_q \sum\nolimits_q^{///} a_{ij},$$

$$A^{3\wedge} \equiv \left| \begin{matrix} & 3 & \\ a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{matrix} \right|^{\wedge} = \left| \overset{3}{a_{ij}} \right|^{\wedge} = \bigwedge_{q} \sum\nolimits_{q}^{///} a_{ij} \tag{21}$$

The special cases of LD with the restrictions of the 2nd sort (18) are called LDs with the restrictions of the 3rd sort for a matrix A. They always exist. The analytical representation of LD (21) includes (in a common case, in addition to the operations CL $\vee$ and $\wedge$) the operation $+$, which is different of the CL functions of serial logic and serial LD. In special cases of a single non-zero column or single non-zero line in a matrix A the LD $A^{3\vee}$ passes into a disjunction, and the LD $A^{3\wedge}$ is transformed into a conjunction of the elements of the given column (the given line). LDs of the 3rd sort $A^{3\vee}$, $A^{3\wedge}$ are special cases of LD of the 2nd sort $A^{2\vee}$, $A^{2\wedge}$ (18) for $m = n$ and the restrictions: $b_i = p_i = c_i = 1$, $i = \overline{1,n}$. Therefore LDs of the 3rd sort (21) have properties of LD of the 2nd sort. Moreover they have properties of LD with the restrictions of the 1st sort, see (4), and also a number of some specific properties, for example:

$$A^{3\vee} = (A^{3\mathrm{T}})^{\vee}, \quad A^{3\wedge} = (A^{3\mathrm{T}})^{\wedge}, \tag{22}$$

where "T" means transposition of a matrix. The LD (21) can be decomposed onto the smaller LDs from any column or any line:

$$\begin{aligned} A^{3\vee} &= \bigvee_{i=1}^{n} (a_{ij} + A_{ij}^{3\vee}), \quad j = \overline{1,n}, \\ A^{3\vee} &= \bigvee_{j=1}^{n} (a_{ij} + A_{ij}^{3\vee}), \quad i = \overline{1,n}. \end{aligned} \tag{23}$$

The same can be done for $A^{3\wedge}$. In (23), $A_{ij}^{3\vee}$ is LD, obtained from the LD $A^{3\vee}$ by the elimination of the ith line and jth column, intersection of which hosts the element a_{ij} (a logical complement of the element a_{ij} in LD $A^{3\vee}$). The decomposition of (23) onto the collections of lines and columns is similar:

$$\begin{aligned} A^{3\vee} &= \bigvee_{D_r} (A_{\{D_r,B_r\}}^{3\vee} + A_{D_r,B_r}^{3\vee}), \quad B_r \subset \{1,\ldots,n\}, \\ A^{3\vee} &= \bigvee_{B_r} (A_{\{D_r,B_r\}}^{3\vee} + A_{D_r,B_r}^{3\vee}), \quad D_r \subset \{1,\ldots,n\} \end{aligned} \tag{24}$$

The same can be done for $A^{3\wedge}$. In (24) $A_{\{D_r,B_r\}}^{3\vee}$ is LD, obtained by the selection, in the LD $A^{3\vee}$, the elements being on the intersection of lines

$D_r = (i_1, \ldots, i_r)$ and columns $B_r = (j_1, \ldots, j_r)$; $A^{3\vee}_{D_r, B_r}$ is LD, obtained from the LD $A^{3\vee}$ by eliminating the lines D_r and columns B_r. The sequential decomposition of LD (21) allows us to calculate them. Thus, the least complexity turns out with the use of decompositions (24).

Now we can consider the main matrix A (13) and the matrix of restrictions $B = \|b_{ij}\|$; both matrices have the same order. Let us operate with the sums of elements of both matrices of sorts $\sum'_q a_{ij}$, $\sum'_q b_{ij}$, $q = 1, 2, \ldots$ (as well as in LD of the 1st sort of matrices A, B). Functions of the form

$$A^{4\vee} \equiv \left|\begin{matrix} a_{11} & \overset{4}{\cdots} & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix}\right|^{\vee} = \left|\overset{4}{a_{ij}}\right|^{\vee} = \bigvee_{q} \sum\nolimits'_{q} \underset{\sum'_q b_{ij} \le b}{a_{ij}}\,,$$

$$A^{4\wedge} \equiv \left|\begin{matrix} a_{11} & \overset{4}{\cdots} & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix}\right|^{\wedge} = \left|\overset{4}{a_{ij}}\right|^{\wedge} = \bigwedge_{q} \sum\nolimits'_{q} \underset{\sum'_q b_{ij} \le b}{a_{ij}} \tag{25}$$

are called LDs with the restrictions of the 4th sort for a matrix A. Such LDs exist only for $B^{1\wedge} \le b$. In the specific case of single non-zero column in a matrix A the LD $A^{4\vee}$ is transformed into disjunction and the LD $A^{4\wedge}$ becomes the CL conjunction of some elements of this column (satisfying the restrictions). In general, the analytical expression of LD (25) includes the CL operations $\vee$ and $\wedge$ as well as the operation $+$. The expressions of LD of the 4th sort obey all laws of CL. These LD have properties (15) of LD of the 1st sort and the following specific properties:

1) rearrangements of two tines (columns) of a main matrix A simultaneously with rearrangement of appropriate lines (columns) of a matrix of limitations B does not change the value of LD;
2) the common for all elements of a column addend can be taken out for the sign of LD;
3) LD with column from equal elements is equal to sum of this element and the CL disjunction of logical adjuncts of all elements of a column.

Here, the logical adjunct $A^{4\overset{\vee}{\wedge}}_{ij}$ of the element a_{ij} in LD $A^{4\overset{\vee}{\wedge}}$ is LD obtained from $A^{4\overset{\vee}{\wedge}}$ by the elimination of the jth column and decrease of the boundary b on b_{ij}. Concerning the LDs $A^{4\vee}$ and $A^{4\wedge}$, we can say that it is possible to decompose them onto smaller LDs from any column which agrees to the 1st formula of (23), and also to decompose them onto any collection of columns. Using decompositions allows us to calculate LD.

The LD with restriction on sum of elements are used in the analytical representations of algorithms of static optimization, such as problems of optimal distribution of limited resources. Such algorithms can be expressed by the disjunction and conjunction of CL because there we compute maximum and minimum of two variables, i.e. make elementary acts of optimization. The importance of LD in this case lies in a compact description of the algorithms when problems are high-dimensional; the different types of the restrictions on sums of elements of LD correspond to different classes of static optimization. For example, the LD with the restriction of the 3rd sort (21) is an analytical representation of algorithm for the problem on assignment of n candidates on n positions, when a matrix of efficiency of the candidates i for the positions j is $|a_{ij}|$, i.e. every candidate receives a position, and every vacancy is filled by a candidate. In this situation, the decomposition of LD (23) allows us to decrease the problem of dimension from $n \times n$ to $(n-1) \times n$ or $n \times (n-1)$. The LDs with restrictions on sum of elements and their application to the problems of static optimization are invented by V. I. Levin in [2, 3, 6, 7].

4. Logical determinants with a domain restriction

In the rectangular $n \times m$-size matrix A (13) we shall consider every possible descending step paths from a block (1, 1) into a block (n, m). Let us designate by $\sum_q^1 a_{ij}$ a sum of elements of the matrix A along the path of the length q, and $\sum_q^2 a_{ij}$ a sum of elements outside the path. Functions of the form

$$A^\vee \equiv |a_{ij}|^\vee = \bigvee_q \sum\nolimits_q^1 a_{ij}, \quad A^\wedge \equiv |a_{ij}|^\wedge = \bigwedge_q \sum\nolimits_q^2 a_{ij} \tag{26}$$

are called disjunctive and conjunctive LDs of a matrix A. They are elementary LDs with the restrictions on area of creation of sums of elements. The function

$$A^+ \equiv |a_{ij}|^+ = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \tag{27}$$

Any matrix A always has the LD $A^\vee$, $A^\wedge$, and A^+. In the case of single non-zero element in the q-length path of a matrix A, the LD $A^\vee$ is transformed into the CL disjunction and LD $A^\wedge$ becomes a conjunction of the elements of this matrix. The analytical representation of the LDs $A^\vee$ and $A^\wedge$ contains the CL operations of disjunction and conjunction and algebraic addition; the LD A^+ has only addition. The expressions LD $A^\vee$ and $A^\wedge$ satisfy to the laws of CL. Additionally, $A^\vee$, $A^\wedge$ and A^+ have a number of specific properties similar to the properties of algebraic determinants:

$$|a_{ij}+c|_{n\times m}^{\vee} = |a_{ij}|_{n\times m}^{\vee} + c(m+n-1);$$

$$|a_{ij}+c|_{n\times m}^{\wedge} = |a_{ij}|_{n\times m}^{\wedge} + c(mn-m-n+1);$$

$$|a_{ij}+c|_{n\times m}^{+} = |a_{ij}|_{n\times m}^{+} + cmn;$$

$$|a_{ij}c|^{\vee} = c\,|a_{ij}|^{\vee}, \quad |a_{ij}c|^{\wedge} = c\,|a_{ij}|^{\wedge}, \quad c>0;$$

$$|a_{ij}c|^{\vee} = c\left(|a_{ij}|^{\vee} - \Delta\right), \quad |a_{ij}c|^{\wedge} = c\left(|a_{ij}|^{\wedge} + \Delta\right), \quad c<0, \qquad (28)$$

where $\Delta = \bigvee\limits_{q} \sum_q^1 a_{ij} - \bigwedge\limits_{q} \sum_q^1 a_{ij}$; $\begin{vmatrix} a_{11}+c & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm}+d \end{vmatrix}^{\vee} = |a_{ij}|^{\vee} + c + d$;
$\begin{vmatrix} a_{11}+c & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm}+d \end{vmatrix}^{\wedge} = |a_{ij}|^{\wedge}$; $|a_{ij}c|^{+} = c\,|a_{ij}|^{+}$, $A^\vee + A^\wedge = A^+$.

The rearrangement of each pair of symmetric lines and each pair of symmetric columns does not change the value of LD $A^\vee$ and $A^\wedge$. The LDs $A^\vee$ and $A^\wedge$ can be decomposed on angular elements a_{11} and a_{nm} as

$$A^\vee = a_{11} + (A_{1,-}^{\vee} \vee A_{-,1}^{\vee}), \quad A^\vee = a_{nm} + (A_{m,-}^{\vee} \vee A_{-,n}^{\vee}). \qquad (29)$$

This is similar to $A^\wedge$. In (29) $A_{i,-}^{\vee}$ is LD obtained from $A^\vee$ by the elimination of the ith line (logical complement of the ith line), and $A_{-,j}^{\vee}$ is LD, obtained from $A^\vee$ by the elimination of the jth column (logical complement of the jth column). The decompositions of the LDs $A^\vee$ and $A^\wedge$ onto a collection of elements of lines and columns are possible. Consistently decomposing the LDs $A^\vee$ and $A^\wedge$ in formulas (29) we can calculate LD. But practically it is more convenient to use the formula

$$A_{rk}^{\vee} = (A_{r,k-1}^{\vee} \vee A_{r-1,k}^{\vee}) + a_{rk}, \qquad (30)$$

where $A_{rk}^{\vee}$ is LD of a type $A^\vee$ with r first lines and k first columns of a matrix A. The required LD $A^\vee$ is $A_{nm}^{\vee}$. Therefore, to compute $A^\vee$ we can calculate the angular element $A_{nm}^{\vee}$ in the matrix $A^* = \|A_{rk}^{\vee}\|$ associated with the matrix A. To do it we can use a recurrent relation (30) and wave algorithms of sequential finding of elements of matrix A^*, which starts in the left upper corner of the matrix and stops in the right lower corner. The complexity of such a calculation of LD $A^\vee$ has an upper bound $O(mn)$.

The LD A^+ is calculated directly from (27); its complexity is also $O(mn)$. The LD $A^\wedge$ is calculated from (28) through $A^\vee$ and A^+.

The LDs with domain restrictions are used in analytical representations of problems of dynamic optimization, such as schedule theory; in this situations LD allows to make a compact description of optimality condition even in the cases of high dimensions. The theory of LD with domain restrictions are elaborated in the works [2, 3, 6, 7].

5. Hybrid continuous-valued logics

CL is hybrid if it includes non-logical operation in addition to CL operations. We shall consider two two-place operations

$$\begin{aligned} \vee_p(x_1, x_2) &= x_1 \cdot p + x_2 \cdot \bar{p}, \\ \wedge_p(x_1, x_2) &= x_1 \cdot \bar{p} + x_2 \cdot p, \end{aligned} \qquad x_1, x_2 \in C, \quad p \in \{0, 1\}, \tag{31}$$

where $p = p(y_1, \ldots, y_n)$ is m-place binary predicate with real values of predicate variables $y_1, \ldots, y_m$. In (31) we added two arithmetical operations $\cdot$ $+$, in addition to the negation operation $\bar{p}$. The predicate p in (31) is a control parameter, the choice of which value defines a choice of the concrete function $f(x_1, x_2)$. Thus, for $p = 1(x_1 - x_2)$, the functions $\vee_p$ and $\wedge_p$ are transformed into the CL disjunction and conjunction. A number of possible choices of the functions f increases with a superposition of functions (31) onto subject and predicate variables x_i and y_i. The algebra $\{C, \vee_p, \wedge_p, \bar{\ }\}$ is called a predicate algebra of choice. There are several laws which take place in the algebra $\{C, \vee_p, \wedge_p, \bar{\ }\}$ and which are analogous with the laws of the algebra of CL:

$$\begin{aligned} &\text{Tautology:} && \vee(x, x) = \wedge(x, x) = x, \\ &\text{Commutativity:} && \vee(x_1, x_2) = \wedge(x_2, x_1), \\ &\text{Distributivity:} && f[{}^{\vee}_{\wedge} x_1, x_2] ={}^{\vee}_{\wedge} [f(x_1), f(x_2)], \end{aligned} \tag{32}$$

However, there are also several laws, which are specific only for this algebra.

If we allow the variable x_i to take as continuous values from the set $C = [A, B]$ as discrete values a hybrid logic emerges. It can be based on the threshold operations, which transform continuous variables into discrete variables, and anti-threshold operations, which implement inverse transformation. Thus, for two continuous and one discrete variables these operations are as follows:

$$P(x_1, x_2) = \left\{ \begin{array}{l} 1, x_1 \geq x_2 \\ 0, x_1 < x_2 \end{array} \right| , \quad D_1(y) = \left\{ \begin{array}{l} x_1, y = 1 \\ x_2, y = 0 \end{array} \right| ,$$

$$D_2(y) = \left\{ \begin{array}{l} x_2, y = 1 \\ x_1, y = 0 \end{array} \right| \begin{array}{l} x_1, x_2 \in C \\ y \in \{0, 1\}. \end{array} \tag{33}$$

The algebra $\{C \cup \{0;1\}; P, D_1, D_2\}$ is called an algebra of hybrid logic. Any function derived by a superposition of operations (33) is called a function of hybrid logic. The base operations of hybrid logic and CL are connected by the following ratio

$$D_1P(x_1, x_2) = x_1 \vee x_2, \quad D_2P(x_1, x_2) = x_1 \wedge x_2. \tag{34}$$

Also, hybrid logic can be based on the CL operations $\vee$, $\wedge$, $\bar{\ }$. In that case, it satisfies the CL laws.

If we combine the CL operations $\vee$ and $\wedge$ with the arithmetic operations $+$, $-$, $\cdot$, $:$, the so-called logic-arithmetic algebra may be constructed. Any function $C^n \to C$ as a superposition of six determined operations is called a logic-arithmetic function. In logic-arithmetic algebra the laws of arithmetic, CL and mixed logic-arithmetic laws may be used. For example:

$$\begin{array}{l} a + (b \vee c) = (a + b) \vee (a + c), a + (b \wedge c) = (a + b) \wedge (a + c), \\ a - (b \vee c) = (a - b) \wedge (a - c), a - (b \wedge c) = (a - b) \vee (a - c). \end{array} \tag{35}$$

It is more difficult to define the laws containing operations $\cdot$, $:$. Let symbols $\underset{k}{\vee}$ and $\underset{k}{\wedge}$ mean the operations $\vee$ and $\wedge$, respectively, when the constant k is positive and they mean $\wedge$ and $\vee$, respectively, when the constant k is negative. Then the following laws hold

$$k \cdot (a \vee b) = k \cdot a \underset{k}{\vee} k \cdot b, \quad k \cdot (a \wedge b) = k \cdot a \underset{k}{\wedge} k \cdot b. \tag{36}$$

The predicate algebra of choice solves a problem of simulation of discontinuous functions and facilitate analysis and synthesis of analog and digital devices. This algebra is developed by L. I. Volgin [4]. Hybrid logic is applied in the investigation of the devices, which process a mixture of continuous and discrete signals, e.g. analog-digital encoders. Hybrid logic, in the form discussed in this paper, is offered by P. N. Shimbiriev [5]. The logic-arithmetic algebra aims to describe such systems, which implement both continuous-logic and arithmetic operations, e.g. electric systems and economic systems. This algebra is offered by E. I. Berkovich and V. I. Levin in [2, 3, 7].

6. Complex continuous-valued logics

CL is complex if continuous-valued operations are performed over complex numbers, matrices or intervals.

CL can be generalized to complex-valued logic assuming that basic set C in the quasi-Boolean algebra $\{C, \vee, \wedge, \bar{\ }\}$ is a field of complex-valued numbers. In this case, the center M of the set C is a point $M = 0$ and operation of negation of a complex number a is defined as follows

$$\bar{a} = 2M - a = -a, \quad a \in C. \tag{37}$$

The operations of disjunction $\vee$ and conjunction $\wedge$ of CL are defined in this case in a standard manner: $\vee = \max$, $\wedge = \min$. Since max and min for complex-valued numbers are not defined we have

$$a \vee b = 0,5[a + b + |a - b|], \qquad a \wedge b = 0,5[a + b - |a - b|], \tag{38}$$

where all operations on the right sides of the equations in (38) are generalized to the complex-valued case. The designed complex-valued quasi-Boolean algebra obeys the laws of tautology, commutativity, double negation and descent of negation on addends as well as laws (35) and (38). However, it does not obey the laws of absorptions, of Kleene, operations with constants, of excluded middle, of contradiction, of associative and of de Morgan.

The generalization of CL to a matrix case is also possible. In this case variables are represented by rectangular matrices $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$ of identical dimensions, whose elements a_{ij} and b_{ij} take values from a segment C. In this case, all logical operations are defined item-by-item, for example:

$$A \vee B = \|a_{ij} \vee b_{ij}\|, \quad A \wedge B = \|a_{ij} \wedge b_{ij}\|, \quad \bar{A} = \|\overline{a_{ij}}\|. \tag{39}$$

Therefore all laws of scalar CL are transferred to matrix CL.

If the operations of CL are implemented over random variables from the set C we have probabilistic interpretation of CL. All laws of deterministic CL are well preserved in the probabilistic CL. However all functions of CL such as a superposition of logical operations over random variables become stochastic. The main problem here is to find a probability distribution and moments of the given functions CL, which arguments are distributed by the given laws. If X_i, $i = \overline{1, n}$, is a random variable distributed with the law $F_i(X)$, a disjunction $\bigvee_{i=1}^{n} X_i$, and conjunction $\bigwedge_{i=1}^{n} X_i$ are distributed by the laws

$$F^{\vee}(x) = \prod_{i=1}^{n} F_i(x), \quad F^{\wedge}(x) = 1 - \prod_{i=1}^{n} [1 - F_i(x)]. \tag{40}$$

The distribution of the arbitrary function φ can be found as

1. an expression φ is resulted in a disjunctive normal form;
2. inequalities $\varphi < x$ are calculated by the method of partition; they are resulted in a sort of union of not-intersected systems of inequalities of the arguments $X_1, \ldots, X_n$,
3. a law of distribution $F_\varphi(X) = P(\varphi < X)$ of summation of probabilities of indicated systems of inequalities equal to integrals from products of density function $f_i(x)$ of values x_i is calculated.

If the operations of CL are implemented over interval variables $\tilde{a} = [a_1, a_2]$ from the set C we obtain the so-called interval CL.

The operations can be defined as follows

$$\tilde{a} \vee \tilde{b} = \{a \vee b | a \in \tilde{a}, b \in \tilde{b}\}, \quad \tilde{a} \wedge \tilde{b} = \{a \wedge b | a \in \tilde{a}, b \in \tilde{b}\} \tag{41}$$

In interval CL all laws of standard CL are preserved. However, all CL functions, such as a superposition of logical operations over intervals, become interval functions. Thus, the main problem here is to find an interval of values of the functions using the interval of the values of their variables:

$$\begin{aligned} [a_1, a_2] \vee [b_1, b_2] &= [a_1 \vee b_1, a_2 \vee b_2], \\ [a_1, a_2] \wedge [b_1, b_2] &= [a_1 \wedge b_1, a_2 \wedge b_2], \\ [\overline{a_1, a_2}] &= [\overline{a_2}, \overline{a_1}]\,. \end{aligned} \tag{42}$$

The functions $C^n \to C$, represented in various areas by the various forms of the given algebra CL, are called piecewise functions of CL. The following generalization can be investigated with the help of the serial LDs.

Complex CL is used in investigation of aperiodic processes in electric circuits. This logic, together with matrix CL, is developed by E. I. Berkovich (see review in [7]). Probabilistic and interval CL, aimed to explore systems with uncertain parameters and noise, are invented by V. I. Levin [1, 2, 6–8].

7. Conclusion

The generalizations of continuous-valued logics discussed in the paper can be developed further, however even indicate here fields of logical determinants, algebra of choice, hybrid logics, as well as matrix, probabilistic and interval logics, still bear a full potential for new discoveries. The detailed discussion of the subject, including extensive bibliography, can be found in [1–8].

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Arkady D. Zakrevskij

FINITE PREDICATES WITH APPLICATIONS IN PATTERN RECOGNITION PROBLEMS

We extend the theory of Boolean functions, especially in respect to representing these functions in the disjunctive or conjunctive normal forms, onto the case of finite predicates. So, we show that it is useful to apply the language of Boolean vectors and matrices, developing efficient methods for calculation over finite predicates. This means that finite predicates should be decomposed into some binary units, which will correspond to components of Boolean vectors and matrices and should be represented as combinations of these units. Further, we define probabilities in data bases using Boolean matrices representing finite predicates. We also show that it is natural to try and present knowledge in the most compact form, which allows reducing the time of inference, by which the recognition problems are solved.

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1. Finite predicates and their matrix forms

One of the most important problems of artificial intelligence is the problem of pattern recognition [2, 4]. To solve it, various formal methods were applied, usually based on the theory of Boolean functions [7, 9]. However, they become insufficient when dealing with objects described in terms of multi-valued attributes, so other means should be involved in this case, finite predicates for example [10].

The finite predicates are two-valued functions, which arguments are variables with restricted number of values. Denote these variables by x_1, $x_2, \dots, x_n$. Let them receive values accordingly from finite sets $X_1, \dots, X_n$, which direct product $X_1 \times X_2 \times \dots \times X_n$ generates a space M. The mapping $M \to \{0, 1\}$ of the set M onto the two-element set $\{0, 1\}$ (it is equivalent to $\{$false, true$\}$) is called a *finite* predicate.

If solving practical problems is related to the usage of finite predica-

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tes, it is useful to represent the latter whenever possible in a more compact form. Here it is possible to use experience of the theory of Boolean functions, developed chiefly for the case when the considered functions are represented in the disjunctive normal form (DNF). The most effective methods of minimization of Boolean functions and solution of logical equations are designed just for this form. It is reasonable to extend these methods onto finite predicates.

Accoding to tradition, let us assume that an *elementary conjunction* k represents the characteristic function of some interval I of space M, and this interval is defined as the direct product of nonempty subsets α_i, taken by one from every X_i:

$$I = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_n, \quad \alpha_i \subseteq X_i, \quad \alpha_i \neq \emptyset, \quad i = 1, 2, \ldots, n.$$

This means that an elementary conjunction k is defined as a conjunction of several one-argument predicates $x_i \in \alpha_i$ (x_i receives a value from subset α_i) and is represented by the expression

$$k = (x_1 \in \alpha_1) \wedge (x_2 \in \alpha_2) \wedge \ldots \wedge (x_n \in \alpha_n).$$

The multiplicands, for which $\alpha_i = X_i$ (in this case predicate $x_i \in \alpha_i$ becomes identical to true), can be dropped.

Note that in the simplest case, when all arguments become two-valued, this definition coincides with the definition of elementary conjunction in Boolean algebra.

Similarly, we shall define an *elementary disjunction* d as a disjunction of one-argument predicates distinct from true:

$$d = (x_1 \in \alpha_1) \vee (x_2 \in \alpha_2) \vee \ldots \vee (x_n \in \alpha_n), \quad \alpha_i \subset X_i, \quad i = 1, 2, \ldots, n.$$

If $\alpha_i = \emptyset$, the term $x_i \in \alpha_i$ can be deleted from any elementary disjunction, as representing the identically false expression.

The disjunctive and conjunctive normal forms are defined by the standard way: *DNF* is a disjunction of elementary conjunctions, and *CNF* is a conjunction of elementary disjunctions.

The characteristic functions of elements of space M are naturally represented as *complete* elementary conjunctions, i.e. elementary conjunctions, in which all sets are one-element: $|\alpha_i| = 1$ for all $i = 1, 2, \ldots, n$. Any DNF, composed of complete elementary conjunctions, is called *perfect* (*PDNF*). The number of its terms is equal to the power of characteristic set M_φ of predicate φ, represented by the given PDNF.

Developing efficient methods for calculation over finite predicates, it is useful to apply the language of Boolean vectors and matrices, immediately representable in computer. And it means that all considered objects should be decomposed into some binary units, which will correspond to components of Boolean vectors and matrices and should be represented as combinations of these units.

For representation of such combinations we shall use *sectional Boolean vectors*. They are divided into sections set in one-to-one correspondence with arguments, and the components of these sections are put in correspondence with values of the arguments. Value 1 in component j of section i is interpreted as the expression "variable x_i has value j". The sectional Boolean vectors shall be used for representation of elements and some areas of space M, and collections of such vectors for representation of finite predicates.

Elements of space M, i.e. some concrete sets of values of all arguments, shall be represented by sectional vectors having exactly one 1 in each section, defining in such a way uniquely values accepted by the arguments. The sectional vectors of more general type, which could contain several 1s in each section, have double interpretation. Firstly, they can be understood as elementary conjunctions (conjunctions of one-argument predicates corresponding to intervals of space M, i.e. direct products of nonempty subsets taken by one from $X_1, X_2, \dots, X_n$). Secondly, they can be interpreted as similarly defined elementary disjunctions, which can be regarded as the complements of appropriate elementary conjunctions. Let us call such vectors *conjuncts* and *disjuncts*, respectively. Each section of a conjunct should contain no less than one 1, each section of a disjunct no less than one 0 (otherwise the conjunct degenerates to 0, the disjunct to 1).

The correspondence between elements of sectional vectors, on the one hand, and arguments and their values, on the other hand, is set by a *cliché* – the linear enumeration of arguments and their values. Let us assume that in the considered below examples all vectors are interpreted on a uniform cliché, for example, as follows:

$$\begin{array}{ccccc} a & & b & & c \\ 1\,2\,3 & . & 1\,2\,3\,4 & . & 1\,2 \end{array}$$

Thus, if it is known that vector

$$1\,1\,0\,.\,0\,1\,0\,1\,.\,0\,1$$

represents a conjunct, it is interpreted as a predicate receiving value 1 when

$$((a = 1) \vee (a = 2)) \wedge ((b = 2) \vee (b = 4)) \wedge (c = 2) = 1,$$

and if this vector is regarded as a disjunct, it is interpreted as a predicate accepting value 1 if and only if

$$((a = 1) \vee (a = 2)) \vee ((b = 2) \vee (b = 4)) \vee (c = 2) = 1.$$

Collections of sectional vector-rows can form *sectional Boolean matrices* of two types: conjunctive and disjunctive ones. *Conjunctive matrices* consist of row-conjuncts and are convenient for interpreting as disjunctive normal forms (DNFs) of finite predicates. *Disjunctive matrices* consist of row-disjuncts and are interpreted as conjunctive normal forms (CNFs).

2. Representation of data and knowledge

The main concepts used by solving pattern recognition problems are *world model*, *data* and *knowledge*.

The world model is defined as a set W (called *world* below) of some objects represented by combinations of values of their attributes, which compose the set $X = \{x_1, x_2, \dots, x_n\}$. The attributes could be multi-valued, for example, such as the colour, which can be red, dark blue, green, etc., but should receive only one of these values. The world W is regarded as a subset of space M and is presented by the corresponding predicate φ. Usually $|W| \ll |M|$.

It is natural to define the data as any information about individual objects, and the knowledge about world W as a whole [9, 12]. According to this assumption, we shall consider the data presenting information about the existence of some objects with definite combinations of properties (P) and consider the *knowledge* presenting information about the existence of regular relationships between attributes. These relationships prohibit some other combinations of properties (Q) by equations $k_i = 0$, where k_i is a conjunction over the set of attributes X, or by equivalent to them equations $d_i = 1$ called *disjuncts* below (with elementary disjunction $d_i = \bar{k}_i$). In other words, the knowledge is regarded as the information about the non-existence of objects with some definite (now prohibited) combinations of attribute values. In case when these prohibitions are represented by disjuncts they are called *implicative regularities* [8].

Reflecting availability of the mentioned combinations by the predicates P and Q, one can present the data by affirmations $\exists\, w \in W : P(w)$ with the existential quantifier $\exists$ (there exists), and the knowledge by affirmations $\neg\exists\, w \in W : Q(w)$ with its negation $\neg\exists$ (there does not exist). The latter ones could be easily transformed into affirmations $\forall\, w \in W : \neg Q(w)$ with the generality quantifier $\forall$ (for every).

Suppose that the data present a complete description of some objects where for each attribute its value for a considered object is shown. Usually not all objects from some world W could be described in such a way but only a relatively small part of them which forms a random selection F from W: $|F| \ll |W|$. Selection F can be represented by a set of selected points in space M.

The distribution of these points reflects the regularities inherent in the world: every implicative regularity generates some empty, i.e. free of selected points, interval in the space M. The reverse affirmation suggests itself: maybe any empty interval generates the corresponding regularity. But such an affirmation is a hypothesis which could be accepted if only it is plausible enough. The matter is that an empty interval can appear even if there are no regularities, for instance when $W = M$ (everything is possible) and elements of the set F are scattered in the space M quite at random obeying the law of uniform distribution of probabilities. Thus the problem of plausibility evaluation arises which should be solved on the stage of inductive inference, where some regularities are extracted from the data.

3. Inductive inference

A lot of papers are devoted to the problem of knowledge discovery in data bases [1, 3, 6, etc.]. Inductive inference is used for its solution.

In our case it consists in suggesting hypotheses about regularities represented by those disjuncts, which do not contradict the data. However, these hypotheses could be accepted if only they are reliable enough, and this means that at least these disjuncts should correspond to rather big intervals of space M.

Consider some disjunct. It does not contradict the database if the corresponding interval of space M is empty – if it does not intersect with the random selection F from W. Therefore, it is possible to put forward a hypothesis affirming that the whole world W as well does not contain elements of that interval. However, it is necessary to take into account the possibility that the considered interval has appeared empty quite accidentally. The less is the probability of such possibility, the more reasonable would be to accept the hypothesis.

The formula for evaluation of such a probability is rather complicated. But it can be approximated by the mathematical expectation w of the number of empty intervals of the given size, and the less is that value, the more precise is the approximation.

That expectation w was evaluated in [10] for the case of two-valued attributes, as a function of parameters:

m is the number of elements in the random selection F,

n is the number of binary attributes,

k is the rank of the regarded disjunct (the number of variables in it), determining the size of considered intervals.

The following formula was proposed to calculate it:

$$w(m, n, k) = C_n^k 2^k (1 - 2^{-k})^m,$$

where C_n^k is the number of different k-element subsets of an n-element set.

In order to evaluate the indicated probability for the case of many-valued attributes, we shall carry out the following imaginary experiment. Suppose that the selection F is formed during m steps, on each of which one element is selected from space M at random.

Considering a disjunct, we shall count up the probability p that it will be satisfied with an accidentally selected element of space M (this element will not enter the corresponding interval):

$$p = 1 - \prod_{i=1}^{n} (r_i/s_i),$$

where n denotes the number of attributes, s_i – the number of values of attribute x_i, r_i is the number of those of them, which do not enter the disjunct. For example, the probability p for disjunct 00.1000.101 is equal to $1 - 2/2 \cdot 3/4 \cdot 1/3 = 3/4$.

Let's divide all conceivable disjuncts into classes D_i, consisting of disjuncts with equal values of p, number these classes in ascending order p and introduce the following characteristics:

q_i is the number of disjuncts in class D_i,

p_i is the value of parameter p for elements of class D_i.

The expectation w_i of the number of disjuncts from class D_i, which do not contradict the considered random selection, is

$$w_i = q_i p_i^m,$$

and the similar expectation for the union of classes $D_1, D_2, \ldots, D_k$ is

$$w_k^+ = \sum_{i=1}^{k} w_i.$$

Just this value can be used for the quantitative estimation of hypotheses plausibility. Any disjunct not contradicting to the data can be accepted as a regularity only when this value is small enough. In this case it is impossible to explain the emptiness of the corresponding interval by an accident; hence we have to admit that the disjunct represents some regularity reflected in the database.

4. Knowledge base and its simplification

After extracting regularities from a database, a knowledge base is created playing the main role during recognition of new objects of the researched subject area. It is natural to try and present knowledge in the most compact form, which will allow reducing the time of inference, by which the recognition problems are solved.

The knowledge base is created as a disjunctive matrix $\boldsymbol{D}$, representing CNF of some finite predicate. Therefore its compression is performed as minimization of this finite predicate. Minimizing a predicate we obtain its most compact description. Usually that means finding its *shortest* DNF, which contains a minimum number of terms. This task can be formulated as the task of finding a shortest minor cover of a Boolean matrix.

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be some rows of a disjunctive matrix $\boldsymbol{D}$, and $\boldsymbol{p}$ and $\boldsymbol{q}$ – some of its columns. Let's assume, that vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are in ratio $\boldsymbol{a} \geq \boldsymbol{b}$ if this ratio is fulfilled component-wise (for example, $011.0010.101 \geq 010.0010.100$).

The following rules of reduction allow simplifying a disjunctive matrix $\boldsymbol{D}$ by deleting some rows or columns.

Rule 1. If $\boldsymbol{u} \geq \boldsymbol{v}$, row $\boldsymbol{u}$ is deleted.

Rule 2. If row $\boldsymbol{u}$ contains complete (without zeros) domain (section), it is deleted.

Rule 3. If column $\boldsymbol{p}$ is empty (without ones), it is deleted.

Rule 4. If a row exists containing ones only in one domain, all columns of that domain which contain zeros in the given row are deleted.

The enumerated rules form a set of basic equivalence transformations of the disjunctive matrix (not changing the represented predicate). Alongside with the given rules one more transformation can be applied for simplification of matrix $\boldsymbol{D}$. Its use can change the set of solutions, but does not disturb the property of consistency: any consistent matrix remains consistent, any inconsistent – remains inconsistent.

Rule 5. If $\boldsymbol{p} \geq \boldsymbol{q}$ and the columns $\boldsymbol{p}$ and $\boldsymbol{q}$ belong to the same domain, the column $\boldsymbol{q}$ is deleted.

5. Resolution rules

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be some disjuncts, $\boldsymbol{D}$ and $\boldsymbol{C}$ be disjunctive matrices, specifying some CNFs, and $E(\boldsymbol{u})$, $E(\boldsymbol{v})$, $E(\boldsymbol{D})$, $E(\boldsymbol{C})$ be their characteristic sets, i.e. collections of elements of space M, presenting the solutions for $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{D}$ and $\boldsymbol{C}$, accordingly. Besides, let $\bar{\boldsymbol{u}}$ be the vector obtained from $\boldsymbol{u}$ by its component-wise negation, and $\boldsymbol{D} \wedge \boldsymbol{u}$ be the matrix obtained from $\boldsymbol{D}$ by component-wise conjunction of its each row with vector $\boldsymbol{u}$.

Let us say that disjunct $\boldsymbol{v}$ follows from disjunct $\boldsymbol{u}$ (it is its logical conclusion), denoting it as $\boldsymbol{u} \to \boldsymbol{v}$, if and only if $E(\boldsymbol{u}) \subseteq E(\boldsymbol{v})$. Similarly, $\boldsymbol{D} \to \boldsymbol{u}$ if and only if $E(\boldsymbol{D}) \subseteq E(\boldsymbol{u})$, $\boldsymbol{D} \to \boldsymbol{C}$ if and only if $E(\boldsymbol{D}) \subseteq E(\boldsymbol{C})$, etc.

It is easy to show that $\boldsymbol{u} \to \boldsymbol{v}$ if and only if vector $\boldsymbol{v}$ covers vector $\boldsymbol{u}$.

The following problem is formulated in the mode typical for the logic inference theory. A disjunctive matrix $\boldsymbol{D}$ and a disjunct $\boldsymbol{u}$ are given. The question is to find out, whether $\boldsymbol{u}$ follows from $\boldsymbol{D}$.

Affirmation 1

Disjunct $\boldsymbol{u}$ logically follows from disjunctive matrix $\boldsymbol{D}$ if and only if matrix $\boldsymbol{D} \wedge \bar{\boldsymbol{u}}$ is inconsistent.

The procedure of checking CNF for consistency is useful for conversion of a disjunctive matrix to an irredundant form, which could be sometimes a good approximation to the optimum solution.

A disjunctive matrix is called *irredundant* when at deleting of any row or at changing value 1 of some element for 0 it turns to a matrix not equivalent to the initial one. One can make any disjunctive matrix irredundant by applying operations of these two types while it is possible, i.e. while after their execution the matrix remains equivalent to the initial one.

It is obvious that a row can be deleted from matrix $\boldsymbol{D}$ if it is a logical conclusion of the remaining set. And the check of this condition is circumscribed above.

Sometimes a row cannot be deleted, but it is possible to change value 1 of some of its component for 0, having reduced by that the number of 1s in the matrix.

Affirmation 2

Element d_i^{jk} of disjunctive matrix $\boldsymbol{D}$ can change its value 1 for 0 if and only if a disjunct follows from $\boldsymbol{D}$, which can be obtained from row $\boldsymbol{d}_i$ by replacement of domain $\boldsymbol{d}_i^j$ by other one, where $d_i^{jk} = 0$ and the remaining components have value 1.

6. Deductive inference in pattern recognition

Consider now the disjunctive matrix $\boldsymbol{D}$ as a system of regularities, which are obligatory for all elements of a subject area (class), formally identified with some sets of values of attributes, i.e. with elements of the space M. Thus we shall consider every disjunct representing a particular tie between attributes bounding the set of "admittable" objects.

Let us assume that regarding an object from the researched class we receive the information about values of some attributes. It is convenient to define as a *quantum* of such an information the elementary prohibition $x_j \neq k$: the value of attribute x_j is distinct from k. Having received several such quanta, we can present the total information by a sectional Boolean vector $\boldsymbol{r}$, in which the components corresponding to elementary prohibitions, take value 0, the remaining – value 1. This vector sets some elementary conjunction r and is interpreted as the conjunctive equation $r = 1$. Let us call it a *conjunct*.

For example, conjunct

$$\boldsymbol{r} = 1\ 1\ 1\ .\ 0\ 0\ 1\ 1\ .\ 0\ 1$$

is interpreted as equation

$$((b = 3) \vee (b = 4)) \wedge (c = 2) = 1.$$

This means that the considered object cannot have value 1 or 2 of attribute b, and also value 1 of attribute c. In other words, vector $\boldsymbol{r}$ sets an interval where the object is localized, it is known only that the element of space M representing this object is somewhere inside the indicated interval.

A problem of recognition arises in this situation, consisting in further localization of the object by the way of deductive inference [12, 13, 15]. The information contained in matrix $\boldsymbol{D}$ is used for that. It represents a system of disjunctive equations to which the objects of the given class should be submitted [11].

The best solution of this problem could be achieved via simplifying of this system by its "tuning" onto the interval represented by vector $\boldsymbol{r}$. This operation is performed by deleting values 1 in columns of matrix $\boldsymbol{D}$, corresponding to those components of conjunct $\boldsymbol{r}$, which have value 0.

Affirmation 3

A disjunctive matrix $\boldsymbol{D}$ in aggregate with a conjunct $\boldsymbol{r}$ is equivalent to the disjunctive matrix $\boldsymbol{D}^* = \boldsymbol{D} \wedge \boldsymbol{r}$.

The operation of deleting 1s in some columns could be followed by further reducing the disjunctive matrix by means of standard conversions of equivalence.

The "interval" localization of the object is of interest at recognition, when some more components of vector $\boldsymbol{r}$ can change value 1 for 0. Such a localization well corresponds to the traditional formulation of the problem of recognition, when the values of some selected (target) attributes are searched. The process of such localization could be reduced to search of separate elementary prohibitions, when questions of the following type are put forward: whether it follows from matrix $\boldsymbol{D}^*$, that the considered object cannot have value k of attribute x_j?

Obviously, at the positive answer to this question a disjunct follows logically from matrix $\boldsymbol{D}^*$, represented by the sectional Boolean vector $\boldsymbol{s}(j,k)$, in which all components of domain j, except number k, have value 1, and all rest components have value 0.

Affirmation 4

The value of component r^{jk} of vector $\boldsymbol{r}$ can be changed from 1 to 0 if and only if the disjunctive matrix $\boldsymbol{D}^* \wedge \bar{\boldsymbol{s}}(j,k)$ is inconsistent.

Regard an example with variables a, b, c, receiving values accordingly from sets $A = \{1,2,3\}$, $B = \{1,2,3,4\}$, $C = \{1,2\}$. Let

$$\boldsymbol{D} = \begin{bmatrix} 001.0010.00 \\ 000.0011.01 \\ 010.1100.10 \\ 001.0000.01 \end{bmatrix}$$

and suppose it is known that some object of the considered class has value 1 of attribute c. Then

$$\boldsymbol{r} = [111.1111.10]$$

and

$$\boldsymbol{D}^* = \begin{bmatrix} 001.0010.00 \\ 000.0011.00 \\ 010.1100.10 \\ 001.0000.00 \end{bmatrix}.$$

If we are interested in attribute b, it is possible at once to initiate check of its values and to find out, for example, that b cannot have value 1, because

$$s(b,1) = 0\ 0\ 0\ .\ 0\ 1\ 1\ 1\ .\ 0\ 0,$$

and matrix $\boldsymbol{D}^* \wedge \bar{\boldsymbol{s}}(b,1)$ takes value

$$\begin{bmatrix} 001.0000.00 \\ 000.0000.00 \\ 010.1000.10 \\ 001.0000.00 \end{bmatrix}.$$

It is obvious that it is inconsistent, because there is a row containing only zeros.

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