STATICALLY DETERMINATE PLANAR BAR STRUCTURES

Internal forces

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INTERNAL FORCES

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Preface

There are many different types of structures and each structure has a specific function. Some of them are simple, while others are complex. However they must be capable of carrying the loads that they are designed for without collapsing. The main civil engineering structures are buildings, towers or bridges. These structures are very complex to analyze and design. It is important for a structural engineer to recognize the various types of elements that compose a structure and to be able to classify and analyze them. Simple examples of structures and parts of structures can be classified as: beams, columns, frames, trusses or curved members (arches). The engineer should know how loads are carried by structures. That is one of the most important aspects of structural engineering that one needs to study. Current structural analysis are computer based, but the engineer needs to be able to assess the computer-generated results with a simple independent hand computation. The classical hand-computation-based procedures for finding the internal forces of the statically determinate bar structures due to applied loading will be discussed in this textbook.

This textbook is intended for civil engineering students. Within each chapter, the theoretical basis necessary to solve the problems are given (briefly). The computational examples are presented in detail.

I hope that the analysis of the presented examples will help in solving other statically determinate elements of structures and will greatly facilitate the study of statically indeterminate structures in the future.

At the same time, I will be extremely grateful for all the substantive comments regarding this textbook.

Joanna Krętowska

1. Kinematical analysis and static determinacy of planar bar structures

Civil engineering structures are connected to the ground at certain points called supports. When the external loading is applied to the structure, the supports develop reactions which oppose the tendency of the structure to move. The nature and number of reactions depends on the type of support.

The term "structure" refers to a system of members with connected parts used to support a load.

There are three types of planar motion for a rigid body: translation in the x direction, translation in the y direction, and rotation about an axis normal to the x – y plane (φ) (Fig. 1.1.a.). Hence, each body situated in one plane has three degrees of freedom – three independent parameters determining its movement.

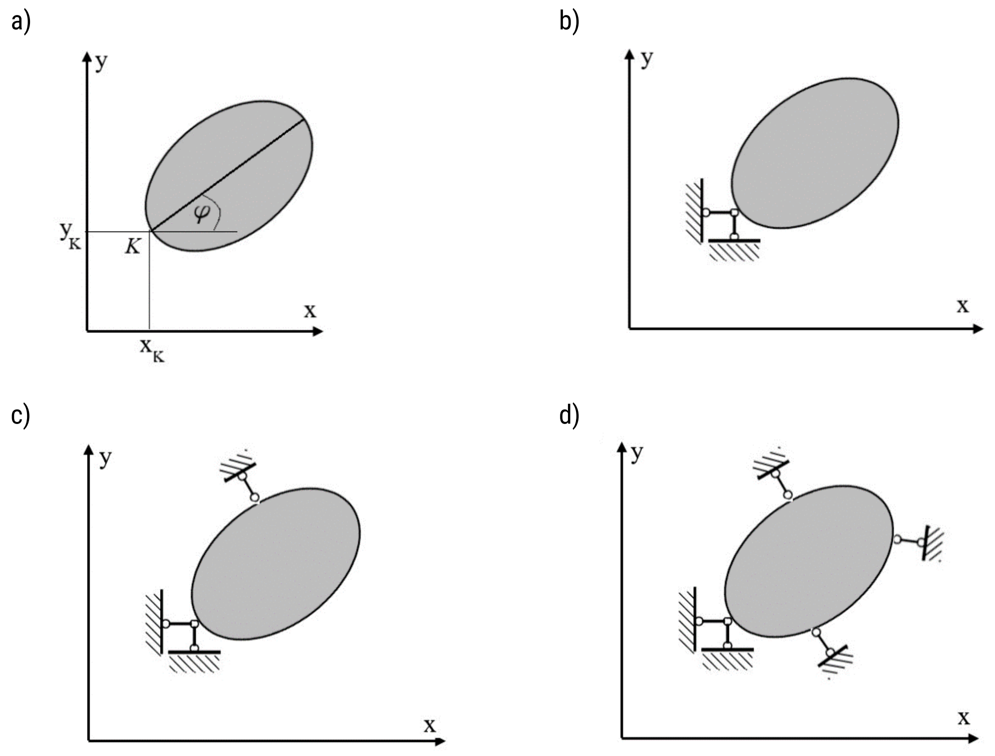


Fig. 1.1. The planar body and the constraints.

A body is said to be stable when body motion is prevented. The possibility of movement can be limited by using three motion constraints. If three specially arranged support links will be put on a planar body, then it will be stable – without any possibility of movement (Fig. 1.1.c.). A structure with an exact number of constraints is called a determinate structure.

If the planar body has less than three constraints, then some movement will be possible. Such a body is called a mechanism. Fig. 1.1.b. presents one degree of freedom mechanism. If more than three support constraints will be put on the planar body, then it is called an indeterminate structure (no possibility of movement – Fig. 1.1.d.).

In the case where there is not one body but several bodies, the degrees of freedom depend on the connections and supports.

1.1. Member connections and supports [1], [2], [3]

Structures are restrained against body motion by supports. When the structure is loaded, reaction forces are developed by the supports.

In this chapter the different structure member connections and supports will be discussed.

Link (constraint) – one-degree-of-freedom support kinematic pin joints, which limit the possibility of movement in one direction (limit one degree of freedom).

Line element (member) – an element whose geometry is essentially one-dimensional, i.e., one dimension is large with respect to the other two dimensions (cables, beams, columns, arches).

Disk (planar body) – two dimensional rigid body.

Planar rigid body can be also formed by at system of line elements (members).

Support links:

- roller support – this support carries only shear forces between jointed members. The roller support allows rotation about the support point and motion parallel to the surface of contact but fully restrains motion in the direction perpendicular to the surface. It can be represented by a single link, which limits the possibility of movement in its direction (Fig. 1.2.) and limit one degree of freedom.

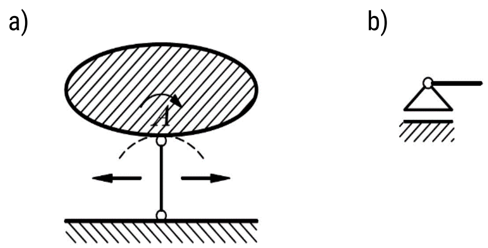


Fig. 1.2.

- hinged (pin) support – this support carries shear and axial forces but not moment force between jointed members. It can be represented by two non-parallel links and limits two degrees of freedom. The hinged support allows rotation about the support point but doesn't allow horizontal and vertical displacements (Fig. 1.3.).

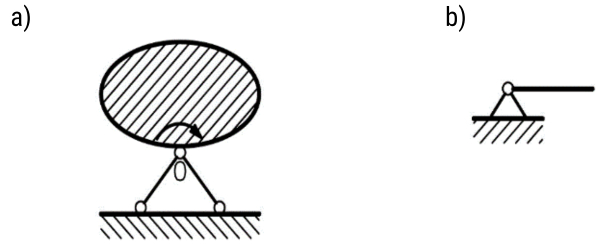


Fig. 1.3.

- support with double parallel links – this support carries moment and shear forces between jointed members. This kind of support doesn't allow vertical displacement of the support point and doesn't allow rotation about the support point. It limits two degrees of freedom (Fig. 1.4.).

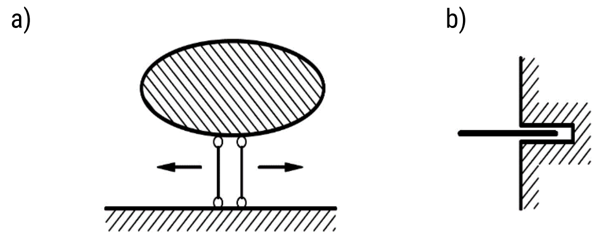


Fig. 1.4.

- fixed support – this support carries moment, shear and axial forces between jointed members. This kind of support is prevented from translating and rotating. It can be represented by three non-parallel links and limits three degrees of freedom (Fig. 1.5.).

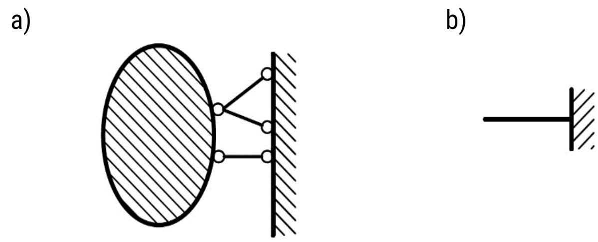


Fig. 1.5.

- hinged (pin) connections (one-degree-of-freedom kinematic pin joints) – this connection carries shear and axial forces but no moment force between jointed members. Hinged connection allows the jointed members to have different rotations but the same displacements. It can be represented by two non-parallel links and limits two degrees of freedom (Fig. 1.6.).

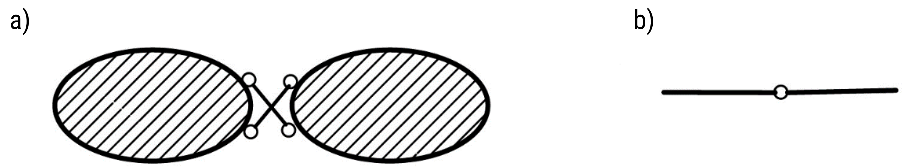


Fig. 1.6.

Hinges that connect m elements can be represented by $2(m - 1)$ links, so it will limit $2(m - 1)$ degrees of freedom.

1.2. Kinematic stability of planar structures [1], [2]

Every two dimensional deformable planar element has three degrees of freedom (two displacements and one rotation). By using different support links, we control these degrees of freedom so the elements cannot move in the limited direction.

In case we have no one planar body but several numbers, the degrees of freedom depends on the body's connection and supports:

$$f = 3t - r - 2s \tag{1.1}$$

where:

f – number of degrees of freedom (mobility),

t – number of planar bodies (disks) (the basic disk – ground (terra) is not counted),

s – number of one-degree-of-freedom kinematic pin joints,

r – number of support links.

The numbers of ***s*** is calculated by the formulae: $s = (m - 1)$, where m is the number of members connected at the pin joint.

If there are one-element-closed-loops:

$$f = 3(t - z) - r - 2s \tag{1.2}$$

where:

z – number of closed loops.

In the example presented in Fig. 1.7.d. and Fig. 1.7.f. we have closed loops composed respectively of 5 elements and of 3 elements (no one-element-closed-loop), so $z = 0$.

In the example presented in Fig. 1.7.b. we have one-element-closed-loop, so z = 1.

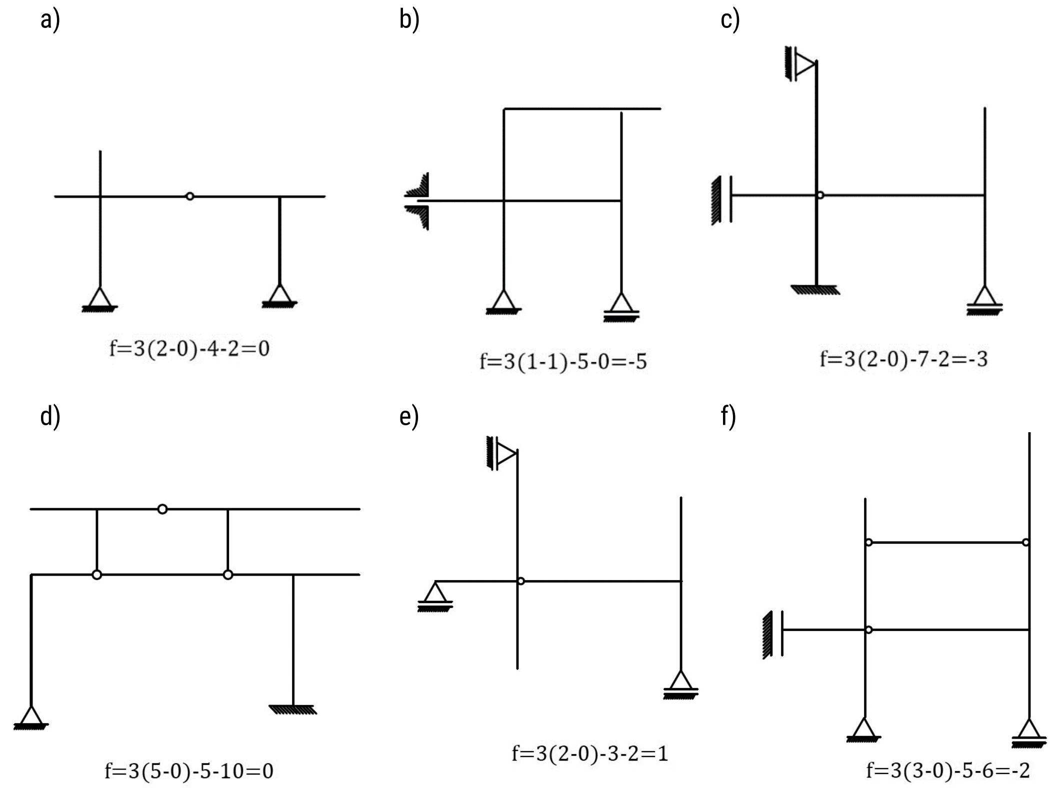


Fig. 1.7.

The number of degrees of freedom **f** may be positive, negative or zero. So we have three different cases for **f**:

f > 0 – the system is mechanism, an unstable system (the structure can't carry any load),

f = 0 – stable system,

f < 0 – stable system, but there are too many support links necessary for the system to be stable.

In general, a planar body with three nonconcurrent coplanar constraints (links) is stable for planar loading (Fig. 1.8.).

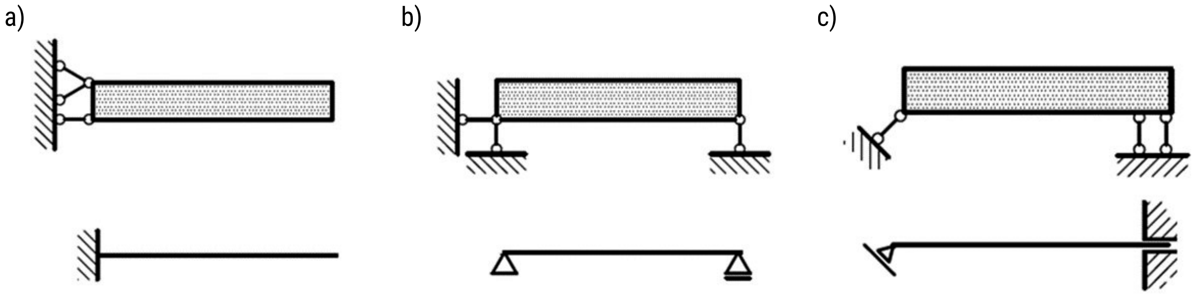


Fig. 1.8.

It is possible for a construction to be determinate and to be mechanism at the same time. This phenomenon is called kinematical instability and such a system – mechanism. So the upper formulas give us information only for the number of links but not for the kinematical stability. That is why a kinematical analysis is needed.

Fig. 1.9. and Fig. 1.10. present these problems. Fig. 1.9. shows kinematical stable bodies. In Fig. 1.9.a. the direction of support links A and B intersect in point O as a rotating point but support link C obstructs this rotation so the structure is stable. Fig. 1.10.a. shows the kinematical unstable body. When the reactive forces are all parallel the translation in the horizontal direction is possible. In Fig. 1.10.b. and Fig. 1.10.c. the direction of the support links intersect at the same point. As a result the rotation is possible and the body is unstable.

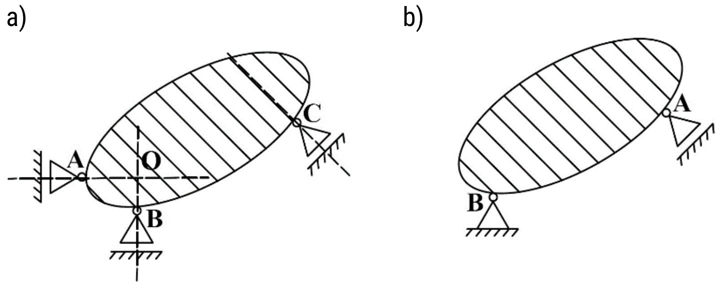


Fig. 1.9.

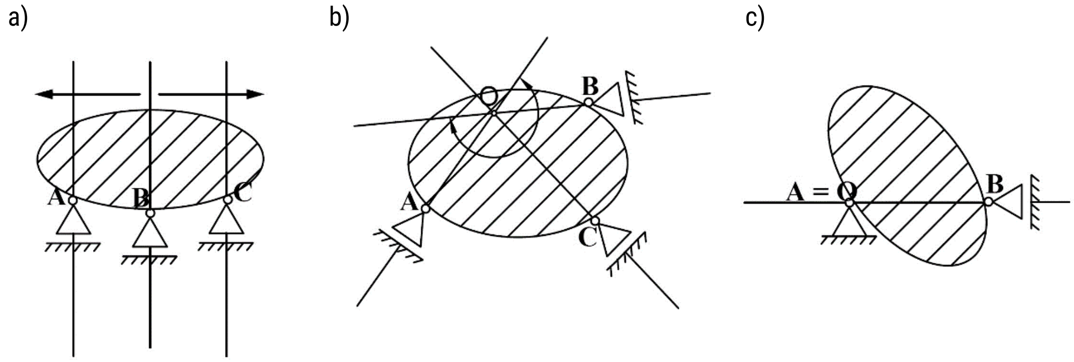


Fig. 1.10.

In the case presented in Fig. 1.11. we have two bodies with the real hinge between them.

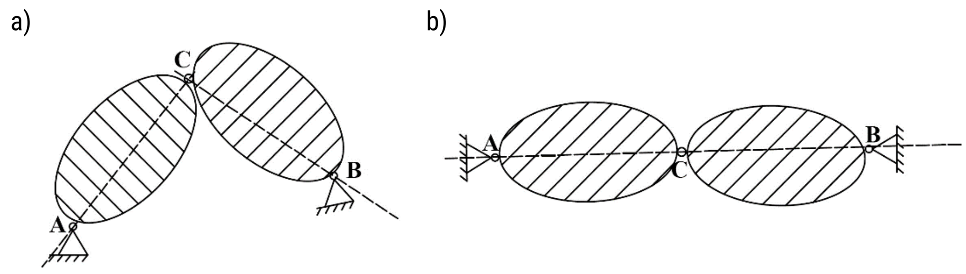


Fig. 1.11.

In case a) the first disk has two support links that intersect in common hinge A and the second one also has two support links that intersect in common hinge B. There is one real hinge C between these two disks. The three hinges A, B and C are not lying at one line, that is why the system is a stable one. In case b) the three hinges are lying at one line and that is the reason the system is unstable.

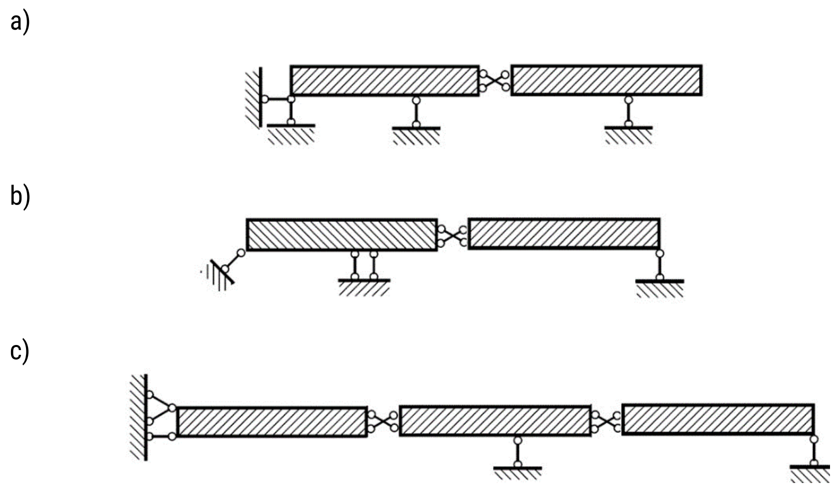


Fig. 1.12.

As we can see above, initial instability occurs when the constraints (links) are insufficient or are not properly arranged to resist applied external forces. In this case, the structure will fail under an infinitesimal load. This condition can be corrected by modifying the location of the supports or by including additional constraints. Fig. 1.12. presents kinematical stable systems.

1.3. Statically determinate planar structures [2]

The aim of structural analysis is to evaluate the external reactions, the deformed shape and internal stresses in the structure. If this can be accomplished by equations of equilibrium, then such structures are known as determinate structures.

However, in many structures it is not possible to determine either reactions or internal stresses or both using equilibrium equations alone. Such structures are known as statically indeterminate structures.

The indeterminacy in a structure may be external, internal or both. A structure is said to be externally indeterminate if the number of support reactions exceeds the number of equilibrium equations.

The degree of static indeterminacy of a planar structure can be expressed by the formula:

$$n = (r + 2s) - 3t \tag{1.3}$$

where:

n – degree of static indeterminacy of planar structures,
t – number of planar bodies (disks) (the disk – ground (terra) is not counted),
s – number of one-degree-of-freedom kinematic pin joints,
$s = (m - 1)$, where m is the number of members connected at the pin joint,
r – number of support links.

The degree of static indeterminacy n may be positive, negative or zero. So we have three different cases for n:

- $n = 0$ – determinate structure – it is possible to analyze the structure by using only equilibrium conditions,
- $n < 0$ – mechanism – in the case of mechanism we don't have a structure carrying any load, the structure is not stable,
- $n > 0$ – indeterminate structure – it is not possible to analyze the structure by using only equilibrium conditions, additional equations are needed.

If there are one-element-closed-loops the degree of static indeterminacy can be calculated by using the formula:

$$n = (r + 2s) - 3(t - z) \tag{1.4}$$

where z – number of one-element-closed-loops.

Computational example

Calculate the degree of static indeterminacy of planar structures presented in Fig. 1.13. degree of static indeterminacy $n = (r + 2s) - 3(t - z)$

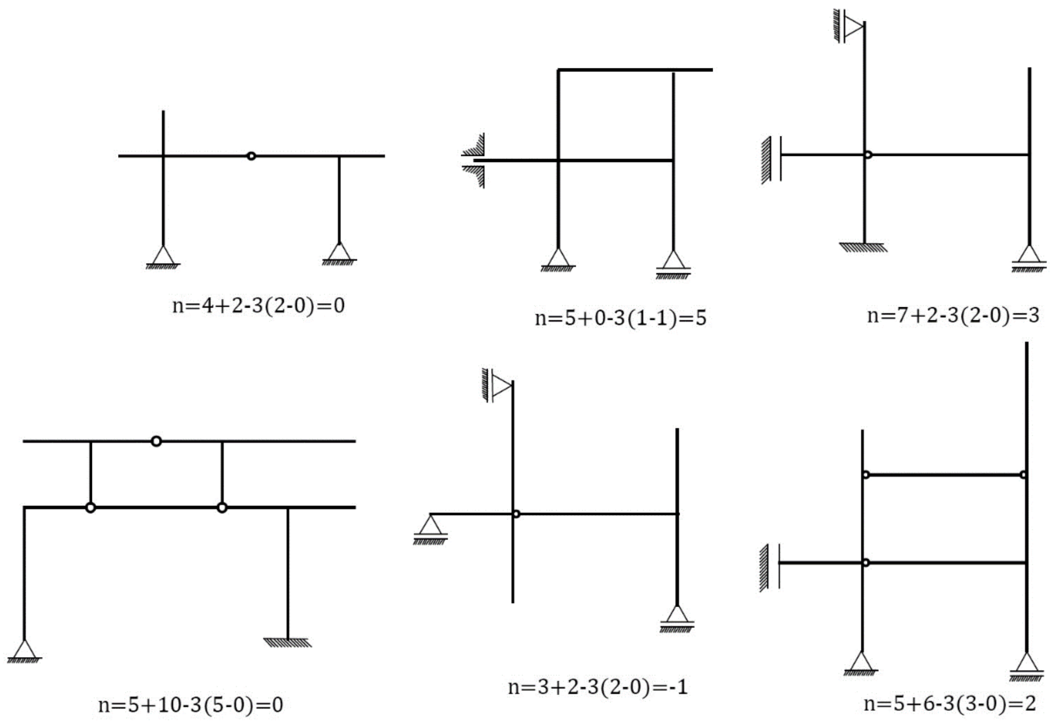


Fig. 1.13.

2. Internal forces in statically determinate planar bar structures

There are many different types of structures and each of them has a specific function. Some structures are simple, while others are complex.

In this chapter, we will consider the procedure of analysis of elementary (statically determinate) structures like simply supported beams, cantilever beams, compound beams, simply supported frames, dyad (three-hinged frame), compound frames, trusses and arches.

When a structure is subjected to an external loading, it responds by developing internal forces which lead to internal stresses. The stresses generate strains, resulting in displacements. To design a structural or mechanical member it is necessary to know the forces acting within the member in order to be sure the material can resist the loading. Internal forces can be determined by using the method of sections [4].

2.1. Internal forces components [1], [3], [4]

The internal forces in a section of a body are those forces which hold together two parts of a given body separated by the section (Fig. 2.1.1.). Both parts of the body remain in equilibrium. It follows that internal forces which exist at a section are equivalent to all external forces acting on the particular part of the body.

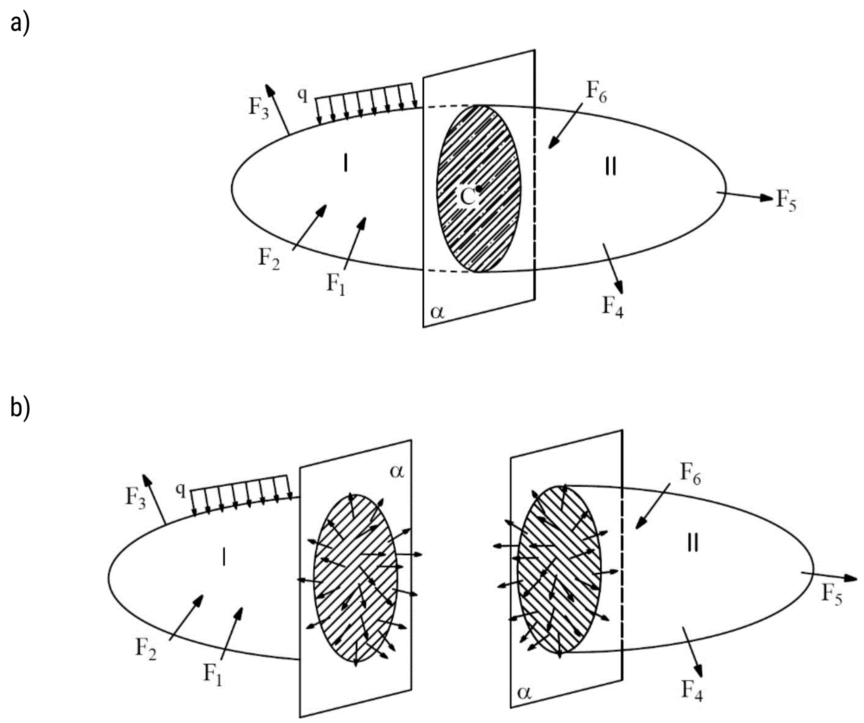


Fig. 2.1.1.

All internal forces in the section can be replaced by a force-couple system $\vec{R}$ and $\vec{M}_C$, in the centroid C of the section α (Fig. 2.1.2.). The resultant vector $\vec{R}$ consists of the axial force $\vec{N}$ (its line of action is perpendicular to the plane α) and shearing force $\vec{V}$ in the plane α ($\vec{V}$ has two components $\vec{V}_y$ and $\vec{V}_z$). Accordingly, $\vec{M}_C$ consists of two components, the first of which is referred to as the torque $\vec{M}_x$ (its line of action is perpendicular to the plane α) and the second is called the bending moment $\vec{M}$ in the plane α (bending moment $\vec{M}$ also has two components $\vec{M}_y$ and $\vec{M}_z$ – Fig. 2.1.3.).

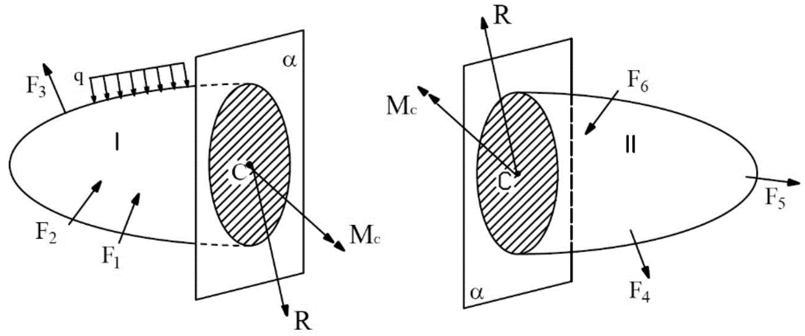


Fig. 2.1.2.

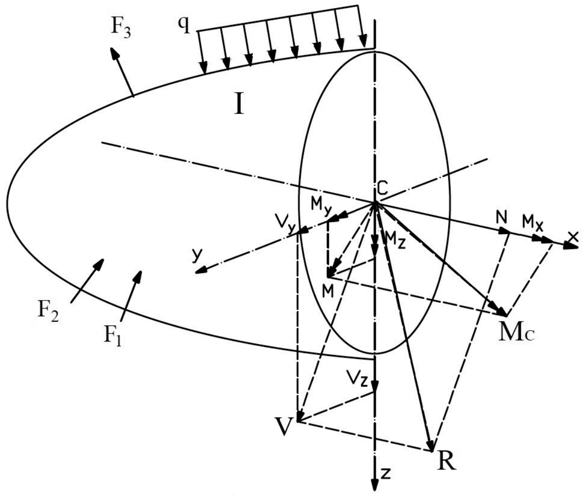


Fig. 2.1.3.

Fig. 2.1.3. shows six internal forces components in three dimensional coordinate system *xyz*, where:

N – normal (axial) force,
V_y – shear (transversal) force in *y* direction,
V_z – shear (transversal) force in *z* direction,
M_x – torque (twisting moment about *x* axis),
M_y – bending moment about *y* axis,
M_z – bending moment about *z* axis.

For three dimensional force systems we can write six equations of equilibrium for the left (or right) part of the body. Solving them we can determine six internal forces components.

2.2. Internal forces in statically determinate planar bar structures [2], [4]

Fig. 2.2.1. presents the simply supported beam under planar load. The support reactions can be calculated using three equations of equilibrium.

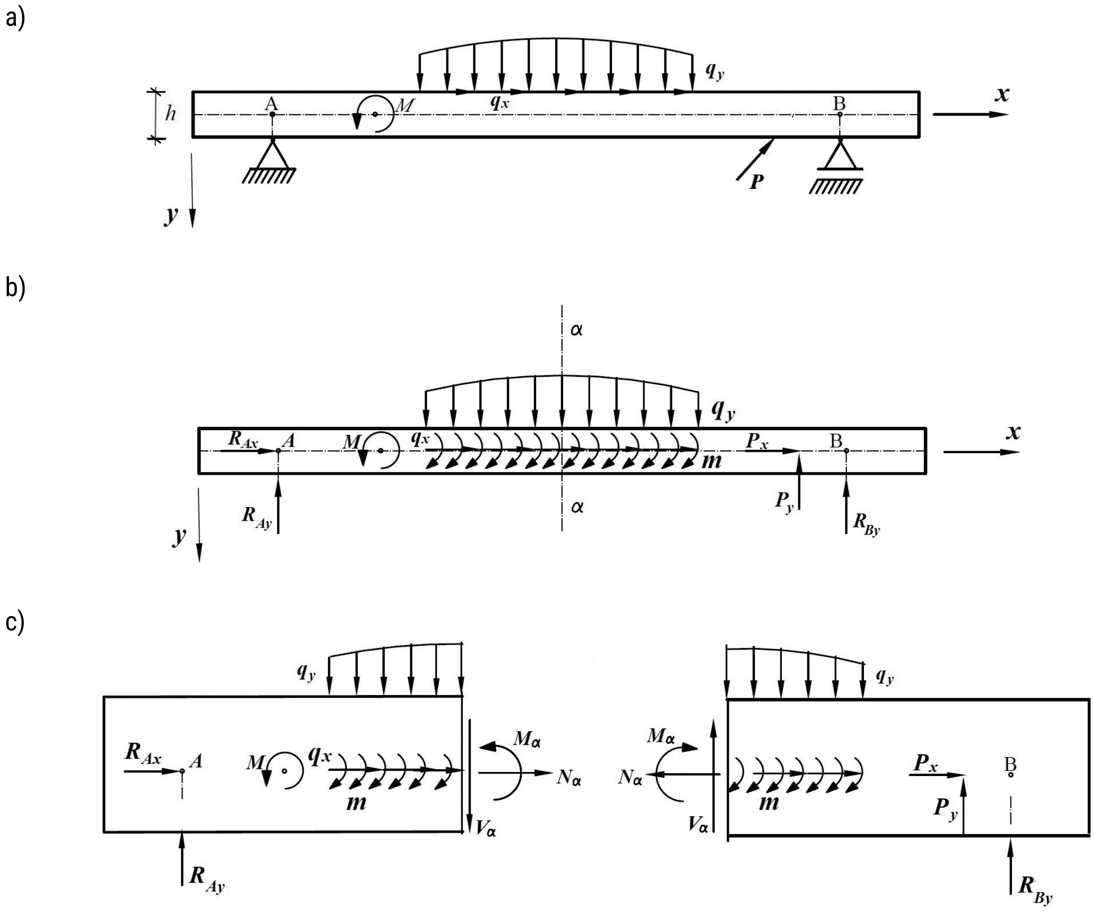


Fig. 2.2.1.

If we cut the body of the beam (Fig. 2.2.1.b. and Fig. 2.2.1.c.) in the section α we will have three internal forces: normal (axial) force N_α, shear (transversal) force V_α and bending moment M_α in the section α. The normal force is said to be positive if it creates tension, a positive shear force will cause the beam segment on which it acts to rotate clockwise, and a positive bending moment will tend to bend the segment on which it acts in a concave upward manner. The internal forces positive positions are shown in Fig. 2.2.1.c.

Actually, if we know the support reactions and loads we just need to compose the three equilibrium equations for the left (or right) part of the beam and we will be able to find the magnitudes of the internal forces. Hence, we have to reduce external loading to the beam axis x (Fig. 2.2.1.b.), so there is located additional distributed moment $m = q_x \frac{h}{2}$.

2.2.1. Relationships between loads, shear and moment. Differential equations of equilibrium [2], [4]

Consider the beam shown in Fig. 2.2.1.1.a. and the differential element (Fig. 2.2.1.1.b.). We use the same sign convention for V_α, N_α and M_α as defined in Chapter 2.2. We take the positive sense of the distributed loading to be "downward" since these loadings are generally associated with gravity.

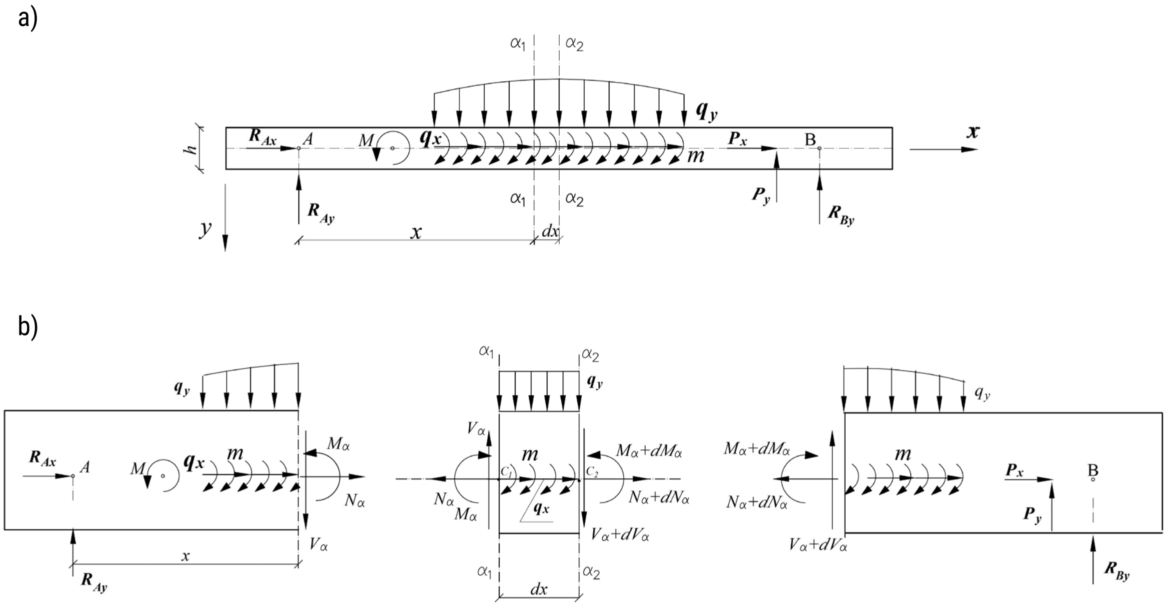


Fig. 2.2.1.1. a) Beam with arbitrary distributed loading, b) Differential beam element [4].

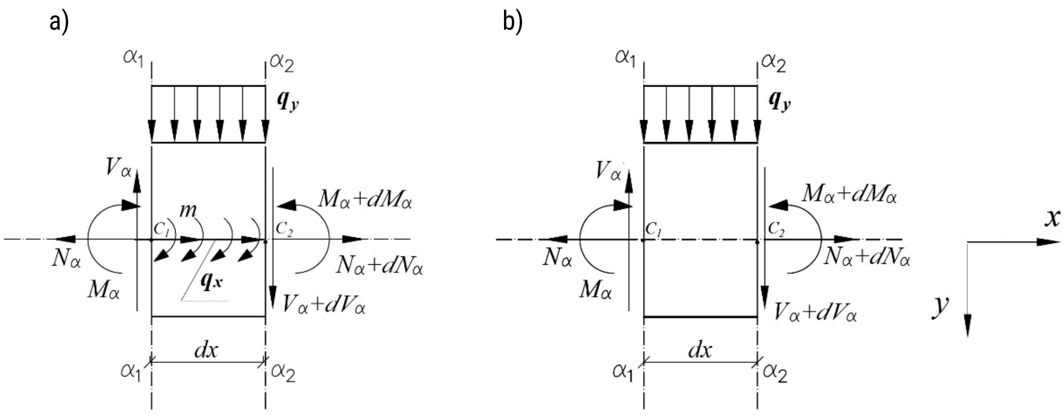


Fig. 2.2.1.2. Differential beam element.

Considering V_α, N_α and M_α to be functions of x, expanding these variables in terms of their differentials, and retaining up to first order terms we have the forces shown in Fig. 2.2.1.2.a. One can write three equations of equilibrium for differential beam elements:

$$\sum F_{iy} = 0; \qquad - V_\alpha + q_y dx + (V_\alpha + dV_\alpha) = 0 \tag{2.1}$$

$$\sum F_{ix} = 0; \qquad -N_\alpha + q_x dx + (N_\alpha + dN_\alpha) = 0 \tag{2.2}$$

$$\sum M_{(C_2)} = 0; \qquad -M_\alpha - V_\alpha dx - m dx + q_y dx \frac{dx}{2} + (M_\alpha + dM_\alpha) = 0 \tag{2.3}$$

Hence, we have:

$$\frac{dV_\alpha}{dx} = -q_y \tag{2.4}$$

$$\frac{dN_\alpha}{dx} = -q_x \tag{2.5}$$

$$\frac{dM_\alpha}{dx} = V_\alpha + m \tag{2.6}$$

In common used bar elements $q_x = 0$ i $m = 0$ (Fig. 2.2.1.2.b.), so in this case the differential equations of equilibrium will have the form:

$$\frac{dT_\alpha}{dx} = -q_y \tag{2.7}$$

$$\frac{dM_\alpha}{dx} = V_\alpha \tag{2.8}$$

$$\frac{d^2M_\alpha}{dx^2} = \frac{dV_\alpha}{dx} = -q_y \tag{2.9}$$

As we can see above, there are connections between the distributed loads, shear and moment functions.

The first equation states that the slope of the shear diagram at a point is equal to the intensity of the distributed load at the point. Likewise, the second equation states that the slope of the moment diagram is equal to the shear at the point.

These two relations are very useful for checking the consistency of the shear and moment diagrams. One can reason about the shape of these diagrams using only information about the loading on a segment of the beam. For example, if shear is constant, moment varies linearly; and if shear varies linearly, moment varies quadratically.

The integral forms can be useful if we want to either compute values at discrete points or determine analytical solutions.

Another useful property that can be established from integral forms relates to the maximum values of the moment. We know from calculus that extreme values of a continuous function are located at points where the first derivative is zero. Applying this theorem to the moment function, $M(x)$, the location x, of an extreme value (either maximum or minimum) of moment can be found by solving equation:

$$\frac{dM_\alpha}{dx} = V_\alpha = 0 \tag{2.10}$$

As we see, the extreme values of moment occur at points where the shear force is zero. We can do the shear diagrams from the applied loading and find the points of zero shear. If we are interested only in peak values of moment, we can calculate its value from the equilibrium conditions.

2.3. Internal forces diagrams – beams [1], [4]

2.3.1. Simple beams

Beams are used extensively in structures (for example in flooring systems of buildings or in bridges). Their longitudinal dimension is large in comparison to their cross-sectional dimensions so they belong to the line (bar) element category. Beams are loaded primarily normal to the longitudinal direction, and carry the loading by bending action. The first step in the analysis of a statically determinate beam is the determination of the reactions. Given the reactions, one can establish the internal forces using equilibrium-based procedures.

- **concentrated P force loading:**

  a) *cantilever beam* (Fig. 2.3.1.)

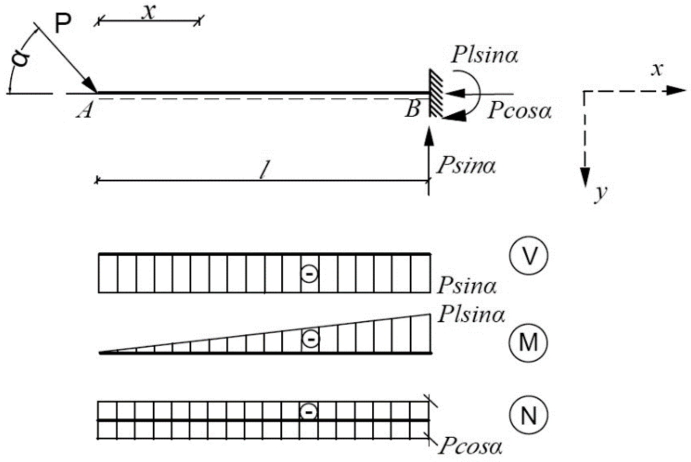


Fig. 2.3.1.

Assume that the beam is cut at point C a distance of x from the left free end and the portion of the beam to the right of C be removed (Fig. 2.3.2.). The portion removed must then be replaced by vertical shearing force V(x) and horizontal normal force N(x) together with a couple M(x) to hold the left portion of the bar in equilibrium under the action of force P. In this case, it is not necessary to determine the support reactions because the left side of the beam is free of supports.

Considering V, N and M are functions of x one can write the three equations of equilibrium for the left part of the beam and find the values of the internal forces.

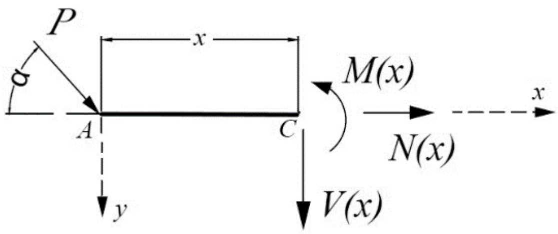


Fig. 2.3.2.

Segment A–B: $0 \leq x < l$

$\sum F_{iy} = 0;$ $\quad V(x) + Psin\alpha = 0 \quad \rightarrow \quad V(x) = -Psin\alpha$

$\sum F_{ix} = 0;$ $\quad N(x) + Pcos\alpha = 0 \quad \rightarrow \quad N(x) = -Pcos\alpha$

$\sum M_{(C)} = 0;$ $\quad M(x) + Psin\alpha \cdot x = 0 \quad \rightarrow \quad M(x) = -Pxsin\alpha$

for $x = 0$ $\quad M(x) = 0$ and for $x = l$ $\quad M(x) = -Plsin\alpha$

Internal forces diagrams are presented in Fig. 2.3.1.

b) *simply supported beam* (Fig. 2.3.3.)

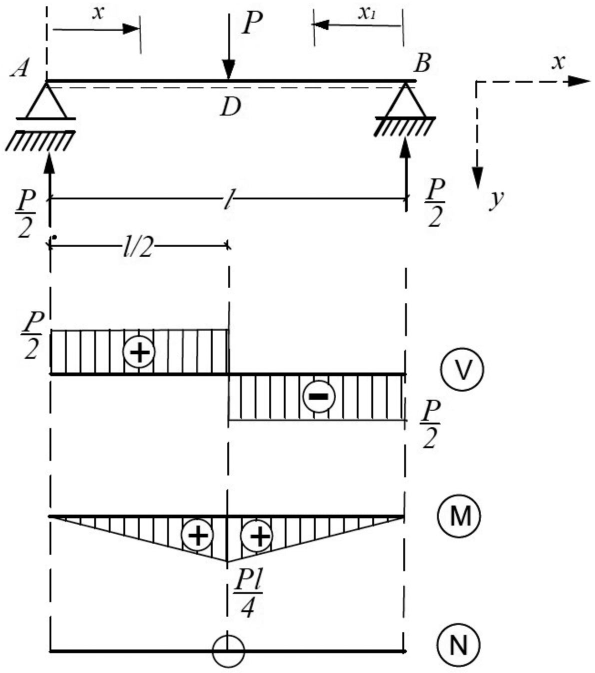


Fig. 2.3.3.

Support reactions are presented in Fig. 2.3.3.

Considering V, N and M are functions of x one has to do the "cut" twice (A–D and B–D segment) and write the three equations of equilibrium for every beam section (separate left or right part) and then calculate the values of the internal forces (Fig. 2.3.4.).

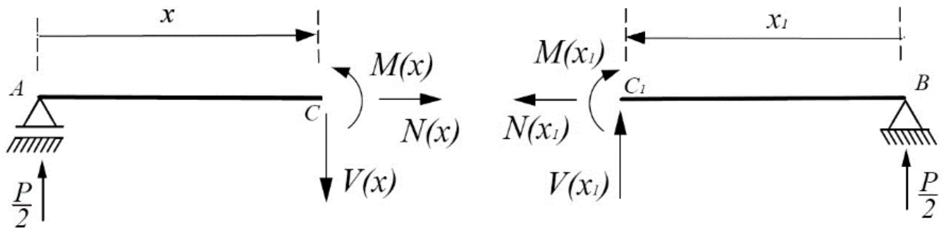


Fig. 2.3.4.

Segment A–D: $\quad 0 \le x < \frac{l}{2}$

$$\sum F_{iy} = 0; \qquad V(x) - \frac{P}{2} = 0 \qquad \rightarrow \qquad V(x) = \frac{P}{2}$$

$$\sum F_{ix} = 0; \qquad N(x) = 0$$

$$\sum M_{(C)} = 0; \qquad M(x) - \frac{P}{2} \cdot x = 0 \qquad \rightarrow \qquad M(x) = \frac{P}{2}x$$

for $x = 0 \quad M(x) = 0$ and for $x = \frac{l}{2} \quad M(x) = \frac{Pl}{4}$

Segment B–D: $0 \le x_1 < \frac{l}{2}$

$\sum F_{iy} = 0;$ $\quad -V(x_1) - \frac{P}{2} = 0 \quad \rightarrow \quad V(x_1) = -\frac{P}{2}$

$\sum F_{ix} = 0;$ $\quad N(x_1) = 0$

$\sum M_{(C_1)} = 0;$ $\quad -M(x_1) + \frac{P}{2} \cdot x_1 = 0 \quad \rightarrow \quad M(x_1) = \frac{P}{2} x_1$

for $x_1 = 0$ $\quad M(x_1) = 0$ and for $x_1 = \frac{l}{2}$ $\quad M(x_1) = \frac{Pl}{4}$

Internal forces diagrams are shown in Fig. 2.3.3.

The normal force is zero because there isn't any horizontal load.

- **concentrated M moment loading:**

a) *cantilever beam* (Fig. 2.3.5.)

It is not necessary to determine the support reactions because the left side of the beam is free of supports. We can cut and separate the left-hand part of the member to determine the internal forces (Fig. 2.3.6.).

Considering that V, N and M are functions of *x,* one can write the three equations for the left part of the beam and find the values of the internal forces.

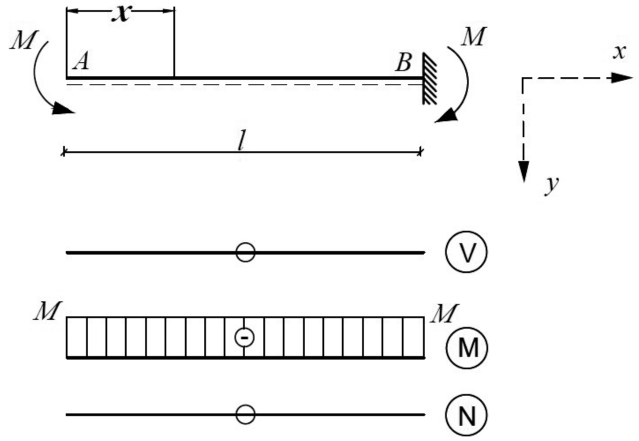


Fig. 2.3.5.

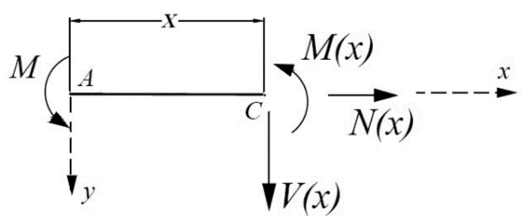


Fig. 2.3.6.

Segment A–B: $0 \le x < l$

$\sum F_{iy} = 0;$ $\quad V(x) = 0$

$\sum F_{ix} = 0;$ $\quad N(x) = 0$

$\sum M_{(C)} = 0;$ $\quad M(x) + M = 0 \quad \rightarrow \quad M(x) = -M$

Internal forces diagrams are shown in Fig. 2.3.5.

b) *simply supported beam* (Fig. 2.3.7.)

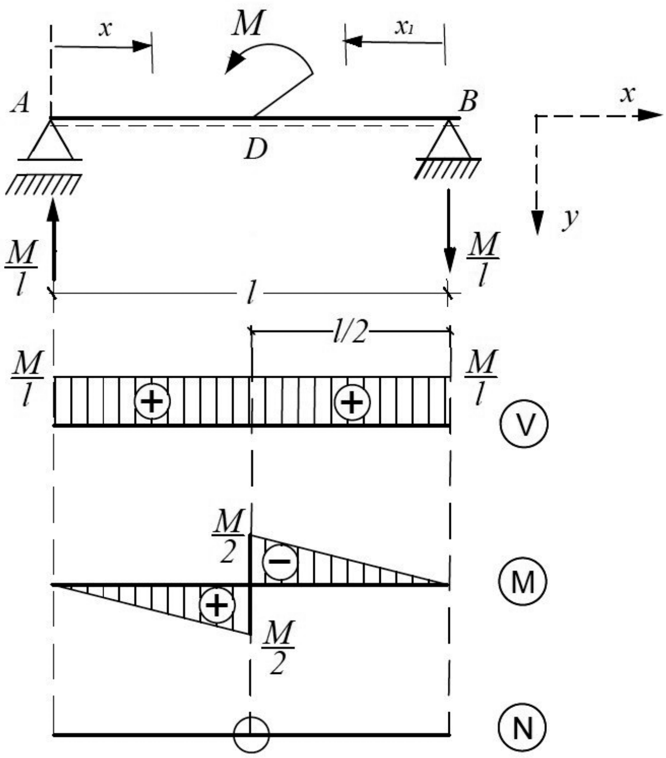


Fig. 2.3.7.

Support reactions are presented in Fig. 2.3.7.

Considering V, N and M are functions of x one has to do the “cut” twice (A–D and B–D) and write the three equations of equilibrium for every beam section (separate left or right part) and then calculate the values of the internal forces (Fig. 2.3.8.).

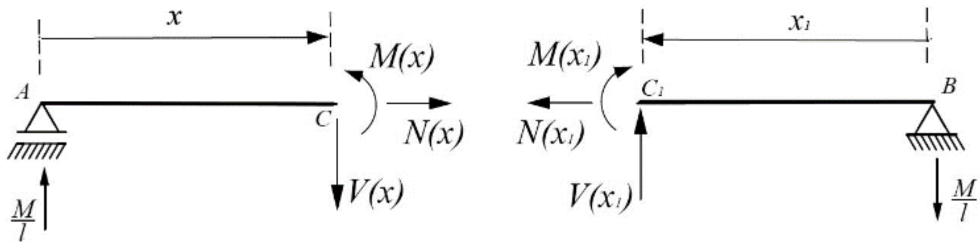


Fig. 2.3.8.

Segment A–D: $0 \leq x < \frac{l}{2}$

$\sum F_{iy} = 0;$ $V(x) - \frac{M}{l} = 0 \rightarrow V(x) = \frac{M}{l}$

$\sum F_{ix} = 0;$ $N(x) = 0$

$\sum M_{(C)} = 0;$ $M(x) - \frac{M}{l} \cdot x = 0 \rightarrow M(x) = \frac{M}{l} x$

for $x = 0$ $M(x) = 0$ and for $x = \frac{l}{2}$ $M(x) = \frac{M}{2}$

Segment B–D: $0 \leq x_1 < \frac{l}{2}$

$\sum F_{iy} = 0;$ $\quad - V(x_1) + \frac{M}{l} = 0 \quad \rightarrow \quad V(x_1) = \frac{M}{l}$

$\sum F_{ix} = 0;$ $\quad N(x_1) = 0$

$\sum M_{(C_1)} = 0;$ $\quad -M(x_1) - \frac{M}{l} \cdot x_1 = 0 \quad \rightarrow \quad M(x_1) = -\frac{M}{l} x_1$

for $x_1 = 0$ $M(x_1) = 0$ and for $x_1 = \frac{l}{2}$ $M(x_1) = -\frac{M}{2}$

Internal forces diagrams are shown in Fig. 2.3.7.

- **uniformly distributed loading q:**

a) *cantilever beam* (Fig. 2.3.9.)

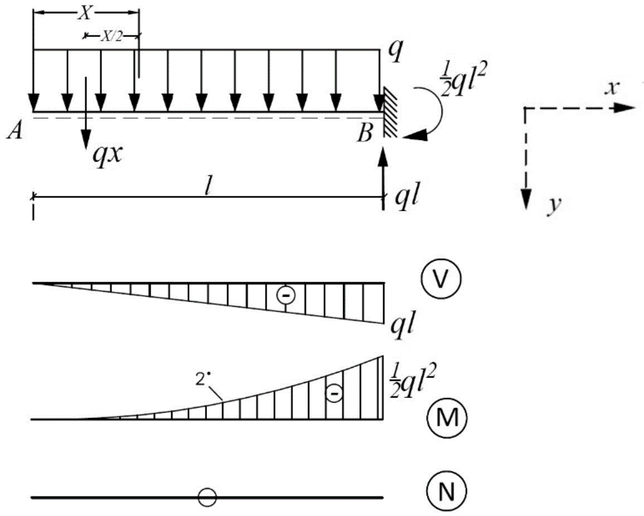


Fig. 2.3.9.

Assume that the beam is cut at point C a distance of x from the left free end and the portion of the beam to the right of C be removed (Fig. 2.3.10.). The portion removed must then be replaced by vertical shearing force V(x) and horizontal normal force N(x) together with a couple M(x) to hold the left portion of the bar in equilibrium under uniformly distributed loading q. In this case, it is not necessary to determine the support reactions because the left side of the beam is free of supports.

Considering V, N and M are functions of x one can write the three equations for the left part of the beam and find the values of the internal forces.

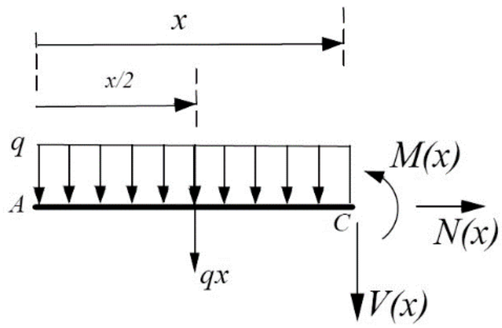


Fig. 2.3.10.

Segment A–B: $\quad 0 \le x < l$

$\sum F_{iy} = 0; \qquad V(x) + qx = 0 \quad \rightarrow \quad V(x) = -qx$

for $x = 0 \quad V(x) = 0; \quad x = l \quad V(x) = -ql$

$\sum F_{ix} = 0; \qquad N(x) = 0$

$\sum M_{(C)} = 0; \qquad M(x) + qx \cdot \frac{x}{2} = 0 \quad \rightarrow \quad M(x) = -\frac{qx^2}{2}$

for $x = 0 \quad M(x) = 0; \quad x = \frac{l}{2} \quad M(x) = -\frac{ql^2}{8}; \quad x = l \quad M(x) = -\frac{ql^2}{2}$

If the uniformly distributed load loads the cantilever beam the internal moment diagram is a parabolic function. Three values are needed for plotting the diagram. The first one can be the free end of the beam, the second one is at the middle of the beam and the last one is at the support.

The shear force diagram is linear and it is sufficient to determine its value at two points – at the free end and at the supported end of the beam.

Internal forces diagrams are shown in Fig. 2.3.9.

b) *simply supported beam* (Fig. 2.3.11.)

Support reactions are presented in Fig. 2.3.11.

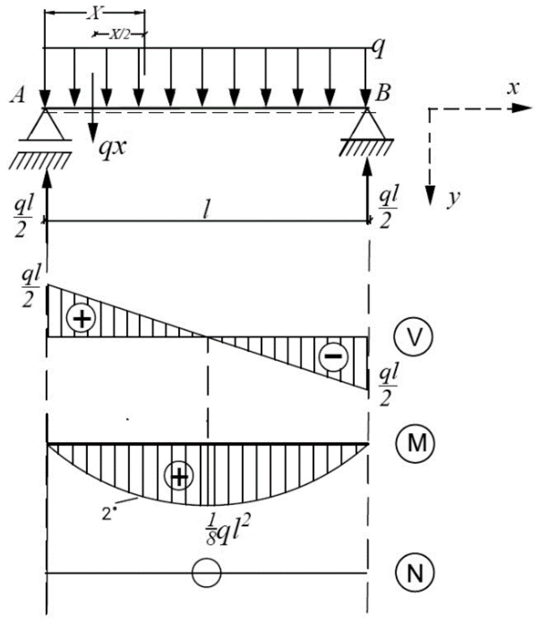


Fig. 2.3.11.

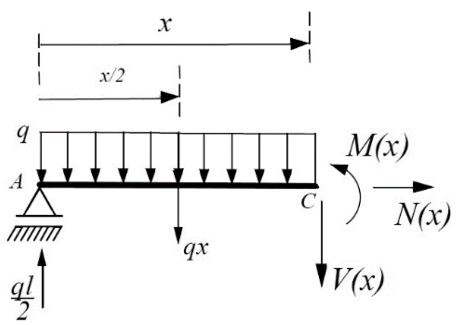


Fig. 2.3.12.

Internal forces equations (Fig. 2.3.12.) have the form:

<u>Segment A–B:</u> $0 \le x < l$

$\sum F_{iy} = 0 \qquad -\frac{ql}{2} + qx + V(x) = 0 \qquad \rightarrow \qquad V(x) = \frac{ql}{2} - qx$

for $x = 0 \quad V(x) = \frac{ql}{2}; \quad x = l \quad V(x) = -\frac{ql}{2}$

$V(x) = 0 \quad \rightarrow \quad \frac{ql}{2} - qx = 0 \quad \rightarrow \quad x = \frac{l}{2}$

$\sum F_{ix} = 0; \qquad N(x) = 0$

$\sum M_{(C)} = 0; \qquad M(x) + qx \cdot \frac{x}{2} - \frac{ql}{2}x = 0 \qquad \rightarrow \qquad M(x) = \frac{ql}{2}x - \frac{qx^2}{2}$

for $x = 0 \quad M(x) = 0; \quad x = l \quad M(x) = 0; \quad x = \frac{l}{2} \quad M(x) = \frac{ql}{2} \cdot \frac{l}{2} - \frac{q\left(\frac{l}{2}\right)^2}{2} = \frac{ql^2}{8}$

Internal forces diagrams are shown in Fig. 2.3.11.

- **triangular distributed loading:**

a) *cantilever beam* (Fig. 2.3.13.)

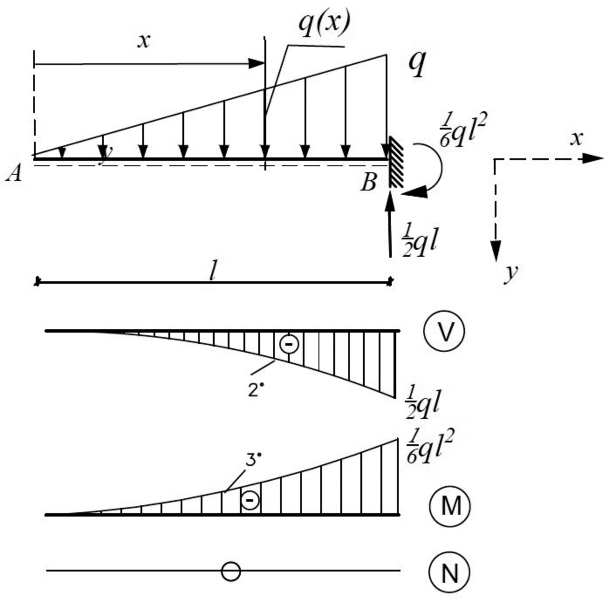


Fig. 2.3.13.

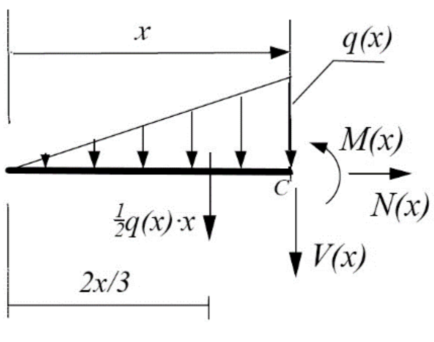


Fig. 2.3.14.

Internal forces equations (Fig. 2.3.14.) have the form:

<u>Segment A–B</u>: $0 \le x < l$

$\sum F_{iy} = 0;$ $\quad V(x) + \frac{1}{2}q(x)x = 0 \quad \rightarrow \quad V(x) = -\frac{1}{2}q(x)x$

We have

$$\frac{q}{l} = \frac{q(x)}{x} \quad \rightarrow \quad q(x) = \frac{qx}{l}$$

thus,

$$V(x) = -\frac{1}{2}q(x)x = -\frac{1}{2}\frac{qx}{l}x = -\frac{1}{2}\frac{qx^2}{l}$$

for $x = 0 \quad V(x) = 0; \quad x = \frac{l}{2} \quad V(x) = -\frac{1}{8}ql; \quad x = l \quad V(x) = -\frac{1}{2}ql;$

$\sum F_{ix} = 0;$ $\quad N(x) = 0$

$\sum M_{(C)} = 0;$ $\quad M(x) + \frac{1}{2}q(x)x \cdot \frac{x}{3} = 0 \quad \rightarrow \quad M(x) = -\frac{1}{2}\frac{qx}{l}x \cdot \frac{x}{3} = -\frac{qx^3}{6l}$

for $x = 0 \quad M(x) = 0; \quad x = \frac{l}{2} \quad M(x) = -\frac{ql^2}{48}; \quad x = l \quad M(x) = -\frac{ql^2}{6}$

Internal forces diagrams are shown in Fig. 2.3.13.

b) *simply supported beam* (Fig. 2.3.15.)

Support reactions are presented in Fig. 2.3.15.

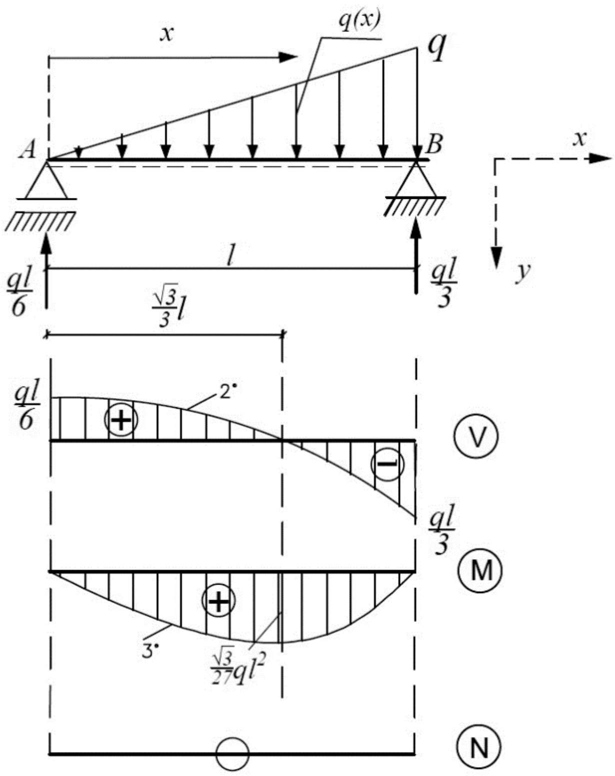


Fig. 2.3.15.

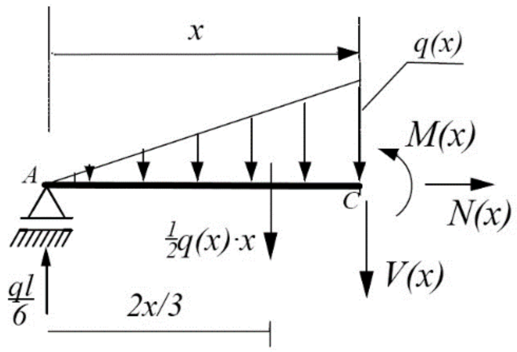


Fig. 2.3.16.

Internal forces equations (Fig. 2.3.16.) have the form:

<u>Segment A–B:</u> $0 \leq x < l$

$\sum F_{iy} = 0;$ $\quad -\frac{ql}{6} + \frac{1}{2}q(x)x + V(x) = 0 \quad \rightarrow \quad V(x) = \frac{ql}{6} - \frac{1}{2}q(x)x$

$\frac{q}{l} = \frac{q(x)}{x} \quad \rightarrow \quad q(x) = \frac{qx}{l}$

thus

$$V(x) = \frac{ql}{6} - \frac{1}{2}q(x)x = \frac{ql}{6} - \frac{1}{2}\frac{qx}{l}x = \frac{ql}{6} - \frac{1}{2}\frac{qx^2}{l}$$

for $x = 0 \quad V(x) = \frac{ql}{6};$

$x = \frac{l}{2} \quad V(x) = \frac{1}{24}ql;$

$x = l \quad V(x) = -\frac{1}{3}ql;$

$V(x) = 0 \quad \rightarrow \quad \frac{ql}{6} - \frac{1}{2}\frac{qx^2}{l} = 0 \quad \rightarrow \quad x = \frac{\sqrt{3}l}{3}$

$\sum F_{ix} = 0;$ $\quad N(x) = 0$

$\sum M_{(C)} = 0;$ $\quad M(x) - \frac{ql}{6}x + \frac{1}{2}q(x)x \cdot \frac{x}{3} = 0 \quad \rightarrow \quad M(x) = \frac{ql}{6}x - \frac{1}{2}\frac{qx}{l}x \cdot \frac{x}{3} = \frac{ql}{6}x - \frac{qx^3}{6l}$

for $x = 0 \quad M(x) = 0;$

$x = l \quad M(x) = 0;$

$x = \frac{\sqrt{3}l}{3} \quad M(x) = \frac{\sqrt{3}ql^2}{27}$

Internal forces diagrams are shown in Fig. 2.3.15.

- **uniformly distributed moment *m*:**
  a) *cantilever beam* (Fig. 2.3.17.)

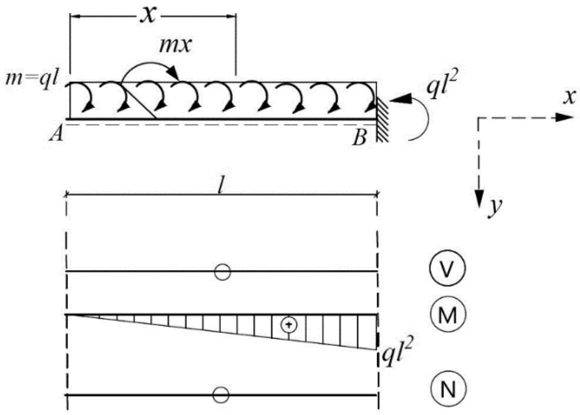


Fig. 2.3.17.

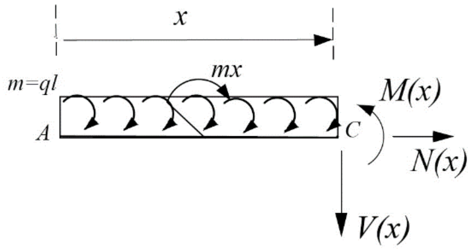


Fig. 2.3.18.

Internal forces equations (Fig. 2.3.18.) have the form:

<u>Segment A–B</u>: $0 \le x < l$

$\sum F_{iy} = 0;$ $V(x) = 0$

$\sum F_{ix} = 0;$ $N(x) = 0$

$\sum M_{(C)} = 0;$ $M(x) - mx = 0 \quad \rightarrow \quad M(x) = mx = qlx$

for $x = 0 \quad M(x) = 0; \quad x = l \quad M(x) = ql^2$

Internal forces diagrams are shown in Fig. 2.3.17.

b) *simply supported beam* (Fig. 2.3.19.)

Support reactions are presented in Fig. 2.3.19.

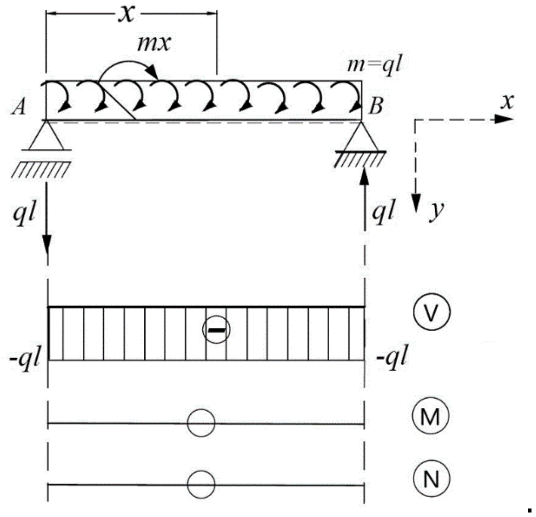


Fig. 2.3.19.

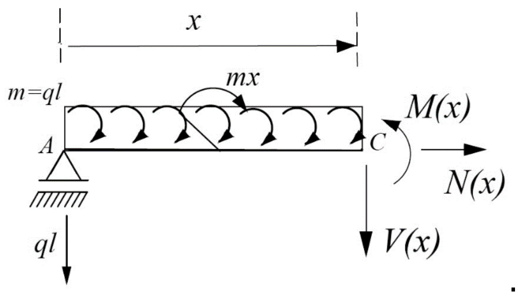


Fig. 2.3.20.

Internal forces equations (Fig. 2.3.20.) have the form:

<u>Segment A–B</u>: $0 \leq x < l$

$\sum F_{iy} = 0;$ $-ql - V(x) = 0$ $\rightarrow$ $V(x) = -\,ql$

$\sum F_{ix} = 0;$ $N(x) = 0$

$\sum M_{(C)} = 0;$ $M(x) - mx + qlx = 0$ $\rightarrow$ $M(x) = qlx - qlx = 0$

Internal forces diagrams are shown in Fig. 2.3.19.

The internal forces analysis in the beams presented above show that:

- normal (axial) force $\boldsymbol{N_\alpha}$ in α section is an algebraic sum of all forces longitudinal components located on the left (or right) side of the section,
- shear (transversal) force $\boldsymbol{V_\alpha}$ in α section is an algebraic sum of all force components normal to the longitudinal direction and located on the left (or right) side of the section,
- bending moment $\boldsymbol{M_\alpha}$ in α section is an algebraic sum of all moments caused by forces located on the left (or right) side of the section calculated about the centroid of the section.

Internal forces – sign convention (Fig. 2.3.21.):

- normal (axial) force $\boldsymbol{N_\alpha}$ – tensile force is positive, compressive force – negative,
- shear (transversal) force $\boldsymbol{V_\alpha}$ – is positive when the algebraic sum of all force components normal to the longitudinal direction and located on the left side of the section is upward (or when the algebraic sum of all force components normal to the longitudinal direction and located on the right side of the section is downward),
- bending moment $\boldsymbol{M_\alpha}$ – is positive if the lower fibers are bended.

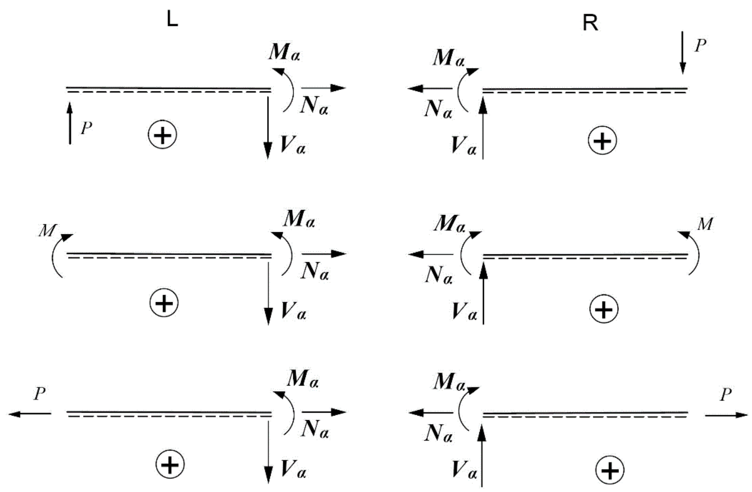


Fig. 2.3.21.

Internal forces diagrams – important points:

We may always determine the internal forces at each characteristic point of the beam.

In addition, if we know some rules we can facilitate the composition of the internal forces diagrams.

Some of these rules are as follow:

- at force load point, the internal moment diagram has a kink and the shear force diagram has a jump,
- at concentrated moment load point the internal moment diagram has a jump,
- if some segment of the beam hasn't any load then the internal moment diagram is linear and the shear force diagram is constant,
- if some segment of the beam is under uniformly distributed load then the internal moment diagram is parabolic of second degree and shear force diagram is linear,

- if some segment of the beam is under triangular distributed loading then the internal moment diagram is third degree curve and the shear force diagram is parabolic of second degree,
- the extreme values of moment occur at points where the shear force function is equal zero. We can do the shear diagrams from the applied loading and find the points of zero shear. If we are interested only in peak values of moment, we can calculate its value from the equilibrium conditions,
- the moment diagram should be plotted at the bended side of the member.

2.3.2. Computational problems – simple beams

Computational example 2.3.1.

For the beam presented in Fig. 2.3.22.a. draw internal forces diagrams.

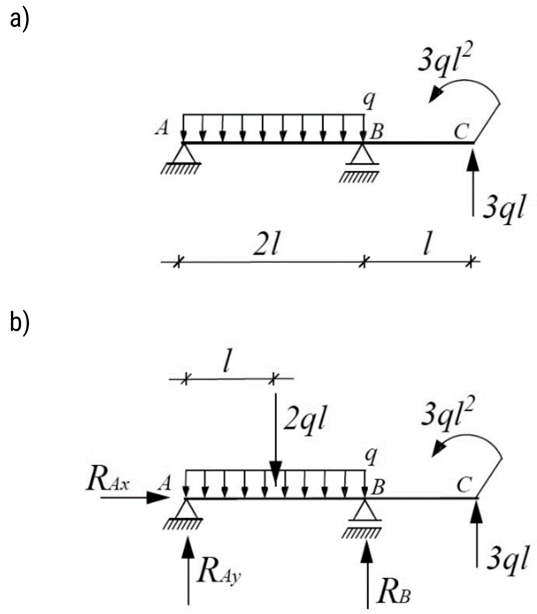


Fig. 2.3.22.

1. Degree of static indeterminacy n = 0
2. Support reactions (Fig. 2.3.22.b.):

$$\sum F_{iy} = 0; \qquad R_{Ay} + R_B - 2ql + 3ql = 0$$

$$\sum F_{ix} = 0; \qquad R_{Ax} = 0$$

$$\sum M_{(A)} = 0; \qquad R_B \cdot 2l + 3ql^2 - 2ql \cdot l + 3ql \cdot 3l = 0$$

hence:

$$R_B = -5ql \qquad R_{Ay} = 4ql$$

3. Internal forces equations (Fig. 2.3.23):

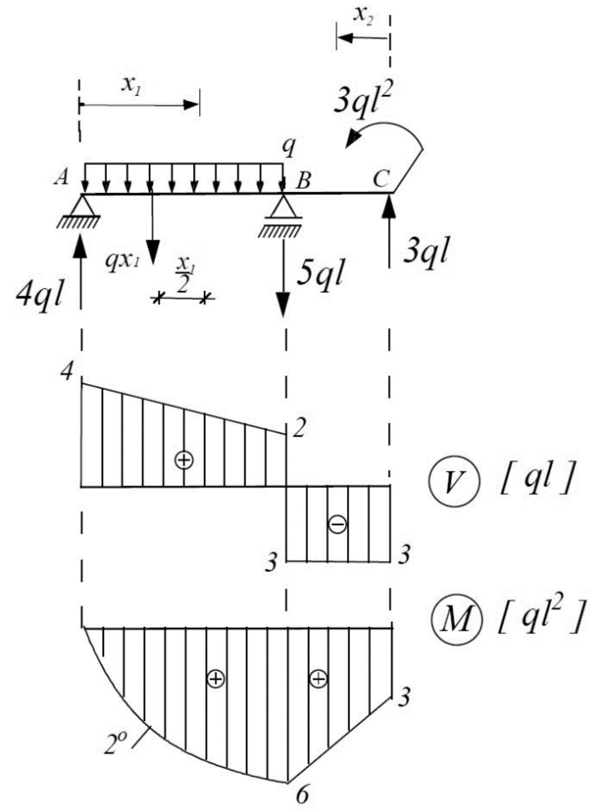


Fig. 2.3.23.

<u>Segment A–B</u> $\quad 0 \le x_1 < 2l$

$$V(x_1) = 4ql - qx_1$$

$$V(x_1 = 0) = 4ql \qquad V(x_1 = 2l) = 2ql$$

$$M(x_1) = 4ql \cdot x_1 - qx_1 \cdot \frac{x_1}{2} = 4ql \cdot x_1 - q \cdot \frac{(x_1)^2}{2}$$

$$M(x_1 = 0) = 0 \qquad M(x_1 = 2l) = 4ql \cdot 2l - q \cdot \frac{(2l)^2}{2} = 6ql^2$$

$$N(x_1) = 0$$

<u>Segment C–B</u> $\quad 0 \le x_2 < l$

$$V(x_2) = -3ql$$

$$M(x_2) = 3ql \cdot x_2 + 3ql^2$$

$$M(x_2 = 0) = 3ql^2 \qquad M(x_2 = l) = 6ql^2$$

Internal forces diagrams (V, M) are shown in Fig. 2.3.23. Diagram $N(x) = 0$.

Computational example 2.3.2.

Draw internal forces diagrams for the beam shown in Fig. 2.3.24.a.

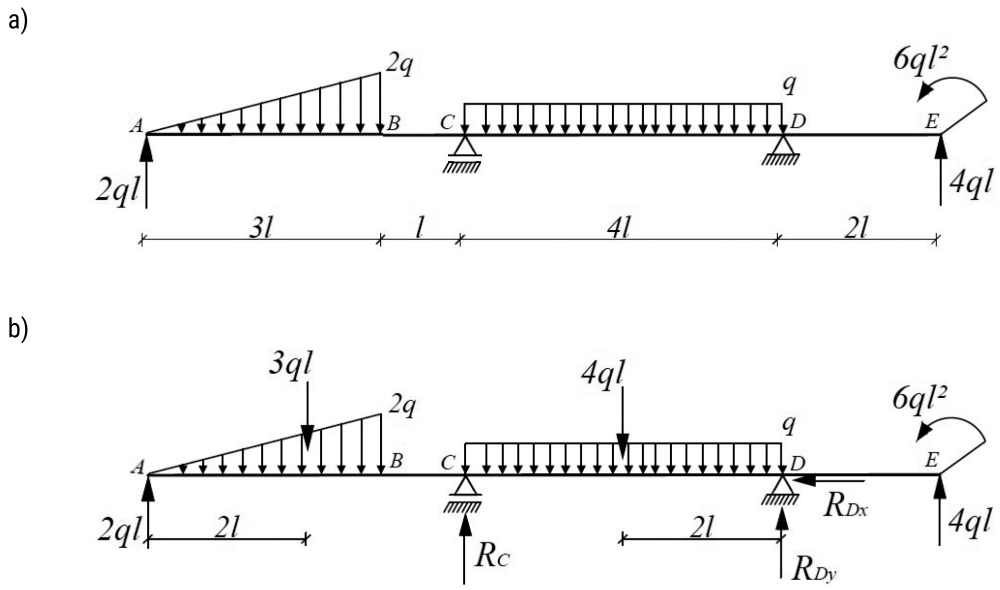


Fig. 2.3.24.

1. Degree of static indeterminacy $n = 0$
2. Support reactions (Fig. 2.3.24.b.):

Since there are no horizontal loads acting on the beam, thus $R_{Dx} = 0$

$$\sum M_{(c)} = 0; \quad -2ql \cdot 4l + 3ql \cdot 2l - 4ql \cdot 2l + R_{Dy} \cdot 4l + 4ql \cdot 6l + 6ql^2 = 0$$

$$R_{Dy} = -5ql$$

$$\sum F_{iy} = 0; \quad R_C + R_{Dy} + 2ql - 3ql - 4ql + 4ql = 0 \qquad R_B = 6ql$$

3. Internal forces equations (Fig. 2.3.25.):

<u>Segment A–B</u> $\quad 0 \leq x_1 < 3l$

$$\frac{q(x_1)}{x_1} = \frac{2q}{3l} \quad \rightarrow \quad q(x_1) = \frac{2qx_1}{3l}$$

$$V(x_1) = 2ql - \frac{1}{2} q(x_1) \cdot x_1 = 2ql - \frac{1}{2} \cdot \frac{2qx_1}{3l} \cdot x_1 = 2ql - \frac{q{x_1}^2}{3l}$$

$$V(x_1 = 0) = 2ql \qquad V\left(x_1 = \tfrac{3}{2} l\right) = \tfrac{5}{4} ql \qquad V(x_1 = 3l) = -ql$$

$$V(x_1) = 0 \quad \rightarrow \; 2ql - \tfrac{1}{2} \cdot \tfrac{2qx_1}{3l} \cdot x_1 = 0 \quad \rightarrow \quad 6ql^2 - q{x_1}^2 = 0 \quad \rightarrow x_1 = \sqrt{6} l$$

$$M(x_1) = 2qlx_1 - \frac{1}{2} q(x_1) \cdot x_1 \cdot \frac{x_1}{3} = 2qlx_1 - \frac{1}{2} \cdot \frac{2qx_1}{3l} \cdot x_1 \cdot \frac{x_1}{3} = 2qlx_1 - q \cdot \frac{(x_1)^3}{9l}$$

$$M(x_1 = 0) = 0 \qquad M(x_1 = 3l) = 3ql^2$$

$$M\left(x_1 = \sqrt{6} l\right) = 2ql \cdot \sqrt{6} l - q \cdot \frac{\left(\sqrt{6} l\right)^3}{9l} = \frac{4}{3} \sqrt{6} ql^2$$

$$N(x_1) = 0$$

Segment B–C $\quad 0 \le x_2 < l$

$V(x_2) = 2ql - 3ql = -ql$

$M(x_2) = 2ql \cdot (3l + x_2) - 3ql \cdot (l + x_2)$

$M(x_2 = 0) = 3ql^2 \qquad M(x_2 = l) = 2ql \cdot (3l + l) - 3ql \cdot (l + l) = 2ql^2$

$N(x_2) = 0$

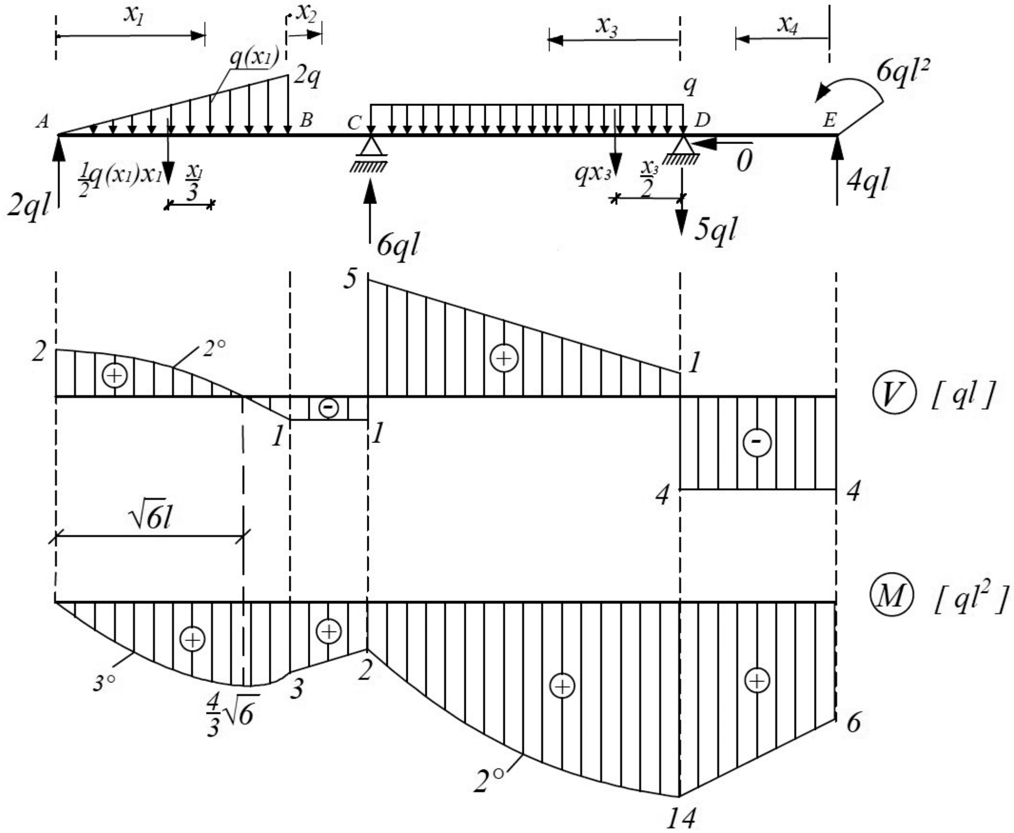


Fig. 2.3.25.

Segment D–C $\quad 0 \le x_3 < 4l$

$V(x_3) = -4ql + 5ql + qx_3$

$V(x_3 = 0) = ql \qquad V(x_3 = 4l) = 5ql$

$M(x_3) = 6ql^2 + 4ql \cdot (2l + x_3) - 5ql \cdot x_3 - qx_3 \cdot \frac{x_3}{2}$

$M(x_3 = 0) = 14ql^2 \qquad M(x_3 = 4l) = 2ql^2$

$N(x_3) = 0$

Segment E–D $\quad 0 \le x_4 < 2l$

$V(x_4) = -4ql$

$M(x_4) = 6ql^2 + 4ql \cdot x_4$

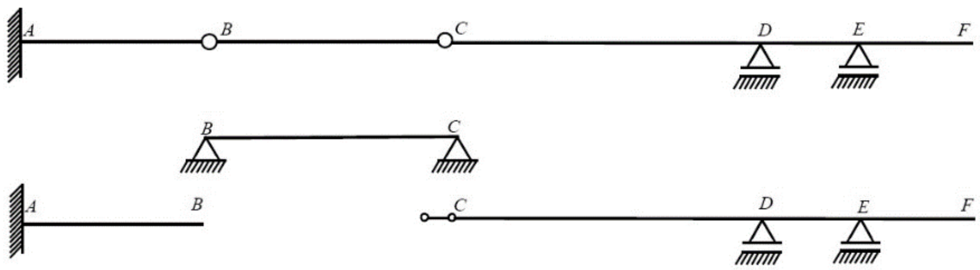
$M(x_4 = 0) = 6ql^2 \qquad M(x_4 = 2l) = 14ql^2$

$N(x_4) = 0$

Internal forces diagrams (V and M) are presented in Fig. 2.3.25. Diagram $N(x) = 0$.

2.3.3. Compound beams

a)



b)

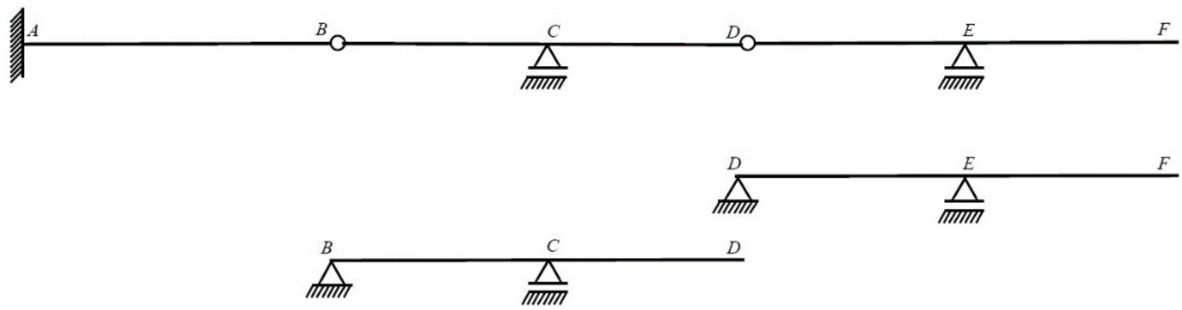


Fig. 2.3.26.

The compound beam is composed of more than one simple beam (Fig. 2.3.26.). The compound beam presented in Fig. 2.3.26.a. is composed of one cantilever beam and two simple beams, all of them lying on one line. Beam BC is based both on beam AB and beam CF. Beam AB is fixed and beam CF is supported on the ground (terra) so these beams we call primary beams. The middle beam is called the secondary beam. The beam shown in Fig. 2.3.26.b. is also composed of one cantilever beam and two simple beams. Beam DF is based on beam BD and on the roller support E. Beam BD is based on beam AB and the roller support C. Beam AB is fixed so we call this beam the primary beam and the two other we call secondary beams.

As we can see in Fig. 2.3.26. the secondary beams transfer the loads to the basics ones. That is why we first analyze the secondary beams and then their support reactions load on the basic beam. We can write internal forces equations or calculate internal forces directly at points separately for every beam. We also draw separately the internal forces diagrams and after that, we join them for the whole compound beam.

2.3.4. Computational problems – compound beams

Computational example 2.3.3.

For the compound beam shown in Fig. 2.3.27.a. draw internal forces diagrams.

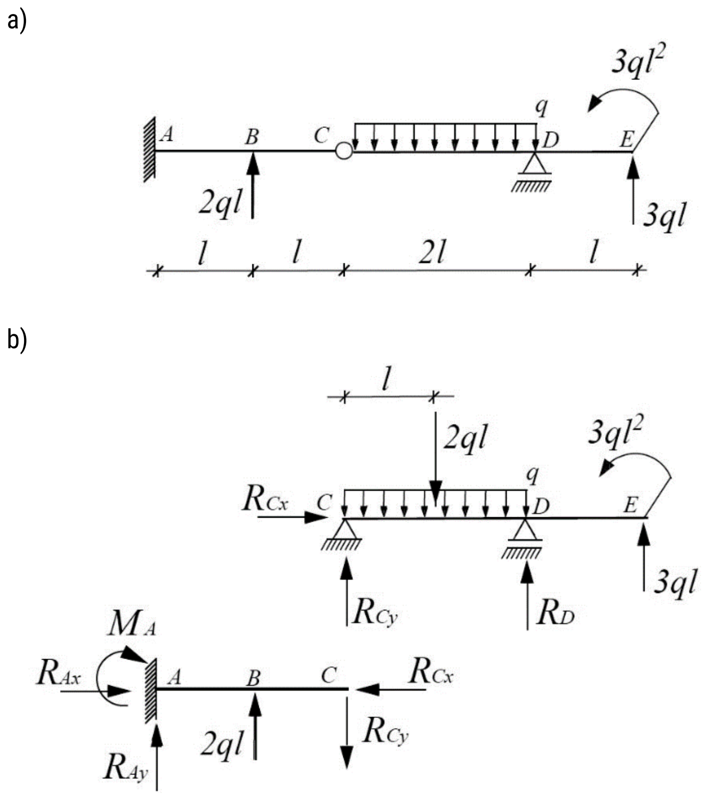


Fig. 2.3.27.

1. Degree of static indeterminacy $n = r + 2s - 3t = 4 + 2 - 3 \cdot 2 = 0$
2. Support reactions (Fig. 2.3.27.b.):

   Since there are no horizontal loads acting on the beam, thus $R_{Ax} = R_{Cx} = 0$

– right beam:

$$\sum M_{(D)} = 0; \quad -R_{Cy} \cdot 2l + 2ql \cdot l + 3ql^2 + 3ql \cdot l = 0 \quad R_{Cy} = 4ql$$

$$\sum F_{iy} = 0; \quad R_{Cy} - 2ql + R_D + 3ql = 0 \quad R_D = -5ql$$

– left beam:

$$\sum M_{(A)} = 0; \quad -M_A + 2ql \cdot l - R_{Cy} \cdot 2l = 0 \quad M_A = -6ql^2$$

$$\sum F_{iy} = 0; \quad R_{Ay} + 2ql - R_{Cy} = 0 \quad R_{Ay} = 2ql$$

3. Internal forces equations (Fig. 2.3.28.):

<u>Segment A–B</u> $\quad 0 \le x_1 < l$

$V(x_1) = 2ql$

$M(x_1) = -6ql^2 + 2ql \cdot x_1$

$M(x_1 = 0) = -6ql^2 \qquad M(x_1 = l) = -4ql^2$

$N(x_1) = 0$

Segment C–B $\quad 0 \le x_2 < l$

$V(x_2) = 4ql$

$M(x_2) = -\,4ql \cdot x_2$

$M(x_2 = 0) = 0 \qquad M(x_2 = l) = -4ql^2$

$N(x_2) = 0$

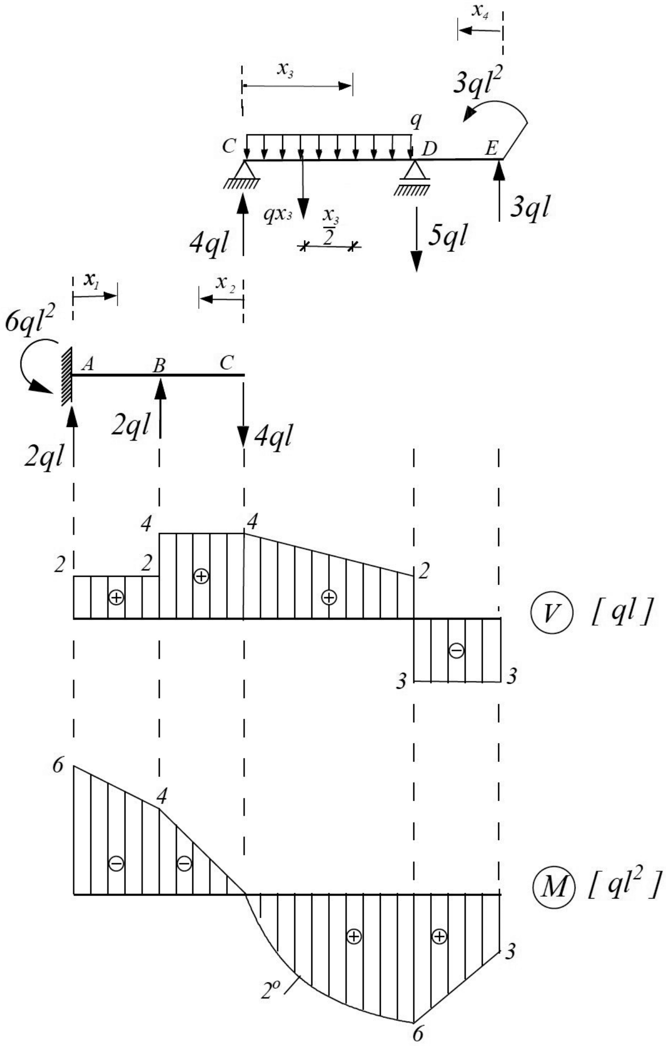


Fig. 2.3.28.

Segment C–D $\quad 0 \le x_3 < 2l$

$V(x_3) = 4ql - qx_3$

$V(x_3 = 0) = 4ql \qquad V(x_3 = 2l) = 2ql$

$$M(x_3) = 4ql \cdot x_3 - qx_3 \cdot \frac{x_3}{2} = 4ql \cdot x_3 - q \cdot \frac{(x_3)^2}{2}$$

$M(x_3 = 0) = 0 \qquad M(x_3 = l) = \frac{7}{2}ql^2 \qquad M(x_3 = 2l) = 6ql^2$

$N(x_3) = 0$

Segment E–D $\quad 0 \le x_4 < l$

$V(x_4) = -3ql$

$M(x_4) = 3ql^2 + 3ql \cdot x_4$

$M(x_4 = 0) = 3ql^2 \qquad M(x_4 = l) = 6ql^2$

$N(x_4) = 0$

Internal forces diagrams (V, M) are shown in Fig. 2.3.28. Diagram $N(x)$ = 0.

Computational example 2.3.4.

For the compound beam shown in Fig. 2.3.29. draw internal forces diagrams.

1. Degree of static indeterminacy $n = r + 2s - 3t = 4 + 2 - 3 \cdot 2 = 0$
2. Support reactions (Fig. 2.3.29.):
   Since there are no horizontal loads acting on the beam, thus $R_{Cx} = R_{Ax} = 0$

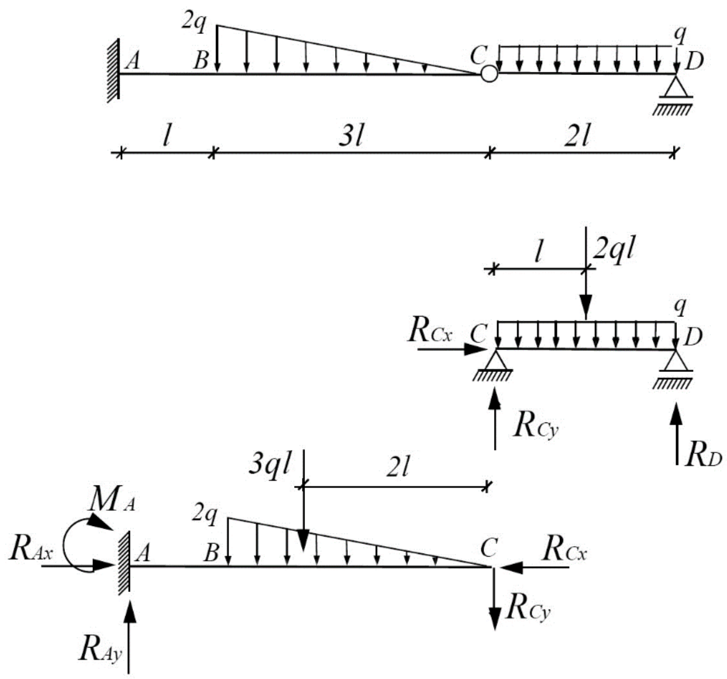


Fig. 2.3.29.

– right beam:

$\sum M_{(C)} = 0; \quad R_D \cdot 2l - 2ql \cdot l = 0 \qquad R_D = ql$

$\sum F_{iy} = 0; \quad R_D - 2ql + R_{Cy} = 0 \qquad R_{Cy} = ql$

This beam is symmetrical and symmetrically loaded so it wasn't necessary to calculate the reactions by the equations of equilibrium. It is obvious that the reactions should be equal:

$R_{Cy} = R_D = ql,$

– left beam:

$$\sum M_{(A)} = 0; \quad -M_A - 3ql \cdot 2l - R_{Cy} \cdot 4l = 0 \qquad M_A = -10\,ql^2$$

$$\sum F_{iy} = 0; \quad -R_{Cy} - 3ql + R_{Ay} = 0 \qquad R_{Ay} = 4ql$$

3. Internal forces equations (Fig. 2.3.30.):

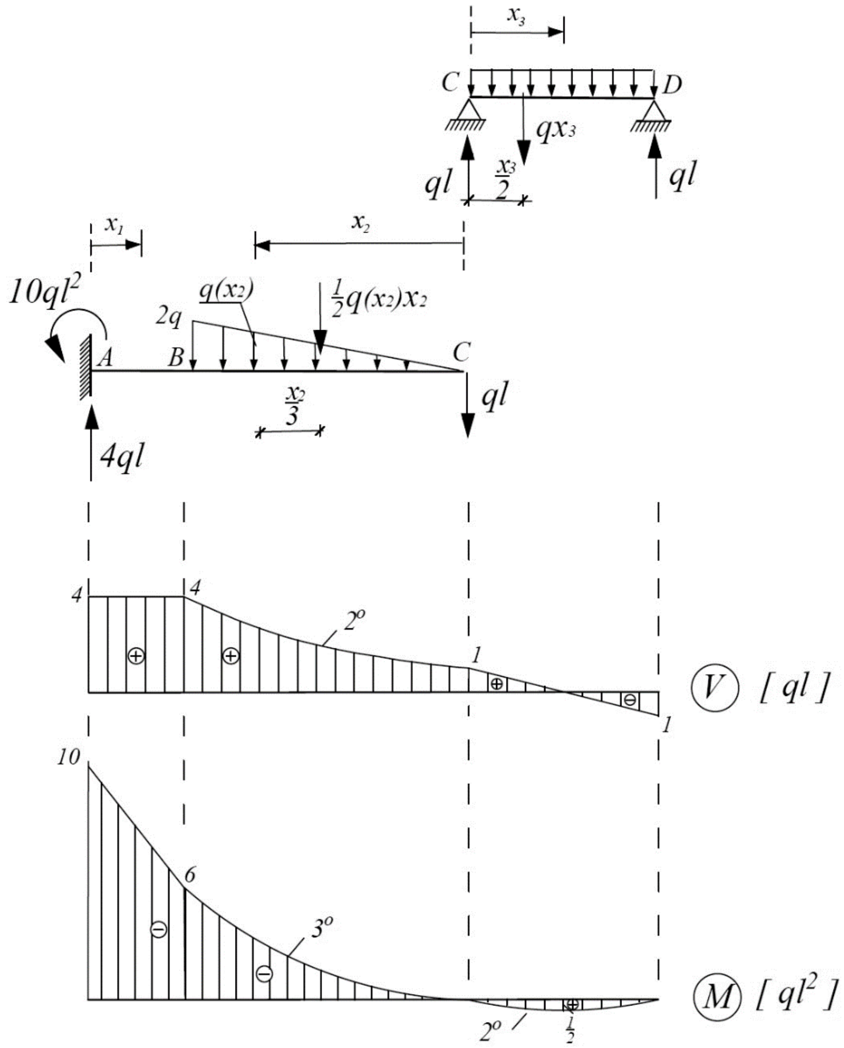


Fig. 2.3.30.

<u>Segment A–B</u> $\quad 0 \le x_1 < l$

$V(x_1) = 4ql$

$M(x_1) = -10ql^2 + 4ql \cdot x_1$

$M(x_1 = 0) = -10ql^2 \qquad M(x_1 = l) = -6ql^2$

$N(x_1) = 0$

<u>Segment C–B</u> $\quad 0 \le x_2 < 3l$

$$\frac{q(x_2)}{x_2} = \frac{2q}{3l} \quad \rightarrow \quad q(x_2) = \frac{2qx_2}{3l}$$

$$V(x_2) = ql + \frac{1}{2}q(x_2) \cdot x_2 = ql + \frac{1}{2} \cdot \frac{2qx_2}{3l} \cdot x_2 = ql + \frac{q{x_2}^2}{3l}$$

$$V(x_2 = 0) = ql \qquad V\left(x_2 = \tfrac{3}{2}l\right) = \tfrac{7}{4}ql \qquad V(x_2 = 3l) = 4ql$$

$$M(x_2) = -ql \cdot x_2 - \tfrac{1}{2}q(x_2) \cdot x_2 \cdot \tfrac{x_2}{3} = -ql \cdot x_2 - \tfrac{1}{2} \cdot \tfrac{2qx_2}{3l} \cdot x_2 \cdot \tfrac{x_2}{3} =$$

$$= -ql \cdot x_2 - q \cdot \frac{(x_2)^3}{9l}$$

$$M(x_2 = 0) = 0 \qquad M\left(x_2 = \tfrac{3}{2}l\right) = -\tfrac{15}{8}ql^2 \qquad M(x_2 = 3l) = -6ql^2$$

$$N(x_2) = 0$$

Segment C–D $\quad 0 \leq x_3 < 2l$

$$V(x_3) = ql - qx_3$$

$$V(x_3 = 0) = ql \qquad V(x_3 = 2l) = -ql$$

$$V(x_3) = 0 \quad \rightarrow \quad ql - qx_3 = 0 \quad \rightarrow \quad x_3 = l$$

$$M(x_3) = ql \cdot x_3 - qx_3 \cdot \frac{x_3}{2} = ql \cdot x_3 - q \cdot \frac{(x_3)^2}{2}$$

$$M(x_3 = 0) = 0 \qquad M(x_3 = 2l) = 0 \qquad M(x_3 = l) = ql \cdot l - q \cdot \tfrac{(l)^2}{2} = \tfrac{1}{2}ql^2$$

$$N(x_3) = 0$$

Internal forces diagrams (V and M) are presented in Fig. 2.3.30. Diagram $N(x) = 0$.

Computational example 2.3.5.

For the compound beam shown in Fig. 2.3.31.a. draw internal forces diagrams.

Degree of static indeterminacy $n = r + 2s - 3t = 4 + 2 - 3 \cdot 2 = 0$

1. Support reactions (Fig. 2.3.31.b.):

Since there are no horizontal loads acting on the beam, thus $R_{Ax} = R_{Bx} = 0$

– left beam:

$$\sum M_{(A)} = 0; \quad -2ql \cdot l + R_{By} \cdot 2l = 0 \qquad R_{By} = ql$$

$$\sum F_{iy} = 0; \qquad R_{Ay} - 2ql + R_{By} = 0 \qquad R_{Ay} = ql$$

This beam is symmetrical and symmetrically loaded so we didn't have to calculate the reactions by the equations of equilibrium. It is obvious that the reactions should be equal:

$$R_{Ay} = R_{By} = ql$$

– right beam:

$$\sum M_{(C)} = 0; \quad R_{By} \cdot 2l + 2ql \cdot l + R_E \cdot 2l + ql^2 = 0 \qquad R_E = -\tfrac{5}{2}ql$$

$$\sum F_{iy} = 0; \qquad -R_{By} + R_C + 2ql + R_E = 0 \qquad R_C = \tfrac{3}{2}ql$$

a)

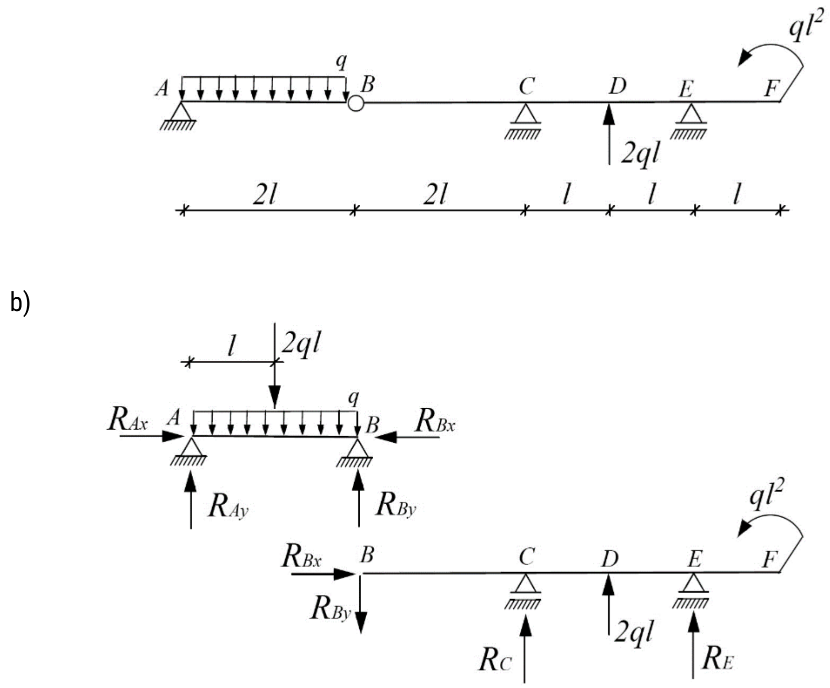


Fig. 2.3.31.

2. Internal forces equations (Fig. 2.3.32.):

<u>Segment A–B</u> $\quad 0 \le x_1 < 2l$

$$V(x_1) = ql - qx_1$$

$$V(x_1 = 0) = ql \qquad V(x_1 = 2l) = -ql$$

$$V(x_1) = 0 \quad \rightarrow \quad ql - qx_1 = 0 \quad \rightarrow \quad x_1 = l$$

$$M(x_1) = ql \cdot x_1 - qx_1 \cdot \frac{x_1}{2} = ql \cdot x_1 - q \cdot \frac{(x_1)^2}{2}$$

$$M(x_1 = 0) = 0 \qquad M(x_1 = 2l) = 0 \qquad M(x_1 = l) = ql \cdot l - q \cdot \frac{(l)^2}{2} = \frac{1}{2}ql^2$$

$$N(x_1) = 0$$

<u>Segment B–C</u> $\quad 0 \le x_2 < 2l$

$$V(x_2) = -ql$$

$$M(x_2) = -\,ql \cdot x_2$$

$$M(x_2 = 0) = 0 \qquad M(x_2 = 2l) = -2ql^2$$

$$N(x_2) = 0$$

<u>Segment C–D</u> $\quad 0 \le x_3 < l$

$$V(x_3) = -ql + \frac{3}{2}ql = \frac{1}{2}ql$$

$$M(x_3) = -ql \cdot (2l + x_3) + \frac{3}{2}ql \cdot x_3$$

$M(x_3 = 0) = -2ql^2 \qquad M(x_3 = l) = -\frac{3}{2}ql^2$

$N(x_3) = 0$

<u>Segment E–D</u> $\quad 0 \le x_4 < l$

$V(x_4) = \dfrac{5}{2}ql$

$M(x_4) = ql^2 - \dfrac{5}{2}ql \cdot x_4$

$M(x_4 = 0) = ql^2 \qquad M(x_4 = l) = -\frac{3}{2}ql^2$

$N(x_4) = 0$

<u>Segment F–E</u> $\quad 0 \le x_5 < l$

$V(x_5) = 0$

$M(x_5) = ql^2$

$N(x_5) = 0$

Internal forces diagrams (V, M) are shown in Fig. 2.3.32. Diagram $N(x) = 0$.

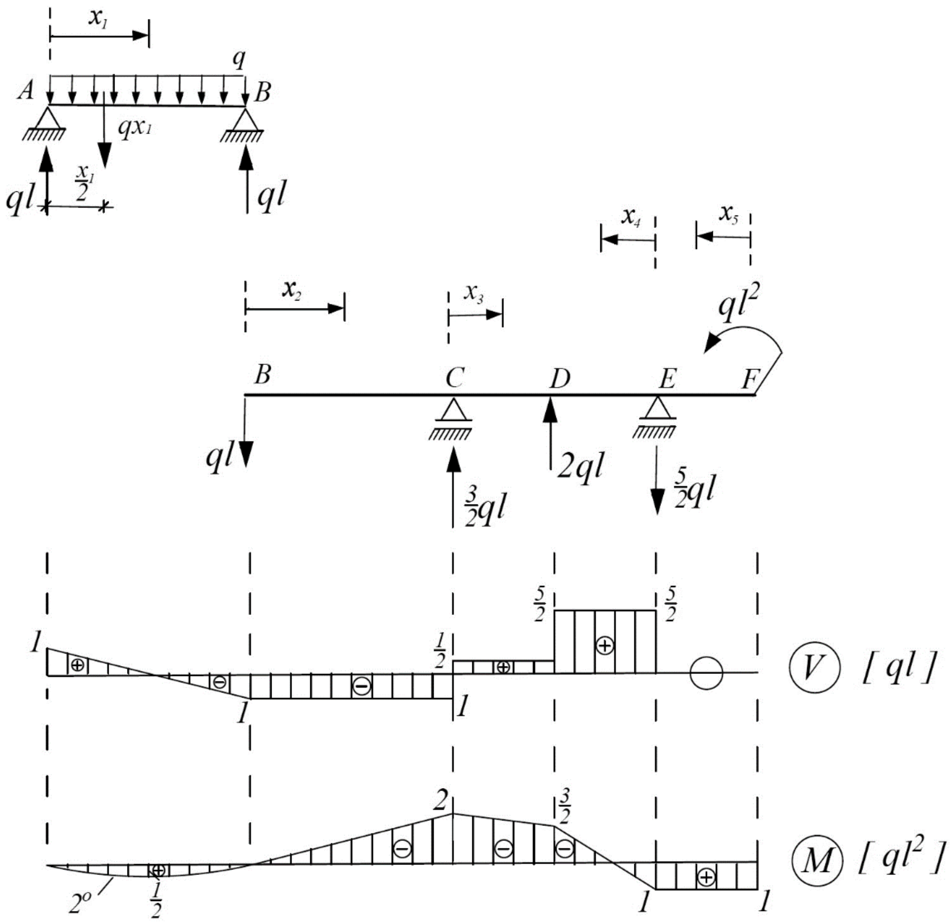


Fig. 2.3.32.

Computational example 2.3.6.

Draw internal forces diagrams for the beam shown in Fig. 2.3.33.

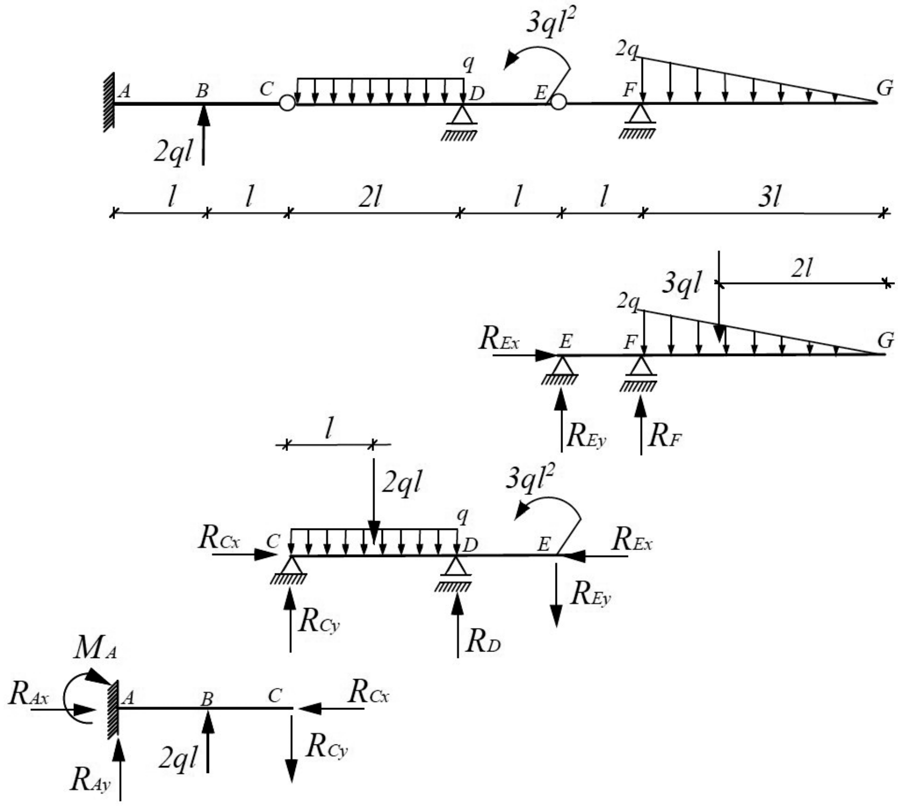


Fig. 2.3.33.

1. Degree of static indeterminacy $n = r + 2s - 3t = 5 + 4 - 3 \cdot 3 = 0$
2. Support reactions (Fig. 2.3.33.):

   Since there are no horizontal loads acting on the beam, thus $R_{Ax} = R_{Cx} = R_{Ex} = 0$

– right beam:

$\sum M_{(F)} = 0; \quad -R_{Ey} \cdot l - 3ql \cdot l = 0 \qquad R_{Ey} = -3\,ql$

$\sum F_{iy} = 0; \quad R_{Ey} + R_F - 3ql = 0 \qquad R_F = 6ql$

– middle beam:

$\sum M_{(D)} = 0; \quad -R_{Cy} \cdot 2l + 2ql \cdot l - R_{Ey} \cdot l + 3ql^2 = 0 \qquad R_{Cy} = 4\,ql$

$\sum F_{iy} = 0; \quad R_{Cy} + R_D - R_{Ey} - 2ql = 0 \qquad R_D = -5ql$

– left beam:

$\sum M_{(A)} = 0; \quad -M_A + 2ql \cdot l - R_{Cy} \cdot 2l = 0 \qquad M_A = -6\,ql^2$

$\sum F_{iy} = 0; \quad -R_{Cy} + 2ql + R_{Ay} = 0 \qquad R_{Ay} = 2ql$

3. Internal forces equations (Fig. 2.3.34.):

<u>Segment A–B</u> $\quad 0 \le x_1 < l$

$V(x_1) = 2ql$

$M(x_1) = -6ql^2 + 2ql \cdot x_1$

$M(x_1 = 0) = -6ql^2 \qquad M(x_1 = l) = -4ql^2$

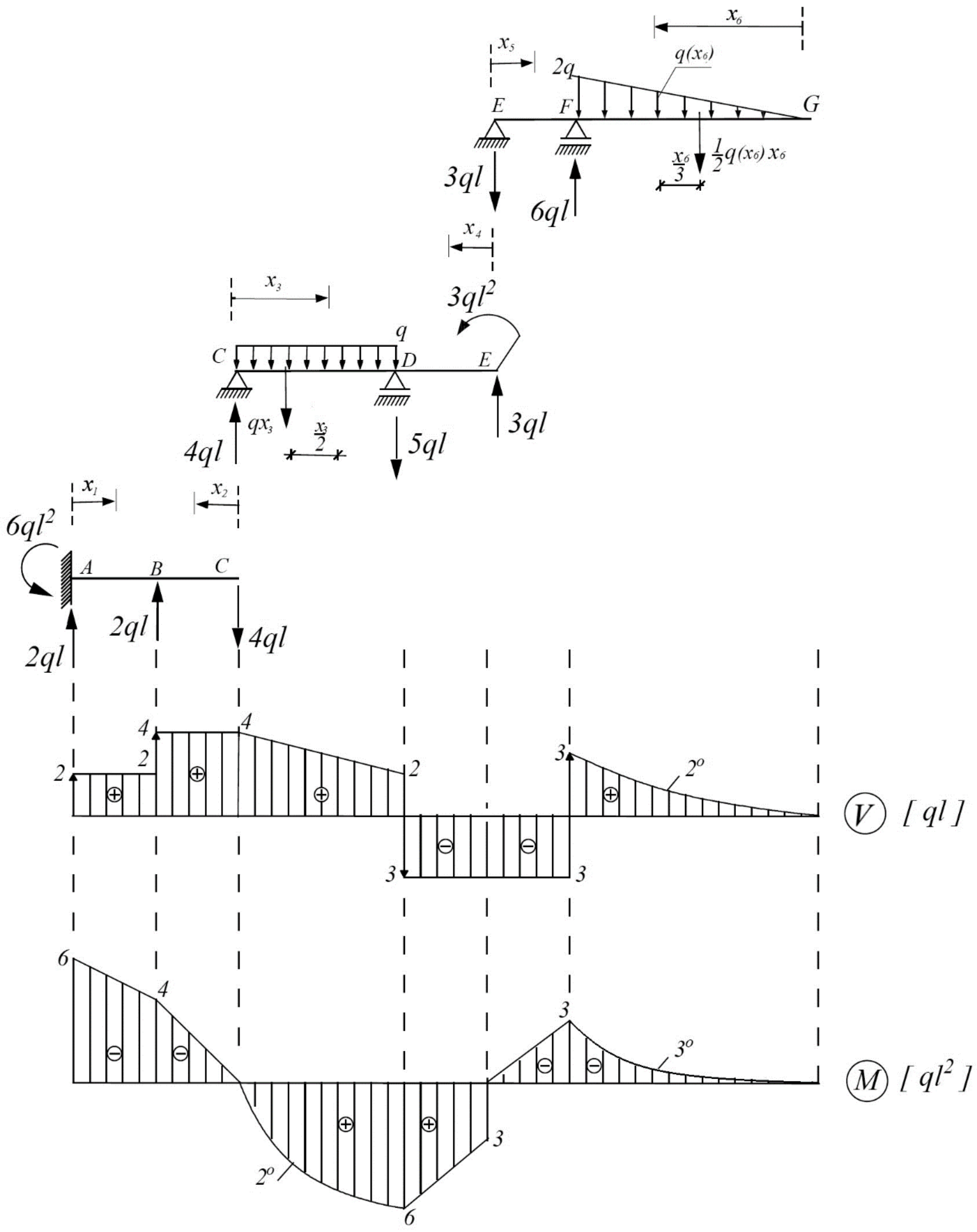


Fig. 2.3.34.

Segment C–B $\quad 0 \leq x_2 < l$

$V(x_2) = 4ql$

$M(x_2) = -4ql \cdot x_2$

$M(x_2 = 0) = 0 \qquad M(x_2 = l) = -4ql^2$

Segment C–D $\quad 0 \leq x_3 < 2l$

$V(x_3) = 4ql - qx_3$

$V(x_3 = 0) = 4ql \qquad V(x_3 = 2l) = 2ql$

$$M(x_3) = 4ql \cdot x_3 - qx_3 \cdot \frac{x_3}{2} = 4ql \cdot x_3 - q \cdot \frac{(x_3)^2}{2}$$

$$M(x_3 = 0) = 0 \qquad M(x_3 = 2l) = 6ql^2$$

Segment E–D $\quad 0 \le x_4 < l$

$$V(x_4) = -3ql$$

$$M(x_4) = 3ql^2 + 3ql \cdot x_4$$

$$M(x_4 = 0) = 3ql^2 \qquad M(x_4 = l) = 6ql^2$$

Segment E–F $\quad 0 \le x_5 < l$

$$V(x_5) = -3ql$$

$$M(x_5) = -3ql \cdot x_5$$

$$M(x_5 = 0) = 0 \qquad M(x_5 = l) = -3ql^2$$

Segment G–F $\quad 0 \le x_6 < 2l$

$$\frac{q(x_6)}{x_6} = \frac{2q}{3l} \quad \rightarrow \quad q(x_6) = \frac{2qx_6}{3l}$$

$$V(x_6) = \frac{1}{2} q(x_6) \cdot x_6 = \frac{1}{2} \cdot \frac{2qx_6}{3l} \cdot x_6 = \frac{q{x_6}^2}{3l}$$

$$V(x_6 = 0) = 0 \qquad V\left(x_6 = \tfrac{3}{2}l\right) = \tfrac{3}{4}ql \qquad V(x_6 = 3l) = 3ql$$

$$M(x_6) = -\frac{1}{2} q(x_6) \cdot x_6 \cdot \frac{x_6}{3} = -\frac{1}{2} \cdot \frac{2qx_6}{3l} \cdot x_6 \cdot \frac{x_6}{3} = -q \cdot \frac{(x_6)^3}{9l}$$

$$M(x_6 = 0) = 0 \qquad M\left(x_6 = \tfrac{3}{2}l\right) = -\tfrac{3}{8}ql^2 \qquad M(x_6 = 3l) = -q \cdot \tfrac{(3l)^3}{9l} = -3ql^2$$

Internal forces diagrams (V, M) are shown in Fig. 2.3.34.

There are no horizontal loads, hence $N(x) = 0$.

Computational example 2.3.7.

Draw internal forces diagrams for the beam shown in Fig. 2.3.35.a.

a)

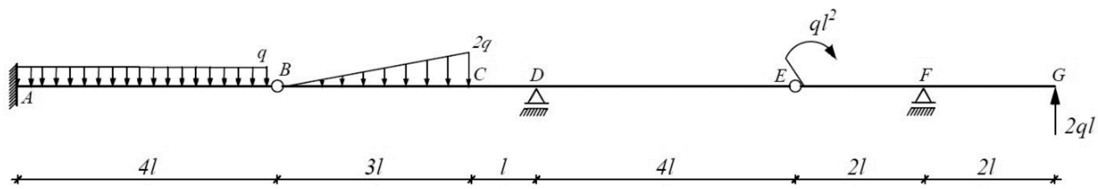


Fig. 2.3.35a.

1. Degree of static indeterminacy $n = r + 2s - 3t = 5 + 4 - 3 \cdot 3 = 0$
2. Support reactions (Fig. 2.3.35.b.):

Since there are no horizontal loads acting on the beam, thus $R_{Ax} = R_{Bx} = R_{Ex} = 0$

– right beam:

b)

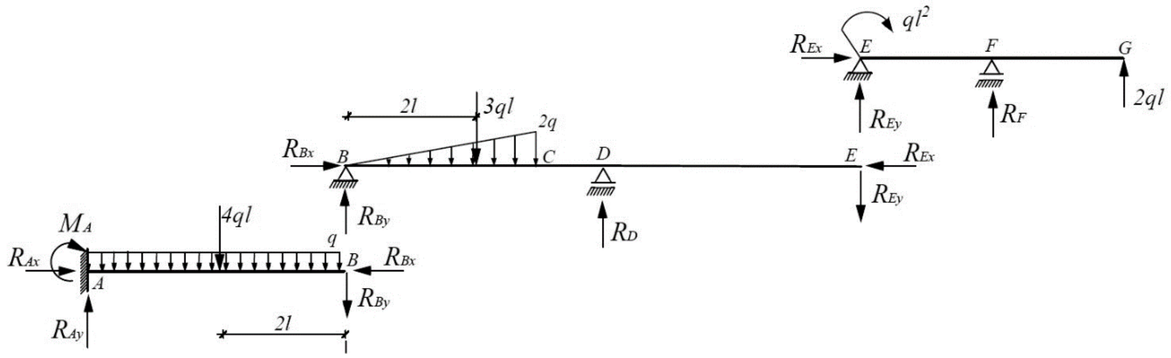


Fig. 2.3.35b.

$\sum M_{(F)} = 0; \quad -R_{Ey} \cdot 2l + 2ql \cdot 2l - ql^2 = 0 \qquad R_{Ey} = \frac{3}{2}\, ql$

$\sum F_{iy} = 0; \qquad R_{Ey} + R_F + 2ql = 0 \qquad R_F = -\frac{7}{2} ql$

– middle beam:

$\sum M_{(D)} = 0; \quad -R_{By} \cdot 4l + 3ql \cdot 2l - R_{Ey} \cdot 4l = 0 \quad R_{By} = 0\, ql$

$\sum F_{iy} = 0; \qquad R_{By} + R_D - R_{Ey} - 3ql = 0 \qquad R_D = \frac{9}{2} ql$

– left beam:

$\sum M_{(A)} = 0; \quad -M_A - 4ql \cdot 2l - R_{By} \cdot 4l = 0 \quad M_A = -8\, ql^2$

$\sum F_{iy} = 0; \qquad -R_{By} - 4ql + R_{Ay} = 0 \qquad R_{Ay} = 4ql$

3. Internal forces equations (Fig. 2.3.36.):

<u>Segment B–A</u> $\quad 0 \le x_1 < 4l$

$V(x_1) = qx_1$

$V(x_1 = 0) = 0 \qquad V(x_1 = 4l) = 4ql$

$$M(x_1) = -qx_1 \cdot \frac{x_1}{2} = -q \cdot \frac{(x_1)^2}{2}$$

$M(x_1 = 0) = 0 \qquad M(x_1 = 2l) = -2ql^2 \qquad M(x_1 = 4l) = -8ql^2$

$N(x_1) = 0$

<u>Segment B–C</u> $\quad 0 \le x_2 < 3l$

$$\frac{q(x_2)}{x_2} = \frac{2q}{3l} \quad \rightarrow \quad q(x_2) = \frac{2qx_2}{3l}$$

$$V(x_2) = -\frac{1}{2} q(x_2) \cdot x_2 = -\frac{1}{2} \cdot \frac{2qx_2}{3l} \cdot x_2 = -\frac{q{x_2}^2}{3l}$$

$V(x_2 = 0) = 0 \qquad V\left(x_2 = \frac{3}{2} l\right) = -\frac{3}{4} ql \qquad V(x_2 = 3l) = -3ql$

$$M(x_2) = -\frac{1}{2} q(x_2) \cdot x_2 \cdot \frac{x_2}{3} = -\frac{1}{2} \cdot \frac{2qx_2}{3l} \cdot x_2 \cdot \frac{x_2}{3} = -q \cdot \frac{(x_2)^3}{9l}$$

$M(x_2 = 0) = 0 \qquad M\left(x_2 = \frac{3}{2} l\right) = -\frac{3}{8} ql^2 \qquad M(x_2 = 3l) = -q \cdot \frac{(3l)^3}{9l} = -3ql^2$

$N(x_2) = 0$

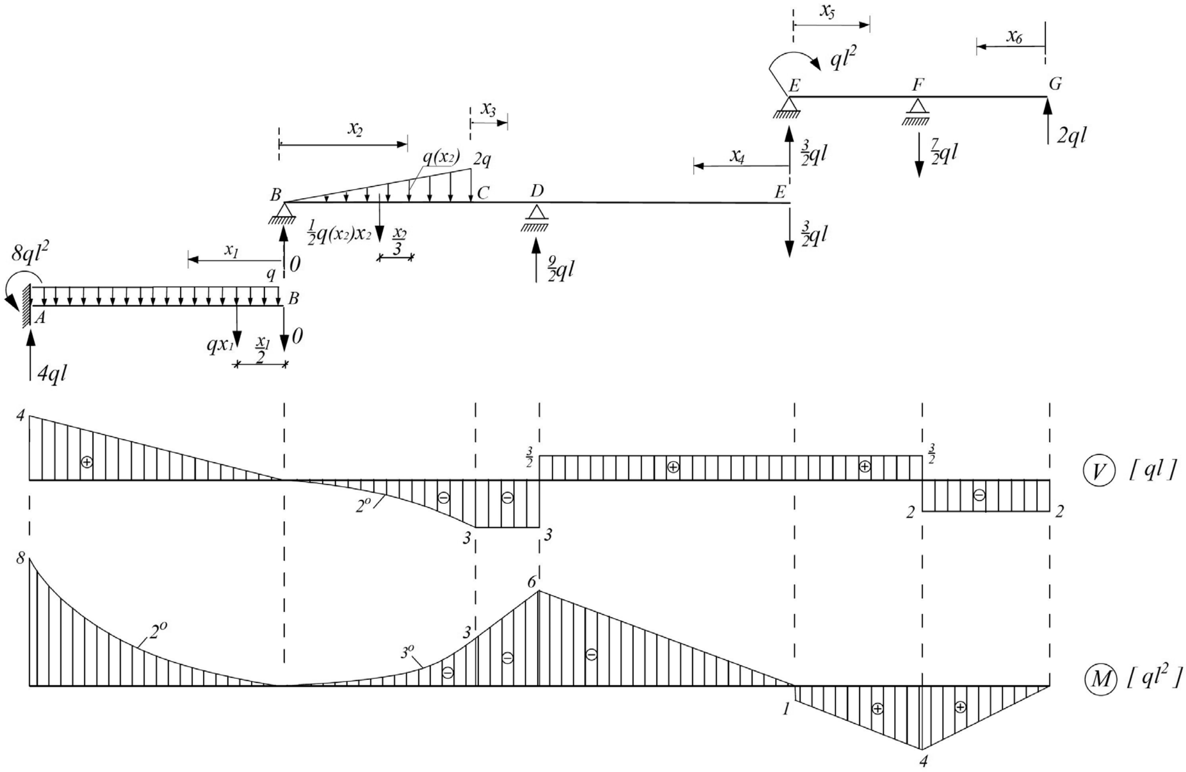


Fig. 2.3.36.

Segment C–D $\quad 0 \le x_3 < l$

$$V(x_3) = -3ql$$

$$M(x_3) = -3ql \cdot (l + x_3)$$

$$M(x_3 = 0) = -3ql^2 \qquad M(x_3 = l) = -6ql^2$$

$$N(x_3) = 0$$

Segment E–D $\quad 0 \le x_4 < 4l$

$$V(x_4) = \frac{3}{2}ql$$

$$M(x_4) = -\frac{3}{2}ql \cdot x_4$$

$$M(x_4 = 0) = 0 \qquad M(x_4 = 4l) = -6ql^2$$

$$N(x_4) = 0$$

Segment E–F $\quad 0 \le x_5 < 2l$

$$V(x_5) = \frac{3}{2}ql$$

$$M(x_5) = ql^2 + \frac{3}{2}ql \cdot x_5$$

$$M(x_5 = 0) = ql^2 \qquad M(x_5 = 2l) = 4ql^2$$

$$N(x_5) = 0$$

<u>Segment G–F</u> $0 \leq x_6 < 2l$

$V(x_6) = -2ql$

$M(x_6) = 2ql \cdot x_6$

$M(x_6 = 0) = 0 \qquad M(x_6 = 2l) = 4ql^2$

$N(x_6) = 0$

Internal forces diagrams (V and M) are presented in Fig. 2.3.36. Diagram $N(x) = 0$.

Computational example 2.3.8.

For the compound beam shown in Fig. 2.3.37. draw internal forces diagrams.

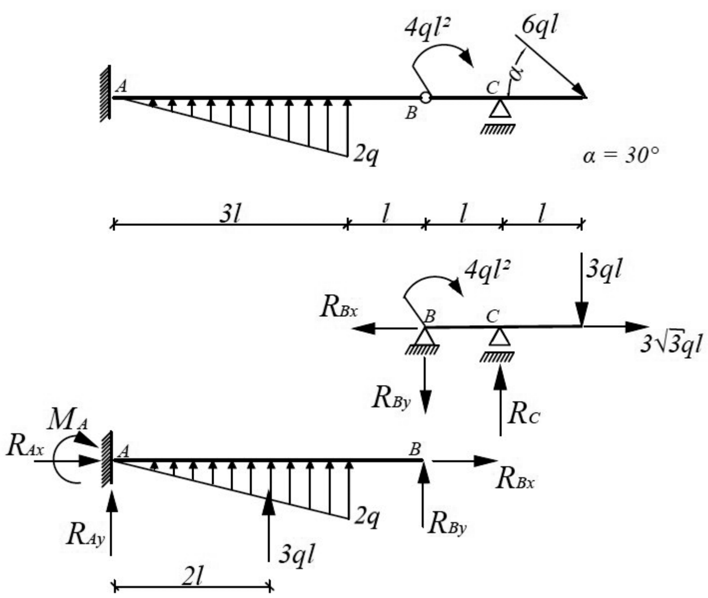


Fig. 2.3.37.

1. Degree of static indeterminacy $n = r + 2s - 3t = 4 + 2 - 3 \cdot 2 = 0$
2. Support reactions (Fig. 2.3.37.):

– right beam:

$\sum M_{(B)} = 0; \quad -4ql^2 - 3ql \cdot 2l + R_C \cdot l = 0 \qquad R_C = 10\,ql$

$\sum F_{iy} = 0; \quad -R_{By} + R_C - 3ql = 0 \qquad R_{By} = 7ql$

$\sum F_{ix} = 0; \quad -R_{Bx} + 3\sqrt{3}ql = 0 \qquad R_{Bx} = 3\sqrt{3}ql$

– left beam:

$\sum M_{(A)} = 0; \quad -M_A + 3ql \cdot 2l + R_{By} \cdot 4l = 0 \qquad M_A = 34\,ql^2$

$\sum F_{iy} = 0; \quad R_{Ay} + R_{By} + 3ql = 0 \qquad R_{Ay} = -10ql$

$\sum F_{ix} = 0; \quad R_{Ax} + R_{Bx} = 0 \qquad R_{Ax} = -3\sqrt{3}ql$

Internal forces diagrams are presented in Fig. 2.3.38.

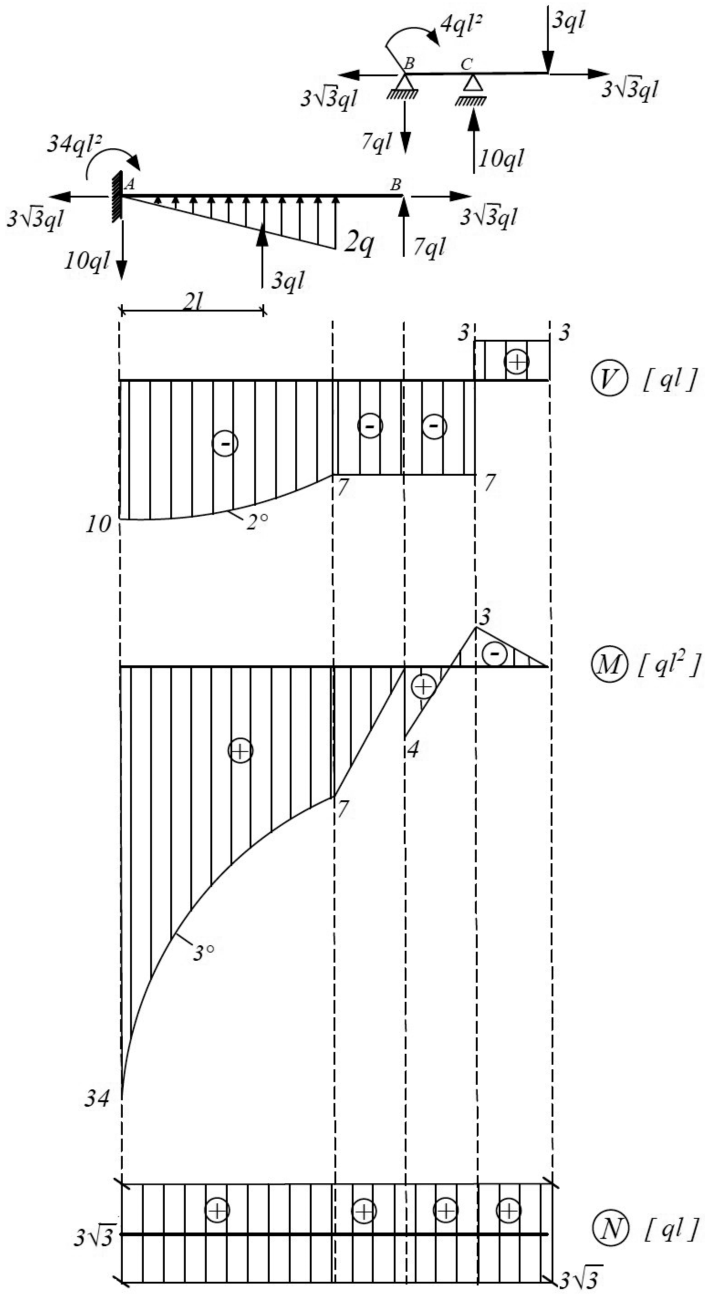


Fig. 2.3.38.

Computational example 2.3.9.

For the compound beam shown in Fig. 2.3.39.a. draw internal forces diagrams.

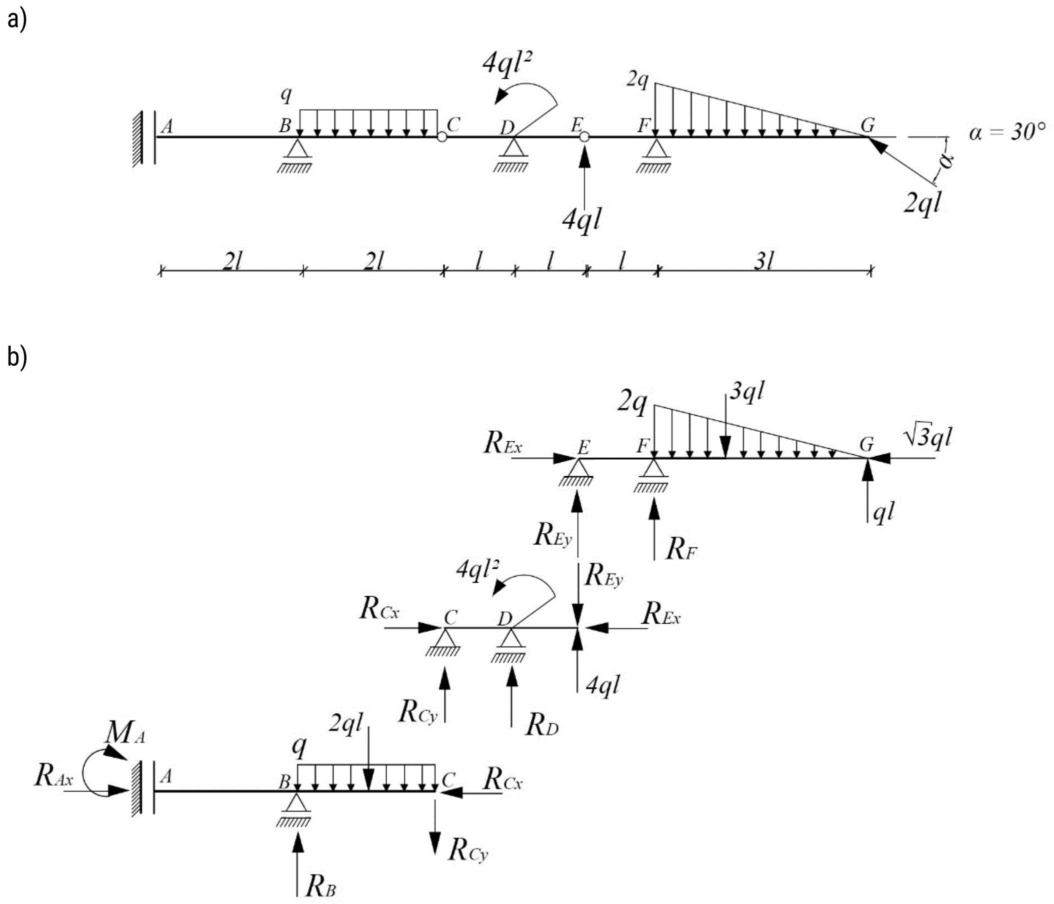


Fig. 2.3.39.

1. Degree of static indeterminacy $n = r + 2s - 3t = 5 + 4 - 3 \cdot 3 = 0$
2. Support reactions (Fig. 2.3.39.b.):

– right beam:

$$\sum M_{(F)} = 0; \qquad ql \cdot 3l - 3ql \cdot l - R_{Ey} \cdot l = 0 \qquad R_{Ey} = 0$$

$$\sum F_{iy} = 0; \qquad R_{Ey} + R_F - 3ql + ql = 0 \qquad R_F = 2ql$$

$$\sum F_{ix} = 0; \qquad R_{Ex} - \sqrt{3}ql = 0 \qquad R_{Ex} = \sqrt{3}ql$$

– middle beam:

$$\sum M_{(D)} = 0; \qquad -R_{Cy} \cdot l + 4\,ql^2 + 4ql \cdot l - R_{Ey} \cdot l = 0 \qquad R_{Cy} = 8ql$$

$$\sum F_{iy} = 0; \qquad R_{Cy} + R_D - R_{Ey} + 4ql = 0 \qquad R_D = -12ql$$

$$\sum F_{ix} = 0; \qquad R_{Cx} - R_{Ex} = 0 \qquad R_{Cx} = \sqrt{3}ql$$

– left beam:

$$\sum M_{(B)} = 0; \qquad -M_A - 2ql \cdot l - R_{Cy} \cdot 2l = 0 \qquad M_A = -18\,ql^2$$

$$\sum F_{iy} = 0; \qquad R_B - R_{Cy} - 2ql = 0 \qquad R_B = 10ql$$

$$\sum F_{ix} = 0; \qquad R_{Ax} - R_{Cx} = 0 \qquad R_{Ax} = \sqrt{3}ql$$

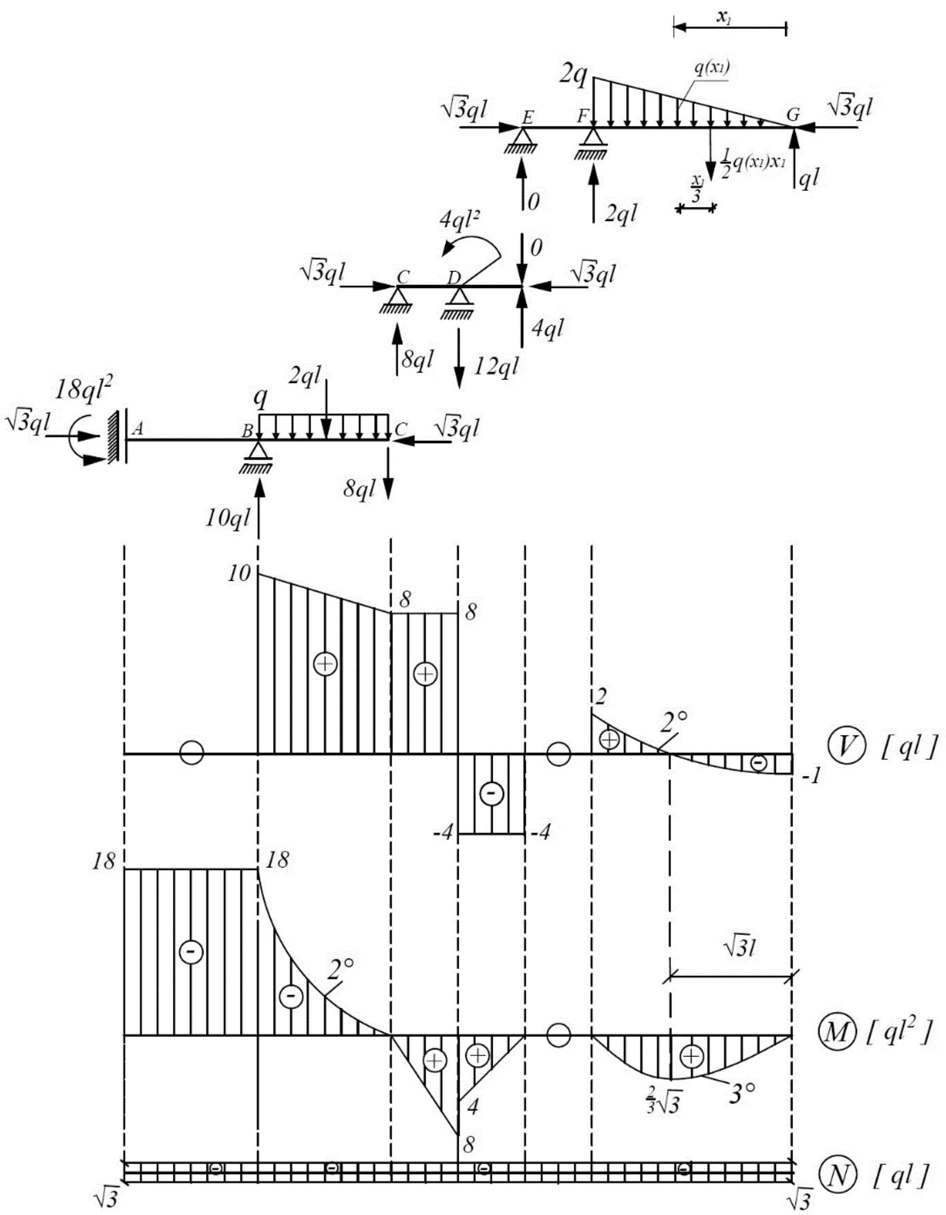


Fig. 2.3.40.

<u>Segment G–F</u> $\quad 0 \le x_1 < 3l$

$$\frac{q(x_1)}{x_1} = \frac{2q}{3l} \quad \rightarrow \quad q(x_1) = \frac{2qx_1}{3l}$$

$$V(x_1) = -ql + \frac{1}{2}q(x_1) \cdot x_1 = -ql + \frac{1}{2} \cdot \frac{2qx_1}{3l} \cdot x_1 = -ql + \frac{q{x_1}^2}{3l}$$

$$V(x_1 = 0) = -ql$$

$$V\left(x_1 = \frac{3}{2}l\right) = -\frac{1}{4}ql$$

$$V(x_1 = 3l) = 2ql$$

$$V(x_1) = 0 \quad \rightarrow -ql + \tfrac{1}{2} \cdot \tfrac{2qx_1}{3l} \cdot x_1 = 0 \quad \rightarrow \quad 3ql^2 - q{x_1}^2 = 0 \quad \rightarrow x_1 = \sqrt{3}l$$

$$M(x_1) = qlx_1 - \frac{1}{2}q(x_1)\cdot x_1 \cdot \frac{x_1}{3} = qlx_1 - \frac{1}{2}\cdot\frac{2qx_1}{3l}\cdot x_1 \cdot \frac{x_1}{3} = qlx_1 - q\cdot\frac{(x_1)^3}{9l}$$

$$M(x_1 = 0) = 0$$

$$M(x_1 = 3l) = 0$$

$$M\left(x_1 = \sqrt{3}l\right) = ql\cdot\sqrt{3}l - q\cdot\frac{\left(\sqrt{3}l\right)^3}{9l} = \frac{2}{3}\sqrt{3}ql^2$$

$$N(x_1) = -\sqrt{3}ql$$

Internal forces diagrams are presented in Fig. 2.3.40.

2.3.5. Review problems – beams

Draw internal forces diagrams for the beams presented below:

Problem 1.

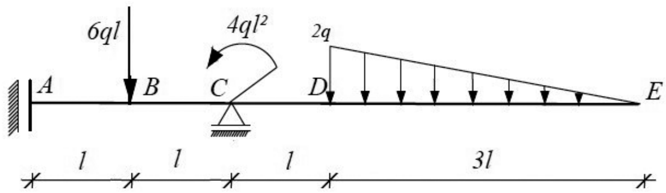


Fig. 2.3.41.

Problem 2.

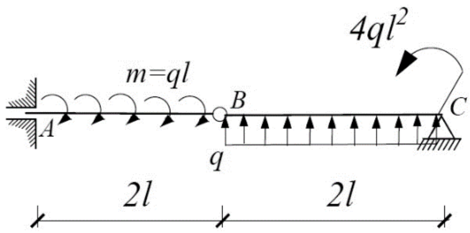


Fig. 2.3.42.

Problem 3.

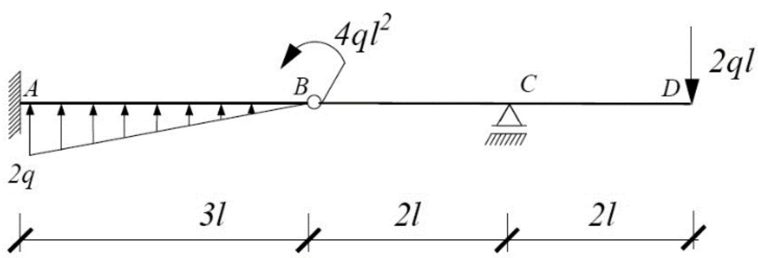


Fig. 2.3.43.

Problem 4.

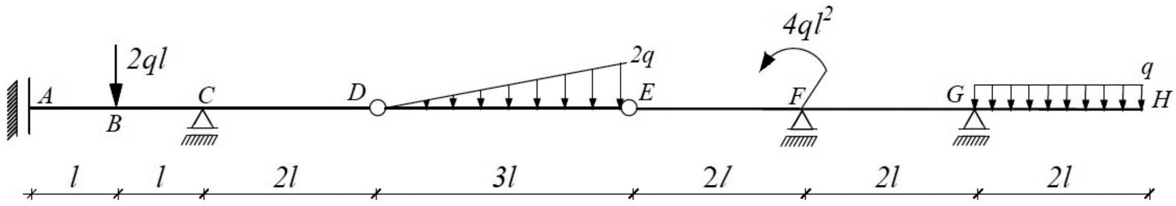


Fig. 2.3.44.

Problem 5.

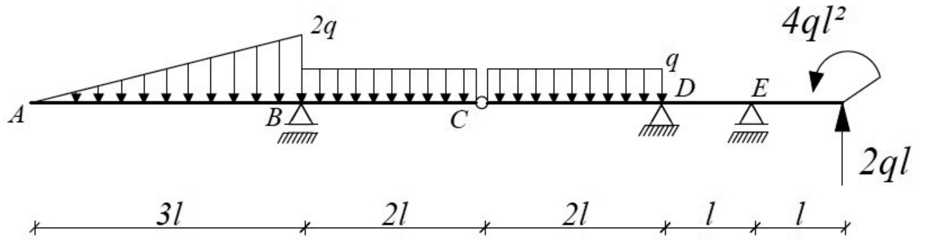


Fig. 2.3.45.

Problem 6.

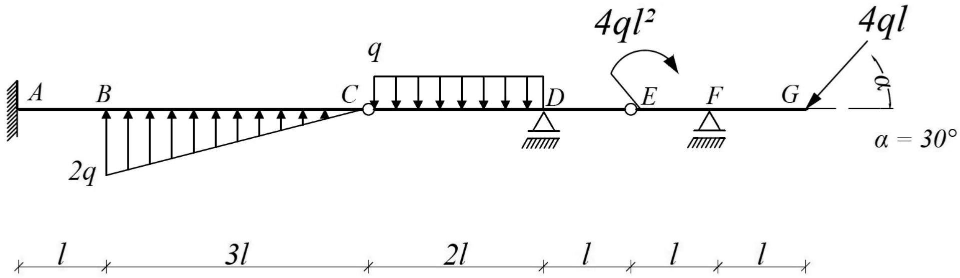


Fig. 2.3.46.

2.4. Internal forces diagrams – planar frames [1], [2], [4]

Frames are structures that consist of vertical, horizontal or inclined members. The vertical members are called columns and the horizontal members are called beams. Frames members can be connected rigidly or by pins (hinges). A planar frame is composed of individual members all of which are located in the same plane. When loaded in this plane, they are subjected to bending, shearing and axial action.

2.4.1. Computational problems – frames

Computational example 2.4.1.

Draw internal forces diagrams for the frame presented in Fig. 2.4.1.

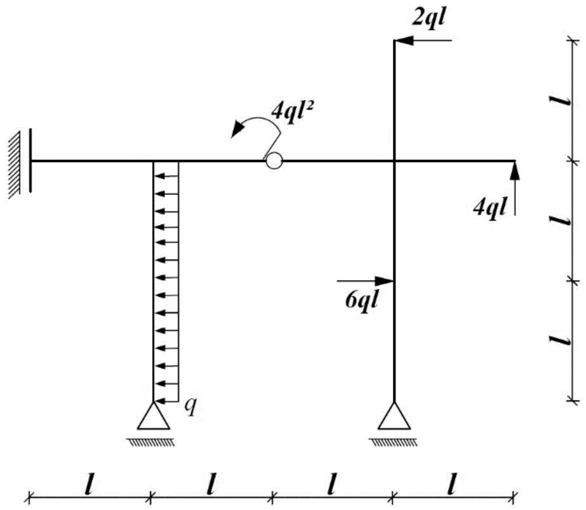


Fig. 2.4.1.

1. Degree of static indeterminacy $n = r + 2s - 3t = 4 + 2 - 3 \cdot 2 = 0$
2. The support reactions:

Compound frame presented in Fig. 2.4.1. has been decomposed in the hinge C into two simple frames (Fig. 2.4.2.).

– the right frame – equations of equilibrium:

$$\sum M_{(C)} = 0; \quad R_A \cdot l + 2ql \cdot l + 4ql \cdot 2l + 6ql \cdot l = 0 \qquad R_A = -16ql$$

$$\sum F_{iy} = 0; \qquad R_A + 4ql + R_{Cy} = 0 \qquad R_{Cy} = 12ql$$

$$\sum F_{ix} = 0; \qquad -R_{cx} - 2ql + 6ql = 0 \qquad R_{Cx} = 4ql$$

– the left frame – equations of equilibrium:

$$\sum F_{iy} = 0; \qquad R_B - R_{Cy} = 0 \qquad R_B = 12ql$$

$$\sum M_{(H)} = 0; \quad -M_D - R_{Cy} \cdot l - 2ql \cdot l + 4ql^2 = 0 \qquad M_D = -10\, ql^2$$

$$\sum F_{ix} = 0; \qquad R_{Cx} - 2ql + R_{Dx} = 0 \qquad R_{Dx} = -2ql$$

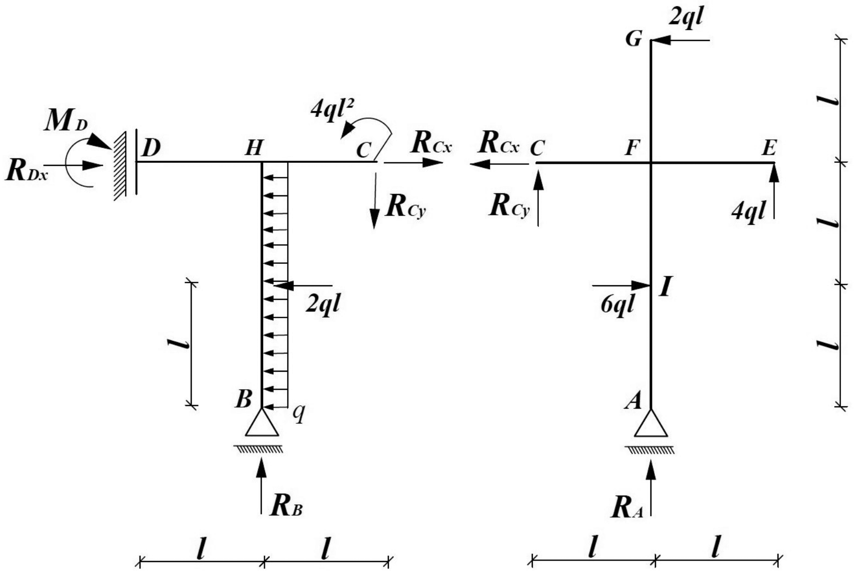


Fig. 2.4.2.

3. Internal forces equations:

Fig. 2.4.3. presents the reactions and loads acting on the frame.
The bottom fibres of the rod members have been marked by using dashed line.

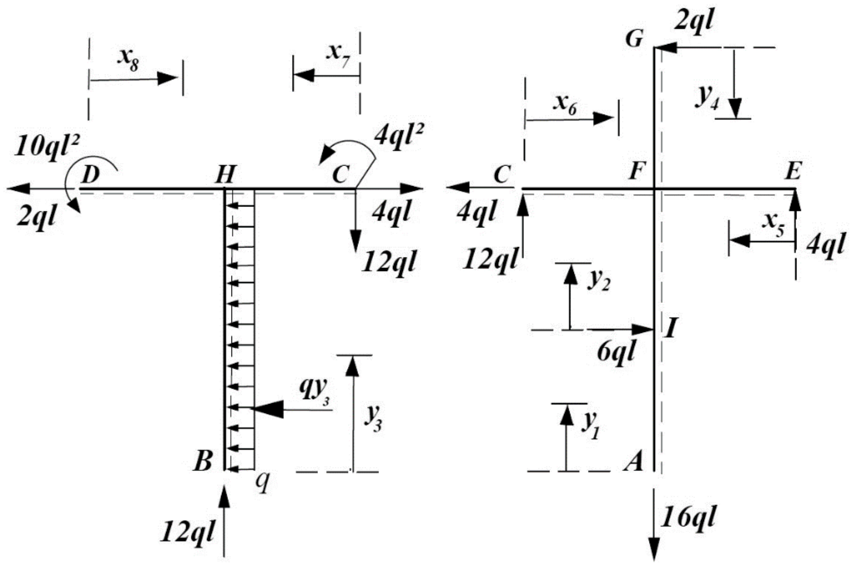


Fig. 2.4.3.

<u>Segment A–I</u> $\quad 0 \le y_1 < l$

$V(y_1) = 0$
$M(y_1) = 0$
$N(y_1) = 16ql$

<u>Segment I–F</u> $\quad 0 \le y_2 < l$

$V(y_2) = -6ql$
$M(y_2) = -6ql \cdot y_2$

$M(y_2 = 0) = 0 \qquad M(y_2 = l) = -6ql^2$

$N(y_2) = 16ql$

<u>Segment B–H</u> $\quad 0 \le y_3 < 2l$

$V(y_3) = qy_3$

$V(y_3 = 0) = 0 \qquad V(y_3 = 2l) = q \cdot 2l = 2ql$

$M(y_3) = qy_3 \cdot \frac{y_3}{2} = q \cdot \frac{(y_3)^2}{2}$

$M(y_3 = 0) = 0 \qquad M(y_3 = 2l) = q \cdot \frac{(2l)^2}{2} = 2ql^2$

$N(y_3) = -12ql$

<u>Segment G–F</u> $\quad 0 \le y_4 < l$

$V(y_4) = -2ql$

$M(y_4) = 2ql \cdot y_4$

$M(y_4 = 0) = 0 \qquad M(y_4 = l) = 2ql \cdot l = 2ql^2$

$N(y_4) = 0$

<u>Segment E–F</u> $\quad 0 \le x_5 < l$

$V(x_5) = -4ql$

$M(x_5) = 4ql \cdot x_5$

$M(x_5 = 0) = 0 \qquad M(x_5 = l) = 4ql^2$

$N(x_5) = 0$

<u>Segment C–F</u> $\quad 0 \le x_6 < l$

$V(x_6) = 12ql$

$M(x_6) = 12ql \cdot x_6$

$M(x_6 = 0) = 0 \qquad M(x_6 = l) = 12ql^2$

$N(x_6) = 4ql$

<u>Segment C–H</u> $\quad 0 \le x_7 < l$

$V(x_7) = 12ql$

$M(x_7) = 4ql^2 - 12ql \cdot x_7$

$M(x_7 = 0) = 4ql^2 \qquad M(x_7 = l) = -8ql^2$

$N(x_7) = 4ql$

<u>Segment D–H</u> $\quad 0 \le x_8 < l$

$V(x_8) = 0$

$M(x_8) = -10ql^2$

$N(x_8) = 2ql$

Internal forces diagrams are presented in Fig. 2.4.4.

Moments verifications – equilibrium of the moments for the F and H nodes (Fig. 2.4.4.).

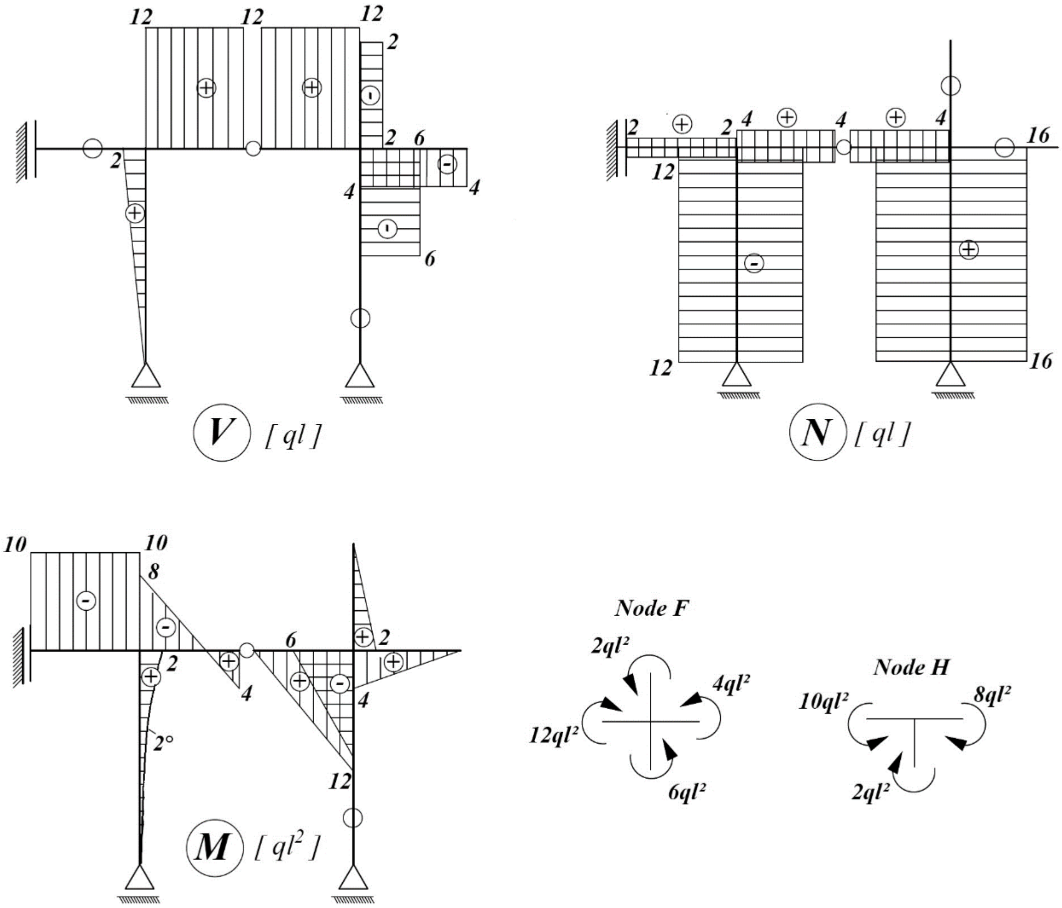


Fig. 2.4.4.

Computational example 2.4.2.

Draw internal forces diagrams for the frame presented in Fig. 2.4.5.

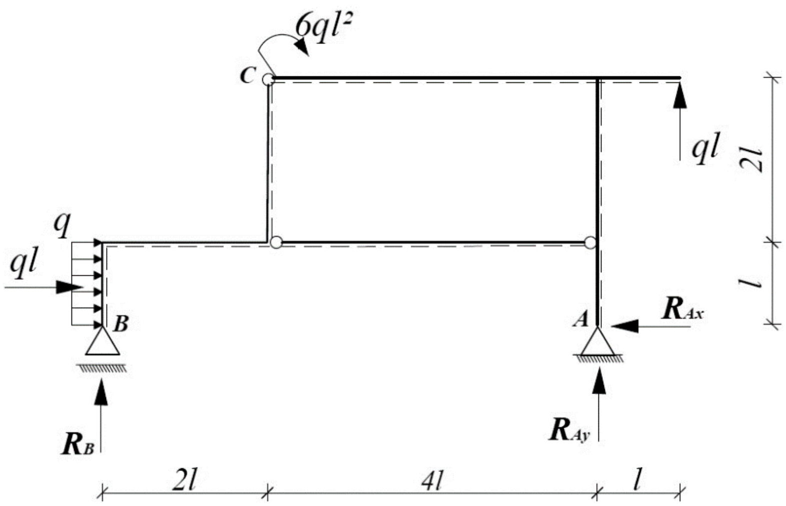


Fig. 2.4.5.

1. Degree of static indeterminacy $n = r + 2s - 3t = 3 + 6 - 3 \cdot 3 = 0$
2. The support reactions:

One of the supports is a roller and another is a pin. Three equations of equilibrium for the planar force system can be used to determine three support reactions (Fig. 2.4.5.).

$\sum M_{(A)} = 0; \qquad -R_B \cdot 6l - ql \cdot \frac{l}{2} + ql \cdot l - 6ql^2 = 0 \qquad R_B = -\frac{11}{12}ql$

$\sum F_{iy} = 0; \qquad R_B + ql + R_{Ay} = 0 \qquad R_{Ay} = -\frac{1}{12}ql$

$\sum F_{ix} = 0; \qquad -R_{Ax} + ql = 0 \qquad R_{Ax} = ql$

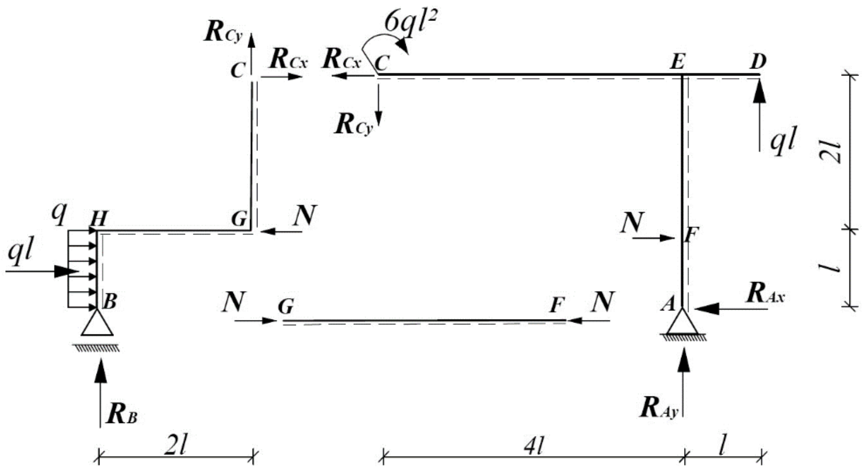


Fig. 2.4.6.

The compound frame presented in Fig. 2.4.5. has been divided at hinges C, G and F into two simple frames and a tie GF (Fig. 2.4.6.). The reactions R_{Cx}, R_{Cy} and N (there is only one force N in the tie) can be calculated by taking into account either the left or right frame.

If we take into account the right one we can use the left part to verify the results. If we take the left one, we use the right for verification.

– the left frame:

$\sum M_{(C)} = 0; \qquad -R_B \cdot 2l + ql \cdot \frac{5l}{2} - N \cdot 2l = 0 \qquad N = \frac{13}{6}ql$

$\sum F_{iy} = 0; \qquad R_B + R_{Cy} = 0 \qquad R_{Cy} = \frac{11}{12}ql$

$\sum F_{ix} = 0; \qquad R_{cx} - N + ql = 0 \qquad R_{Cx} = \frac{7}{6}ql$

3. Internal forces equations

Fig. 2.4.7. presents the frame reactions and loadings.

The bottom fibres of bar members have been marked by using a dashed line.

<u>Segment A–F</u> $\qquad 0 \le y_1 < l$

$V(y_1) = ql$

$M(y_1) = ql \cdot y_1$

$M(y_1 = 0) = 0 \qquad M(y_1 = l) = ql^2$

$$N(y_1) = \frac{1}{12}ql$$

<u>Segment F–E</u> $\qquad 0 \le y_2 < 2l$

$$V(y_2) = ql - \frac{13}{6}ql = -\frac{7}{6}ql$$

$$M(y_2) = ql \cdot (l + y_2) - \frac{13}{6} ql \cdot y_2$$

$M(y_2 = 0) = ql^2 \qquad M(y_2 = 2l) = -\frac{4}{3} ql^2$

$$N(y_2) = \frac{1}{12} ql$$

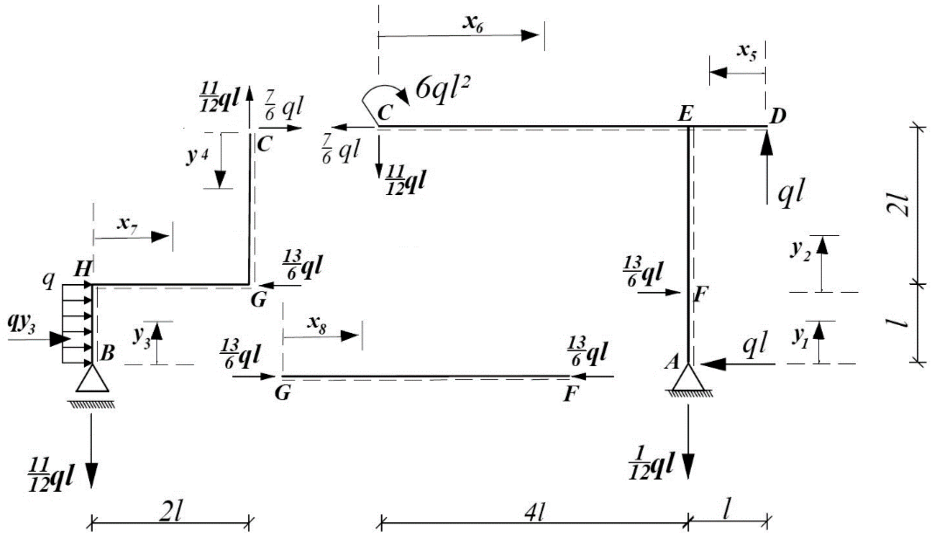


Fig. 2.4.7.

<u>Segment B–H</u> $\quad 0 \le y_3 < l$

$V(y_3) = -qy_3$

$V(y_3 = 0) = 0 \qquad V(y_3 = l) = -ql$

$$M(y_3) = -qy_3 \cdot \frac{y_3}{2} = -q \cdot \frac{(y_3)^2}{2}$$

$M(y_3 = 0) = 0 \qquad M(y_3 = l) = -q \cdot \frac{(l)^2}{2} = -\frac{1}{2} ql^2$

$$N(y_3) = \frac{11}{12} ql$$

<u>Segment C–G</u> $\quad 0 \le y_4 < 2l$

$$V(y_4) = \frac{7}{6} ql$$

$$M(y_4) = -\frac{7}{6} ql \cdot y_4$$

$M(y_4 = 0) = 0 \qquad M(y_4 = 2l) = -\frac{7}{3} ql^2$

$$N(y_4) = \frac{11}{12} ql$$

<u>Segment D–E</u> $\quad 0 \le x_5 < l$

$V(x_5) = -ql$

$M(x_5) = ql \cdot x_5$

$M(x_5 = 0) = 0 \qquad M(x_5 = l) = ql^2$

$N(x_5) = 0$

Segment C–E $\quad 0 \leq x_6 < 4l$

$$V(x_6) = -\frac{11}{12}ql$$

$$M(x_6) = 6ql^2 - \frac{11}{12}ql \cdot x_6$$

$$M(x_6 = 0) = 6ql^2 \qquad M(x_6 = 4l) = \frac{7}{3}ql^2$$

$$N(x_6) = \frac{7}{6}ql$$

Segment H–G $\quad 0 \leq x_7 < 2l$

$$V(x_7) = -\frac{11}{12}ql$$

$$M(x_7) = -\frac{11}{12}ql \cdot x_7 - ql \cdot \frac{1}{2}l$$

$$M(x_7 = 0) = -\frac{1}{2}ql^2 \qquad M(x_7 = 2l) = -\frac{7}{3}ql^2$$

$$N(x_7) = -ql$$

Internal forces diagrams are presented in Fig. 2.4.8.

Moments verifications – equilibrium of the moments for the E node (Fig. 2.4.8.).

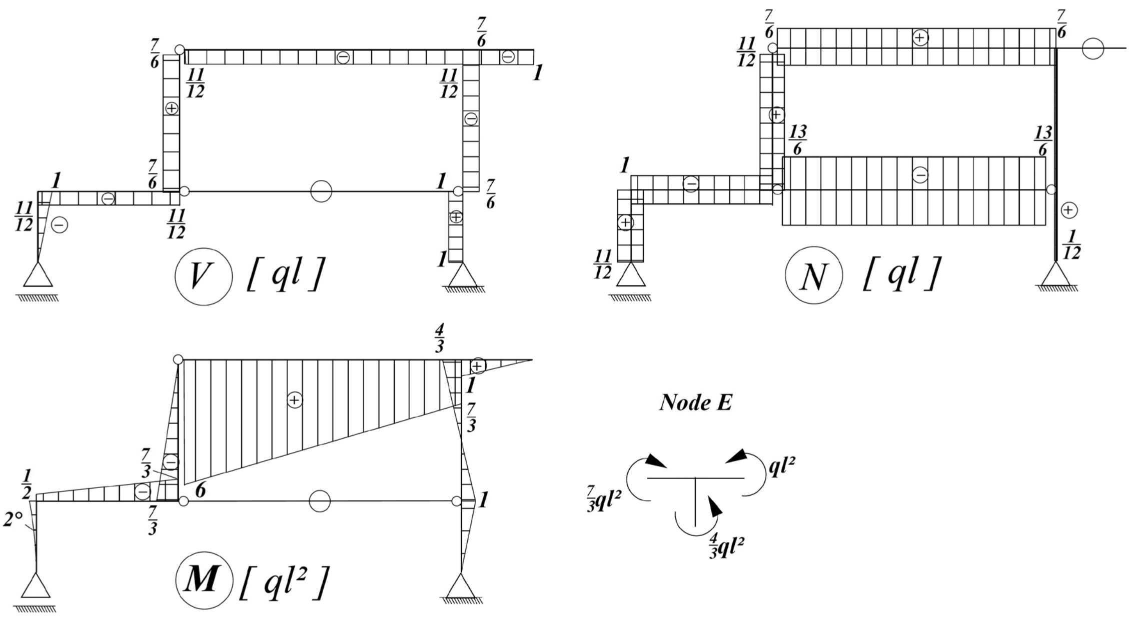


Fig. 2.4.8.

Computational example 2.4.3.

Draw M, V, N diagrams for the frame presented in Fig. 2.4.9.

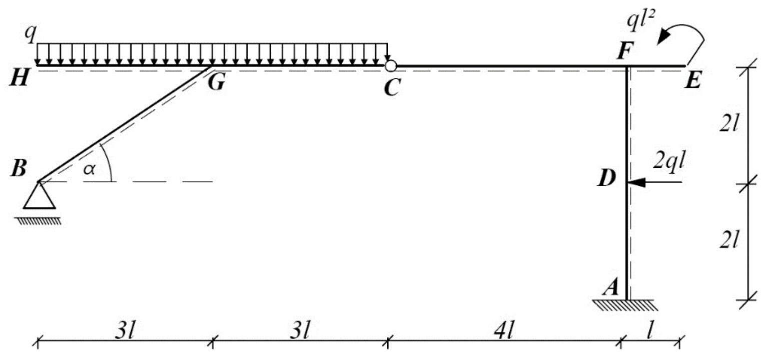


Fig. 2.4.9.

1. Degree of static indeterminacy $n = r + 2s - 3t = 4 + 2 - 3 \cdot 2 = 0$
2. The support reactions:

Compound frame presented in Fig. 2.4.9. has been divided in the hinge C into two simple frames (Fig. 2.4.10.).

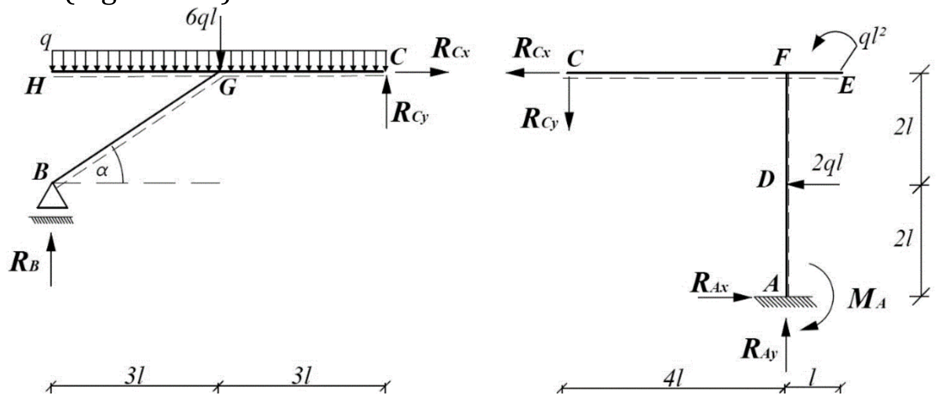


Fig. 2.4.10.

– the left frame – equations of equilibrium:

$\sum M_{(C)} = 0; \qquad -R_B \cdot 6l + 6ql \cdot 3l = 0 \qquad R_B = 3ql$

$\sum F_{iy} = 0; \qquad R_B - 6ql + R_{Cy} = 0 \qquad R_{Cy} = 3ql$

$\sum F_{ix} = 0; \qquad R_{cx} = 0$

– the right frame – equations of equilibrium:

$\sum F_{iy} = 0; \qquad R_{Ay} - R_{Cy} = 0 \qquad R_{Ay} = 3ql$

$\sum M_{(A)} = 0; \quad -M_A + R_{Cy} \cdot 4l + R_{Cx} \cdot 4l + 2ql \cdot 2l + ql^2 = 0 \quad M_A = 17\, ql^2$

$\sum F_{ix} = 0; \qquad -R_{Cx} - 2ql + R_{Ax} = 0 \qquad R_{Ax} = 2ql$

3. Internal forces equations

Fig. 2.4.11. presents the frame reactions and loadings.

<u>Segment A–D</u> $\quad 0 \leq y_1 < 2l$

$V(y_1) = -2ql$

$M(y_1) = 17ql^2 - 2ql \cdot y_1$

$M(y_1 = 0) = 17ql^2 \qquad M(y_1 = 2l) = 13ql^2$

$N(y_1) = -3ql$

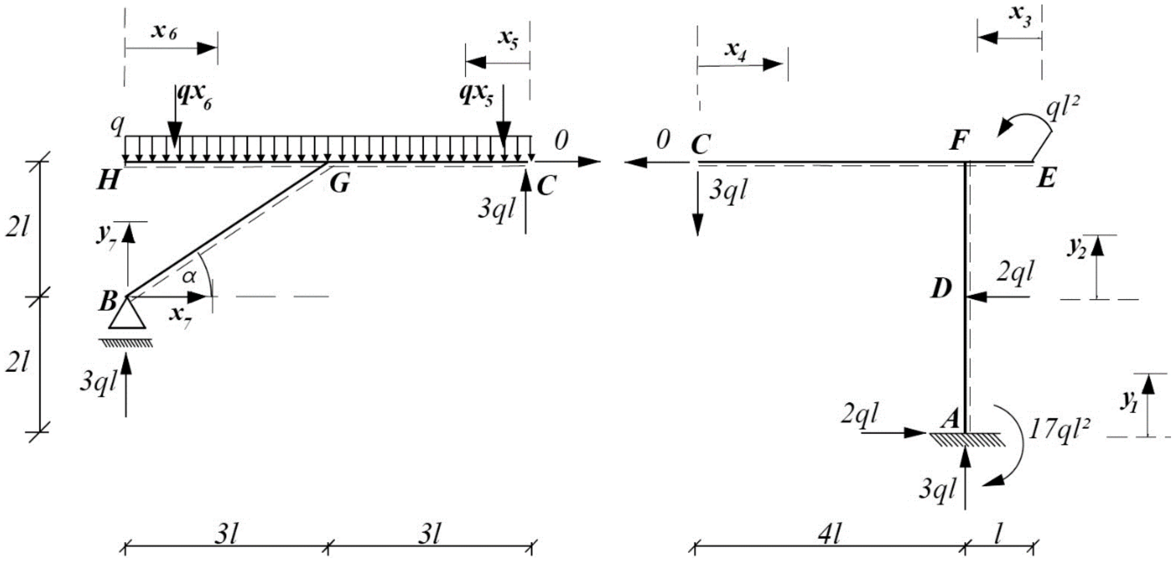


Fig. 2.4.11.

<u>Segment D–F</u> $\quad 0 \le y_2 < 2l$

$V(y_2) = -2ql + 2ql = 0$

$M(y_2) = 17ql^2 - 2ql \cdot (2l + y_2) + 2ql \cdot y_2 = 13ql^2$

$N(y_2) = -3ql$

<u>Segment E–F</u> $\quad 0 \le x_3 < l$

$V(x_3) = 0$

$M(x_3) = ql^2$

$N(x_3) = 0$

<u>Segment C–F</u> $\quad 0 \le x_4 < 4l$

$V(x_4) = -3ql$

$M(x_4) = -3ql \cdot x_4$

$M(x_4 = 0) = 0 \qquad M(x_4 = 4l) = -12ql^2$

$N(x_4) = 0$

<u>Segment C–G</u> $\quad 0 \le x_5 < 3l$

$V(x_5) = -3ql + qx_5$

$V(x_5 = 0) = -3ql \qquad V(x_5 = 3l) = 0$

$$M(x_5) = 3ql \cdot x_5 - qx_5 \cdot \frac{x_5}{2} = 3ql \cdot x_5 - q \cdot \frac{(x_5)^2}{2}$$

$M(x_5 = 0) = 0 \qquad M(x_5 = 3l) = \frac{9}{2}ql^2$

$N(x_5) = 0$

Segment H–G $\quad 0 \le x_6 < 3l$

$$V(x_6) = -qx_6$$

$$V(x_6 = 0) = 0 \qquad V(x_6 = 3l) = -3ql$$

$$M(x_6) = -qx_6 \cdot \frac{x_6}{2} = -q \cdot \frac{(x_6)^2}{2}$$

$$M(x_6 = 0) = 0 \qquad M(x_6 = 3l) = -\tfrac{9}{2}ql^2$$

$$N(x_6) = 0$$

Segment B–G $\quad 0 \le x_7 < 3l, \quad 0 \le y_7 < 2l$

The system of forces in segment B–G is presented in Fig. 2.4.12.

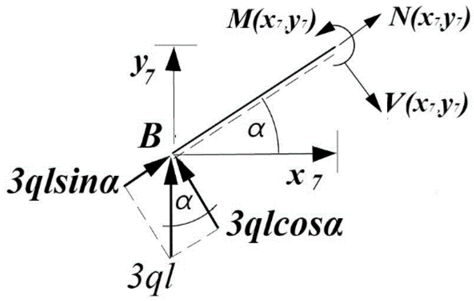


Fig. 2.4.12.

$$cos\alpha = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}, \qquad sin\alpha = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$V(x_7, y_7) = 3qlcos\alpha = 3ql \cdot \frac{3}{\sqrt{13}} = 9\frac{\sqrt{13}}{13}ql$$

$$M(x_7, y_7) = 3ql \cdot x_7$$

$$M(x_7 = 0, y_7 = 0) = 0 \qquad M(x_7 = 3l, y_7 = 2l) = 9ql^2$$

$$N(x_7, y_7) = -3qlsin\alpha = -3ql \cdot \frac{2}{\sqrt{13}} = -6\frac{\sqrt{13}}{13}ql$$

Internal forces diagrams are presented in Fig. 2.4.13.

Moments verifications – equilibrium of the moments for the F and G nodes (Fig. 2.4.13.).

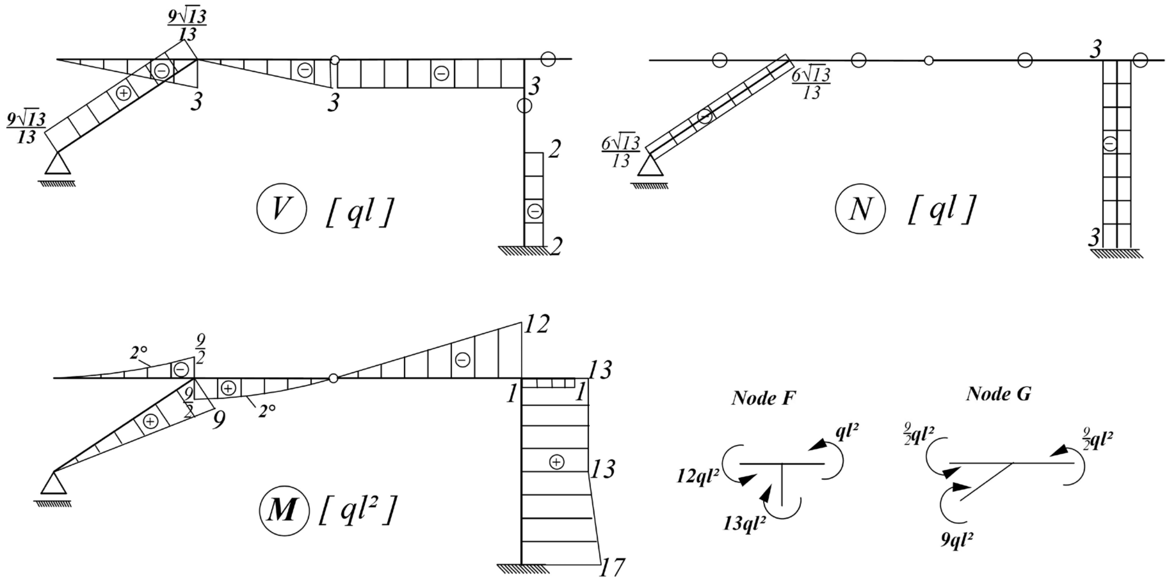


Fig. 2.4.13.

Computational example 2.4.4.

Draw M, V, N diagrams for the frame presented in Fig. 2.4.14.

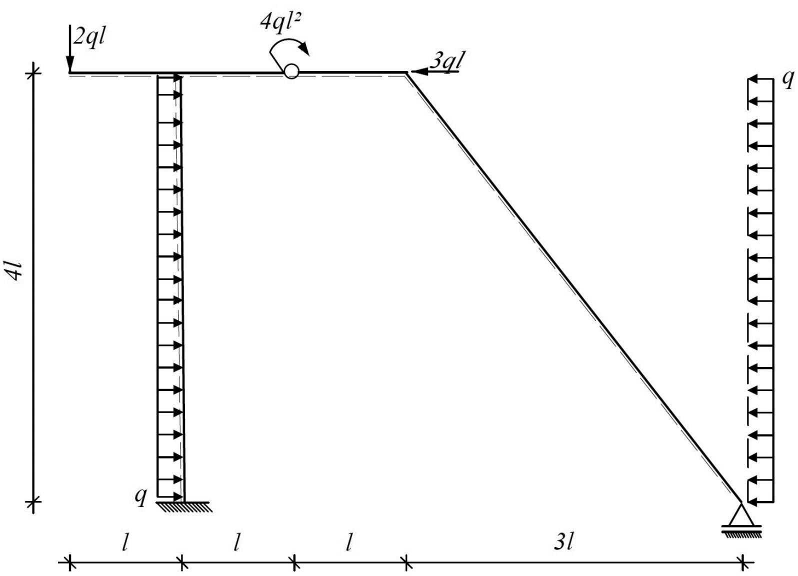


Fig. 2.4.14.

1. Degree of static indeterminacy $n = r + 2s - 3t = 4 + 2 - 3 \cdot 2 = 0$
2. The support reactions:

The compound frame presented in Fig. 2.4.14. has been divided at hinge C into two simple frames (Fig. 2.4.15.).

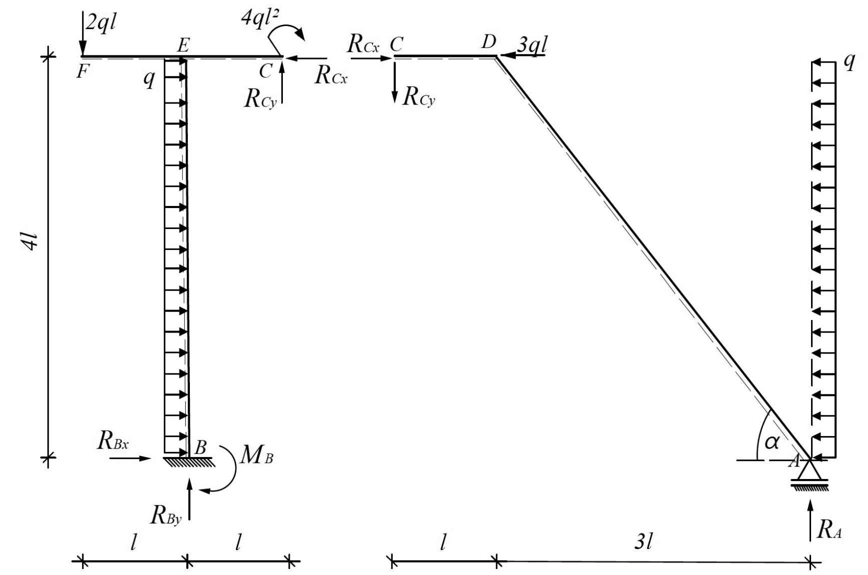


Fig. 2.4.15.

– the right frame:

$\sum M_{(C)} = 0;$ $\quad R_A \cdot 4l - 4ql \cdot 2l = 0$ $\quad R_A = 2ql$

$\sum F_{iy} = 0;$ $\quad R_A - R_{Cy} = 0$ $\quad R_{Cy} = 2ql$

$\sum F_{ix} = 0;$ $\quad -3ql - 4ql + R_{cx} = 0$ $\quad R_{Cx} = 7ql$

– the left frame:

$\sum F_{iy} = 0;$ $\quad R_{By} - 2ql + R_{Cy} = 0$ $\quad R_{By} = 0$

$\sum M_{(B)} = 0;$ $\quad -M_B + R_{Cy} \cdot l - 4ql^2 + R_{Cx} \cdot 4l + 2ql \cdot l - 4ql \cdot 2l = 0$ $\quad M_A = 20\, ql^2$

$\sum F_{ix} = 0;$ $\quad -R_{Cx} + 4ql + R_{Bx} = 0$ $\quad R_{Bx} = 3ql$

Fig. 2.4.16. presents the frame reactions and loadings.

3. Internal forces equations

Segment C–E $\quad 0 \leq x_3 < l$

$V(x_3) = -2ql$

$M(x_3) = 2ql \cdot x_3 - 4ql^2$

$M(x_3 = 0) = -4ql^2$ $\qquad M(x_3 = l) = -2ql^2$

$N(x_3) = -7ql$

Segment F–E $\quad 0 \leq x_4 < l$

$V(x_4) = -2ql$

$M(x_4) = -2ql \cdot x_4$

$M(x_4 = 0) = 0$ $\qquad M(x_4 = l) = -2ql^2$

$N(x_4) = 0$

Segment B–E $\quad 0 \le y_5 < 4l$

$V(y_5) = -3ql - qy_5$

$V(y_5 = 0) = -3ql \qquad V(y_5 = 4l) = -7ql$

$M(y_5) = -20ql^2 + 3ql \cdot y_5 + qy_5 \cdot \frac{y_5}{2} = -20ql^2 + 3ql \cdot y_5 + q \cdot \frac{(y_5)^2}{2}$

$M(y_5 = 0) = -20ql^2$

$M(y_5 = 4l) = 0$

$N(y_5) = 0$

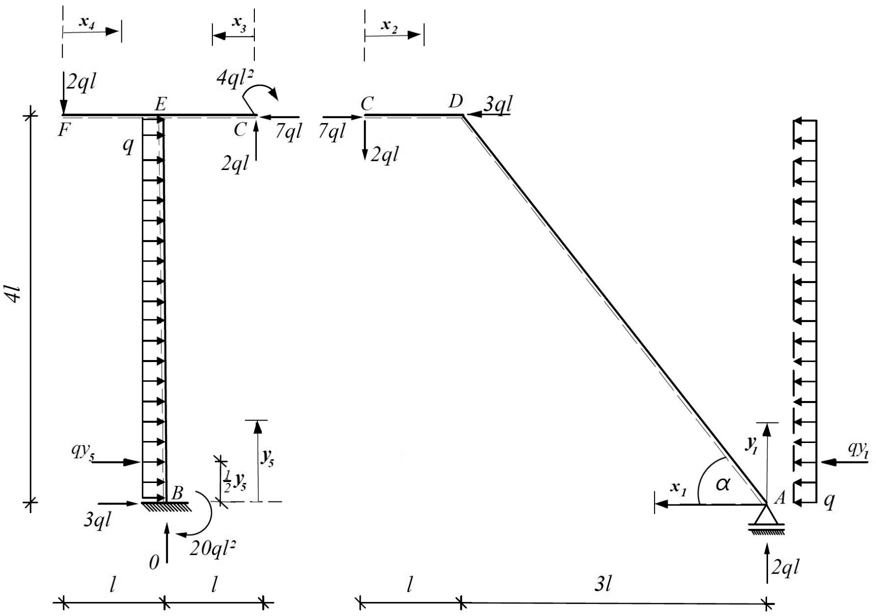


Fig. 2.4.16.

Segment A–D $\quad 0 \le x_1 < 3l, \quad 0 \le y_1 < 4l$

The system of forces in segment A–D is presented in Fig. 2.4.17.

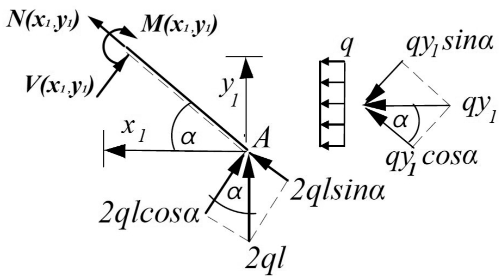


Fig. 2.4.17.

$$cos\alpha = \frac{3}{5}, \quad sin\alpha = \frac{4}{5}$$

$$V(x_1, y_1) = -2qlcos\alpha + qy_1 sin\alpha = -2ql\frac{3}{5} + qy_1\frac{4}{5}$$

$$V(x_1, y_1) = 0 \quad \rightarrow -2ql\frac{3}{5} + qy_1\frac{4}{5} = 0 \ \rightarrow \ y_1 = \frac{3}{2}l \ \rightarrow \ x_1 = \frac{3}{4}y_1 = \frac{9}{8}l$$

$$V(x_1 = 0, \ \ y_1 = 0) = -\frac{6}{5}\,ql$$

$$V(x_1 = 3l, \ \ y_1 = 4l) = 2ql$$

$$M(x_1, y_1) = 2ql \cdot x_1 - qy_1 \cdot \frac{y_1}{2} = 2ql \cdot x_1 - q \cdot \frac{(y_1)^2}{2}$$

$$M(x_1 = 0, \ \ y_1 = 0) = 0$$

$$M(x_1 = 3l, \ \ y_1 = 4l) = -2ql^2$$

$$M\left(x_1 = \frac{9}{8}l, \ \ y_1 = \frac{3}{2}l\right) = \frac{9}{8}\,ql^2$$

$$N(x_1, y_1) = -2qlsin\alpha - qy_1 cos\alpha = -2ql \cdot \frac{4}{5} - qy_1\frac{3}{5}$$

$$N(x_1 = 0, \ \ y_1 = 0) = -\frac{8}{5}\,ql$$

$$N(x_1 = 3l, \ \ y_1 = 4l) = -4ql$$

<u>Segment C–D</u> $\quad 0 \le x_2 < l$

$$V(x_2) = -2ql$$

$$M(x_2) = -2ql \cdot x_2$$

$$M(x_2 = 0) = 0 \qquad M(x_2 = l) = -2ql^2$$

$$N(x_2) = -7ql$$

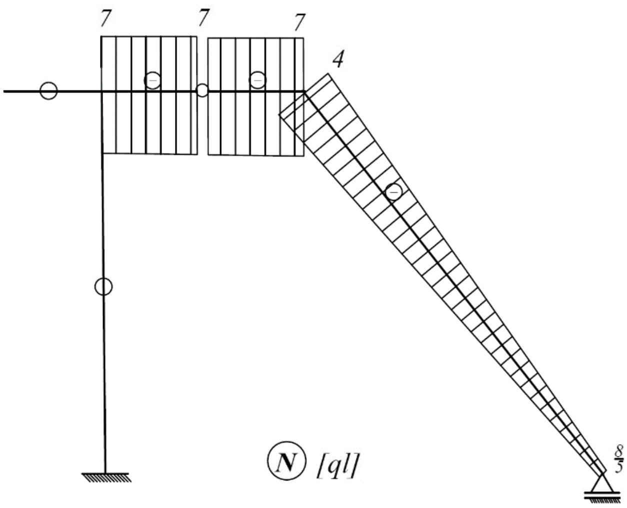


Fig. 2.4.18a.

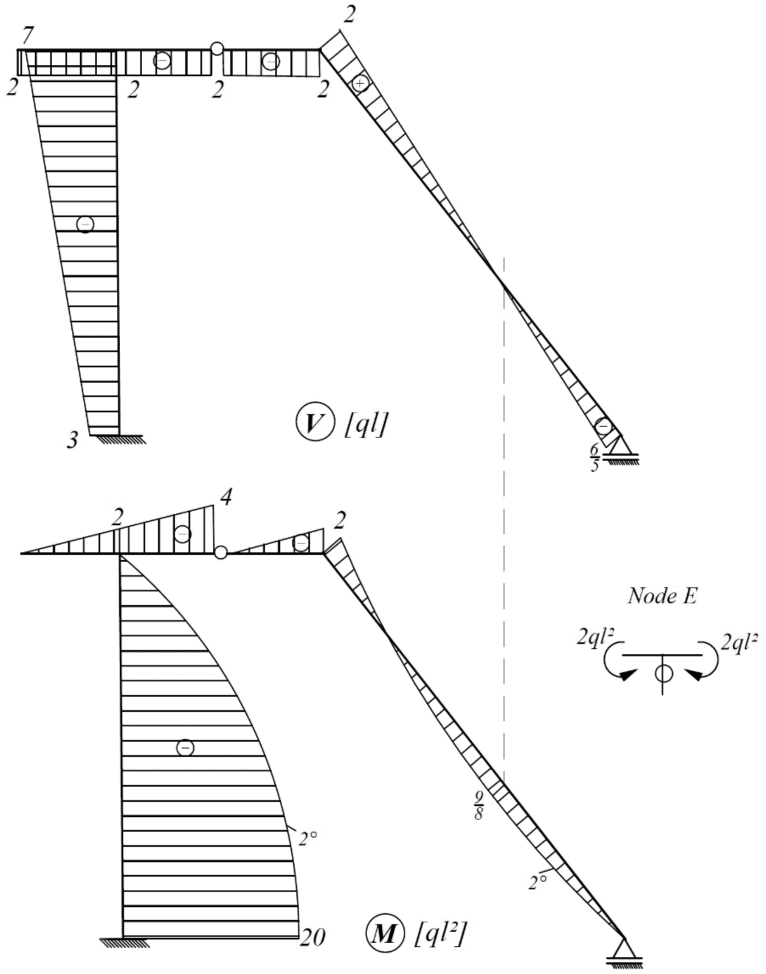


Fig. 2.4.18b.

Internal forces diagrams are presented in Fig. 2.4.18.

Verification of the results – equilibrium of the moments for the E node (Fig. 2.4.18.).

Computational example 2.4.5.

Draw M, V, N diagrams for the frame presented in Fig. 2.4.19.a.

1. Degree of static indeterminacy $n = r + 2s - 3t = 3 + 6 - 3 \cdot 3 = 0$
2. The support reactions:

   Equilibrium equations for the frame (Fig. 2.4.19.b.):

   $\sum F_{ix} = 0;$ $\quad R_{Bx} + 2ql - 3ql = 0$ $\quad R_{Bx} = ql$

   $\sum M_{(A)} = 0;$ $\quad -R_{By} \cdot 7l - 4ql \cdot 10l - 2ql \cdot 6l + 3ql \cdot \frac{3}{2}l - ql^2 + R_{Bx} \cdot l = 0$

   $$R_{By} = -\frac{95}{14}ql$$

   $\sum F_{iy} = 0;$ $\quad R_{By} + 4ql + R_A = 0$ $\quad R_A = \frac{39}{14}ql$

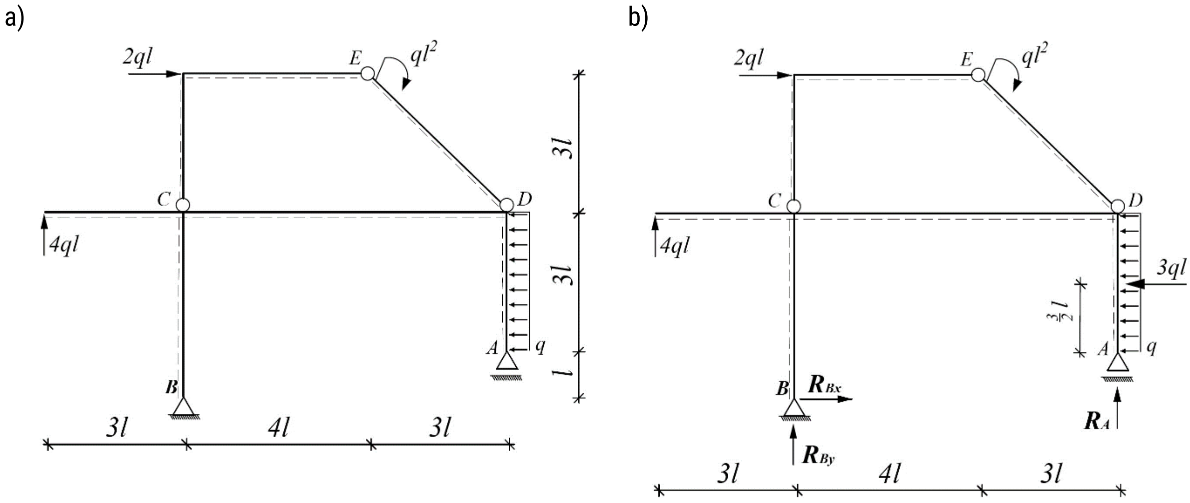


Fig. 2.4.19.

The compound frame presented in Fig. 2.4.19. has been separated at hinges C, D into two frames (Fig. 2.4.20.a.).

The upper frame is at three-hinged frame. The three-hinged frame has two support reactions at each pin support. It is possible to write only three equilibrium equations for the whole upper frame but there are four reactions. It is necessary to find one more equation. The best equation in this case is a moment equation about the middle hinge. We can decompose the upper frame into two simple frames (Fig. 2.4.20.b.) and calculate the moment about hinge E (either for the left or for the right frame). If we take the right one we can use the left part to verify the results. If we take the left one, we use the right for verification.

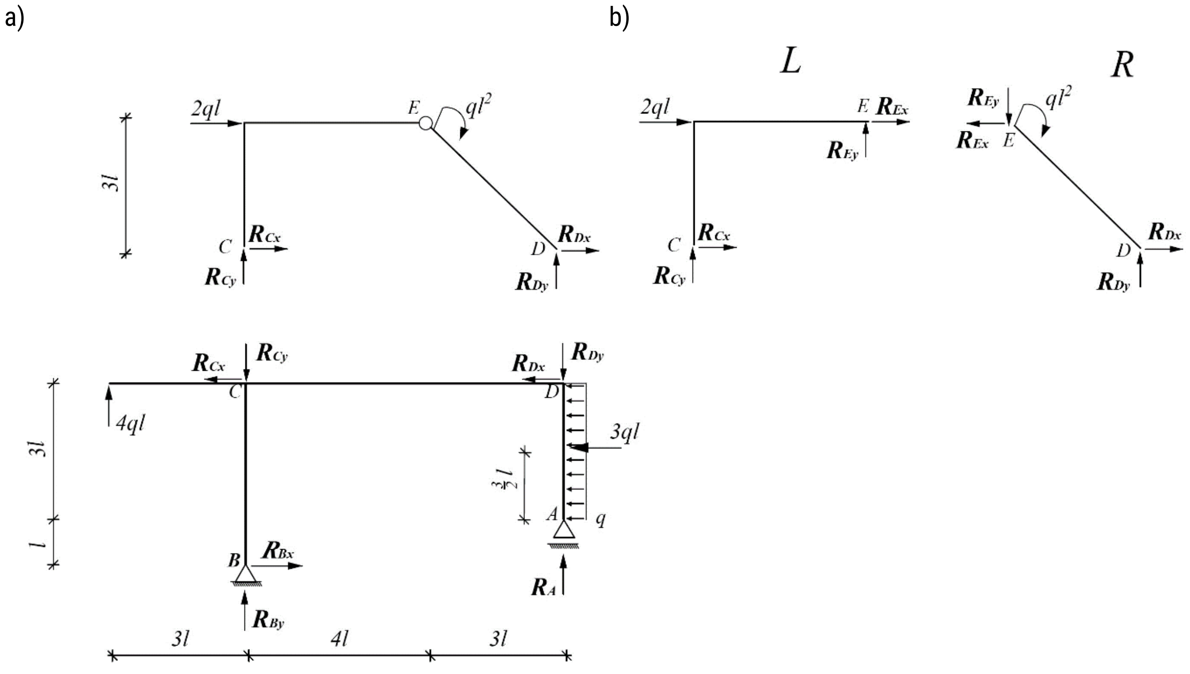


Fig. 2.4.20.

– frame DCE:

$\sum M_{(D)} = 0;$ $\quad -R_{Cy} \cdot 7l - 2ql \cdot 3l - ql^2 = 0$ $\quad R_{Cy} = -ql$

$\sum F_{iy} = 0;$ $\quad R_{Dy} + R_{Cy} = 0$ $\quad R_{Dy} = ql$

$\sum M_E^L = 0;$ $\quad -R_{Cy} \cdot 4l + R_{Cx} \cdot 3l = 0$ $\quad R_{Cx} = -\frac{4}{3}ql$

$\sum F_{iy}{}^L = 0;$ $\quad R_{Ey} + R_{Cy} = 0$ $\quad R_{Ey} = ql$

$\sum F_{ix}{}^L = 0;$ $\quad R_{cx} + R_{Ex} + 2ql = 0$ $\quad R_{Ex} = -\frac{2}{3}ql$

$\sum F_{ix} = 0;$ $\quad R_{cx} + R_{Dx} + 2ql = 0$ $\quad R_{Dx} = -\frac{2}{3}ql$

Fig. 2.4.21. presents the frame reactions and loadings.

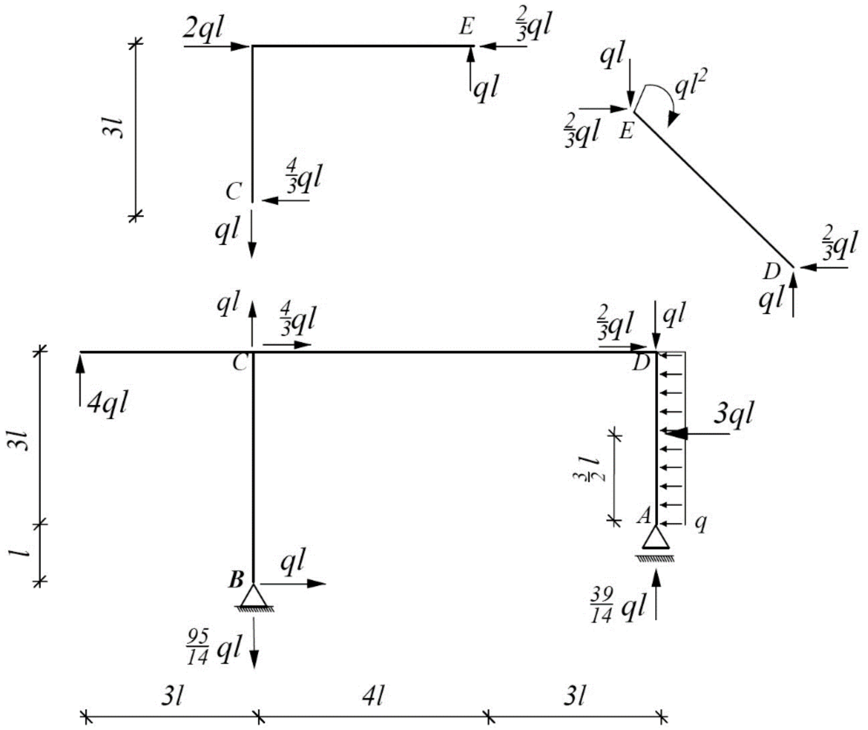


Fig. 2.4.21.

Internal forces diagrams are presented in Fig. 2.4.22.

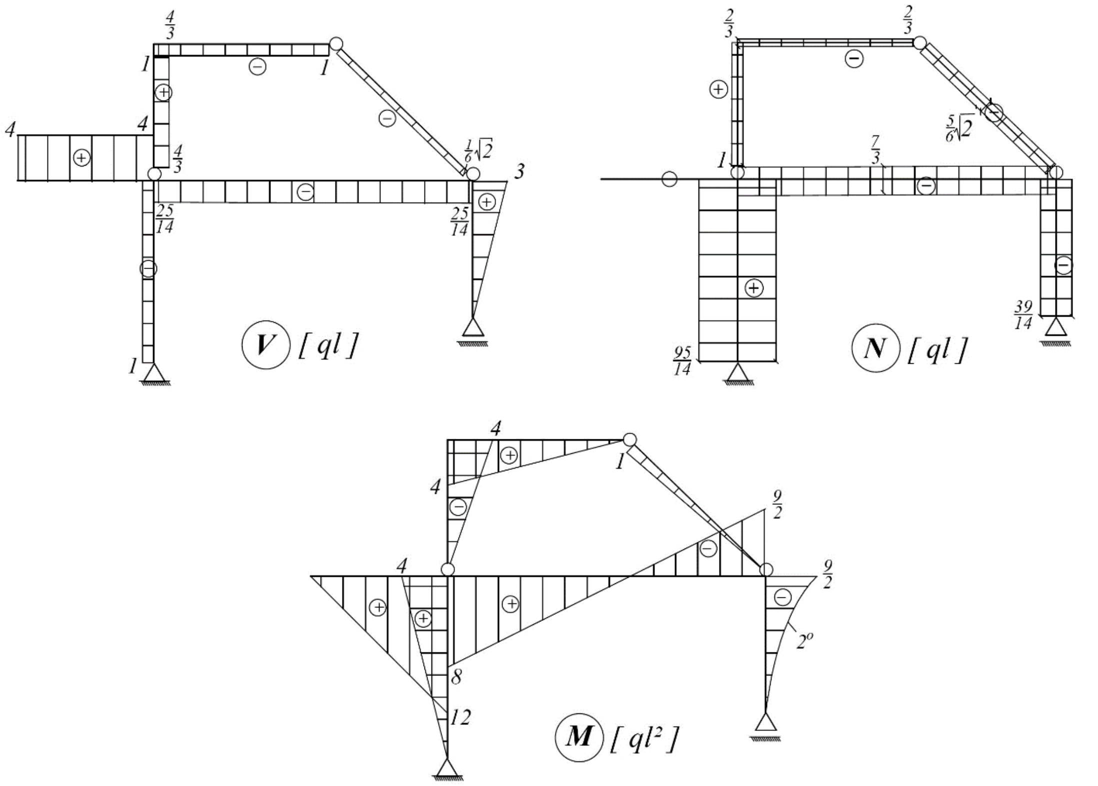


Fig. 2.4.22.

2.4.2. Review problems – frames

Draw internal forces diagrams for the frames presented below:

Problem 1.

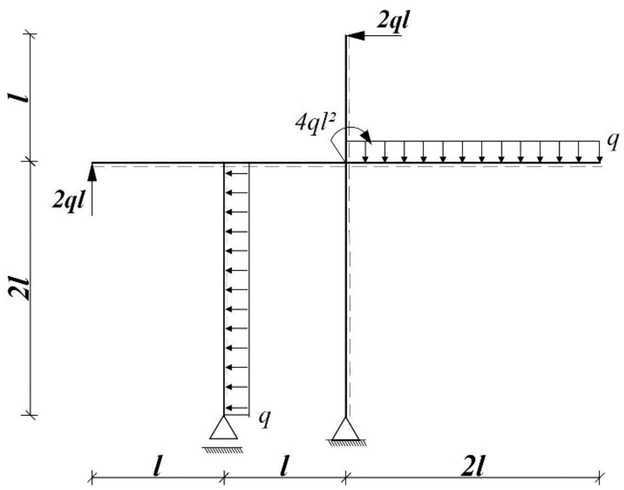


Fig. 2.4.23.

Problem 2.

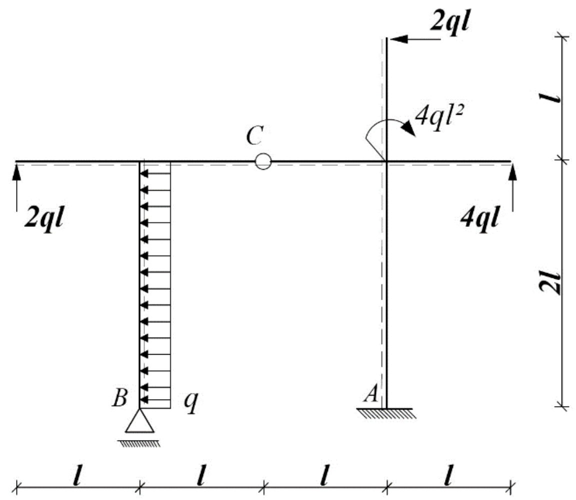


Fig. 2.4.24.

Problem 3.

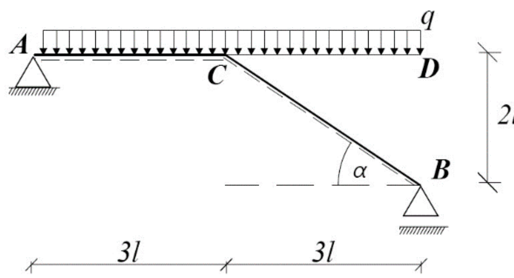


Fig. 2.4.25.

Problem 4.

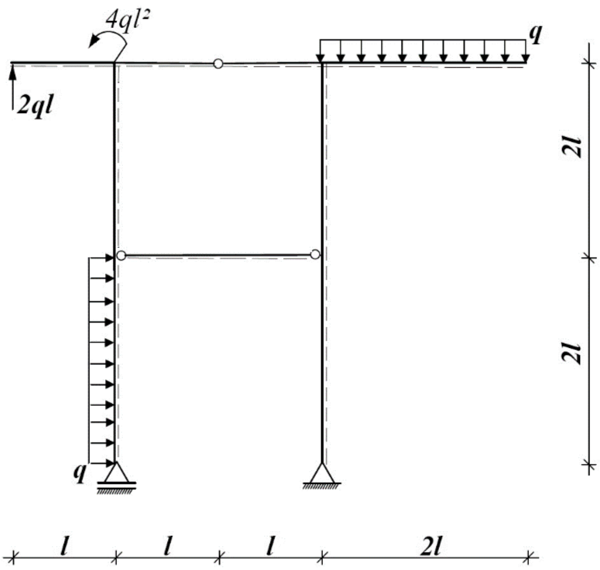


Fig. 2.4.26.

Problem 5.

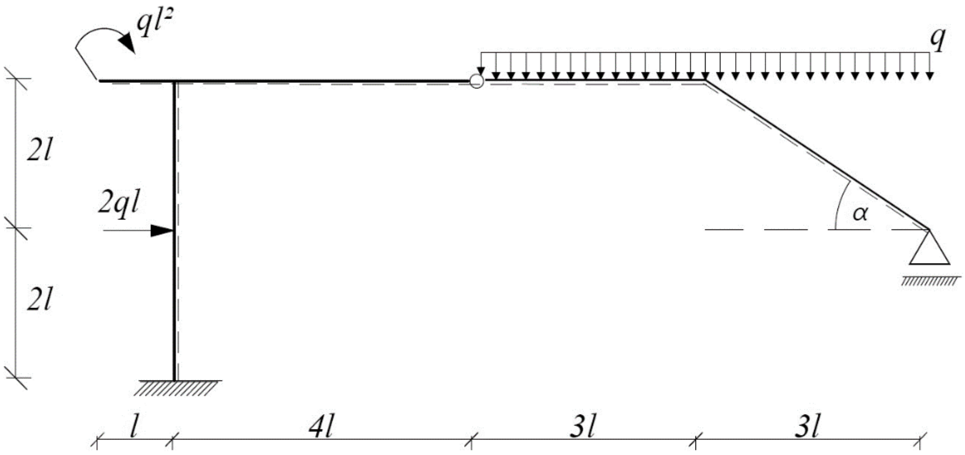


Fig. 2.4.27.

2.5. Statically determinate curved members [1], [2], [4]

2.5.1. Introduction to arches [2]

Arches have been used for a very long time to span large distances in buildings and bridges.

Taking into account the height of the arch, we can divide arches into flat $(f \leq \frac{l}{5})$ and tall $(f \geq \frac{l}{5})$ ones, where f-rise (hight), l-span.

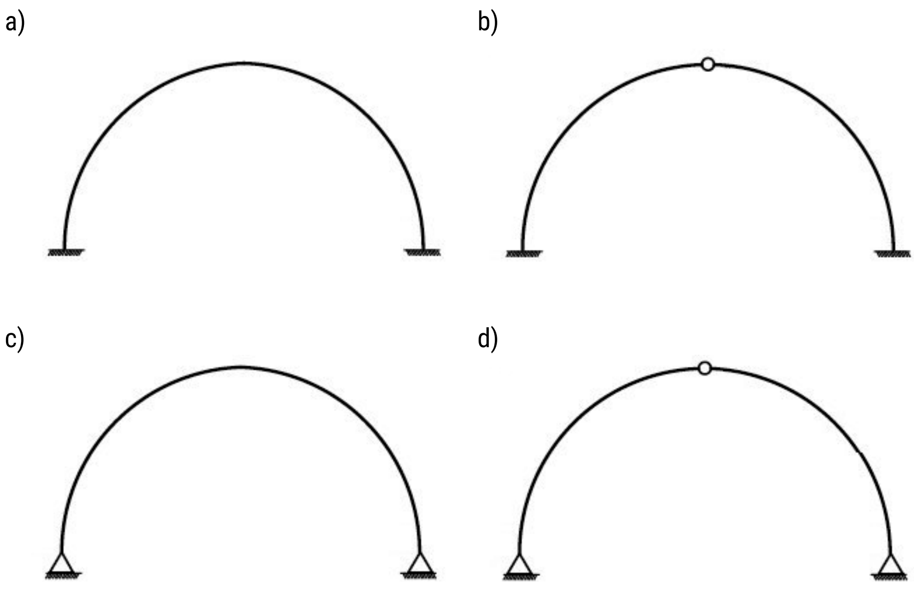


Fig. 2.5.1.

Model of an arch structure is a curved member restrained at its ends with a combination of fixed, hinged, and roller supports. Fig. 2.5.1. illustrates various types of arches due to their supports systems and connections:

- fixed-fixed arch (Fig. 2.5.1.a.),
- single-hinged arch (Fig. 2.5.1.b.),

- two-hinged arch (Fig. 2.5.1.c.),
- three-hinged arch (Fig. 2.5.1.d.).

The three-hinged arch is a statically determinate structure and its reactions and internal forces can be evaluated by static equations of equilibrium. The single-hinged arch, two-hinged arch and fixed-fixed arch are statically indeterminate structures.

Types of arch due to the curvature of the arch:

- parabolic arch (Fig. 2.5.2.):

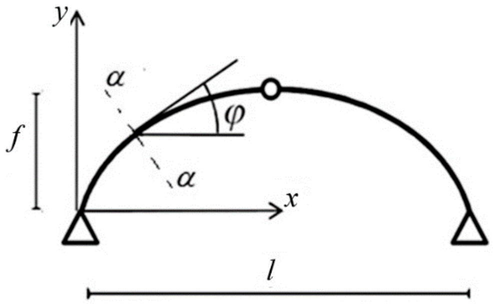


Fig. 2.5.2.

Equation of the central line for a parabolic arch is presented below. Ordinate y of any point of the central line of the parabolic arch (Fig. 2.5.2.) can be calculated by the formula:

$$y = \frac{4f}{l^2} \cdot x \cdot (l - x), \tag{2.11}$$

Hence slope can be expressed by:

$$\frac{dy}{dx} = tg\varphi = \frac{4f}{l^2} \cdot (l - 2x) \Rightarrow \varphi = arctg\left[\frac{4f}{l^2} \cdot (l - 2x)\right] \tag{2.12}$$

- circular arch (Fig. 2.5.3.):

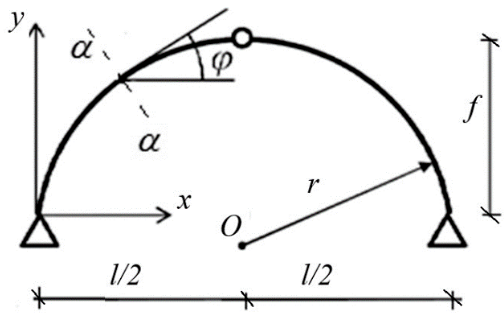


Fig. 2.5.3.

Equation of the central line for circular arch (Fig. 2.5.3.) has the form:

$$y = f - r + \sqrt{r^2 - \left(x - \frac{l}{2}\right)^2} \tag{2.13}$$

and slope can be expressed by:

$$\frac{dy}{dx} = tg\varphi = \frac{l-2x}{2\sqrt{r^2-\left(x-\frac{l}{2}\right)^2}} \Rightarrow \varphi = arctg\left[\frac{l-2x}{2\sqrt{r^2-\left(x-\frac{l}{2}\right)^2}}\right] \tag{2.14}$$

- elliptical arch (Fig. 2.5.4.):

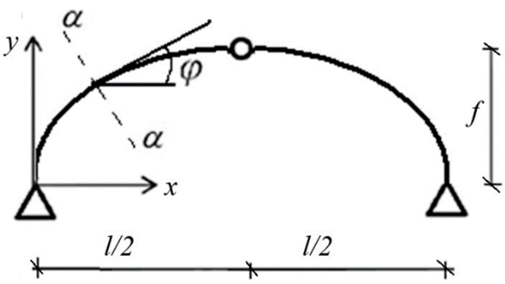


Fig. 2.5.4.

Equation of the central line for elliptical arch is presented below. Ordinate y of any point of the central line of the elliptical arch (Fig. 2.5.4.) can be calculated by the formula:

$$y = \sqrt{f^2 - \frac{4f^2 \cdot \left(x-\frac{l}{2}\right)^2}{l^2}} \tag{2.15}$$

Slope can be written as:

$$\frac{dy}{dx} = tg\varphi = \frac{-f\cdot(2x-l)}{l\cdot\sqrt{-x\cdot(x-l)}} \quad \Rightarrow \quad \varphi = arctg\left[\frac{-f\cdot(2x-l)}{l\cdot\sqrt{-x\cdot(x-l)}}\right] \tag{2.16}$$

2.5.2. Arches – internal forces [1], [4]

In this chapter, the general solution for the internal forces in a planar curved member will be analyzed.

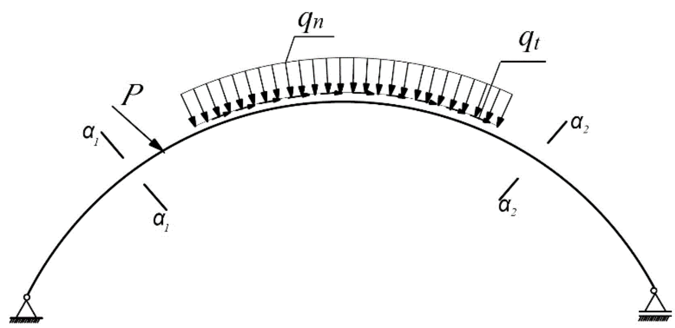


Fig. 2.5.5.

The internal forces in a section of a body are those forces which hold together two parts of a given body separated by the section. Both parts of the body remain in equilibrium.

The basic idea of internal forces has been described in chapter 2.1. and chapter 2.2.

Fig. 2.5.5. presents the simply supported arch under planar load. In a statically determinate arch, we can calculate the support reactions from the three equations of equilibrium (similar to beams or frames).

If we interest the arch with a section α_1, we have three internal forces in this cross-section: normal (axial) force $N_{\alpha 1}$, shear (transversal) force $V_{\alpha 1}$ and bending moment $M_{\alpha 1}$. Their positive positions are shown in Fig. 2.5.6.a. If we make a cross-section α_2, three internal forces will appear in it: normal (axial) force $N_{\alpha 2}$, shear (transversal) force $V_{\alpha 2}$ and bending moment $M_{\alpha 2}$ (Fig. 2.5.6.a.).

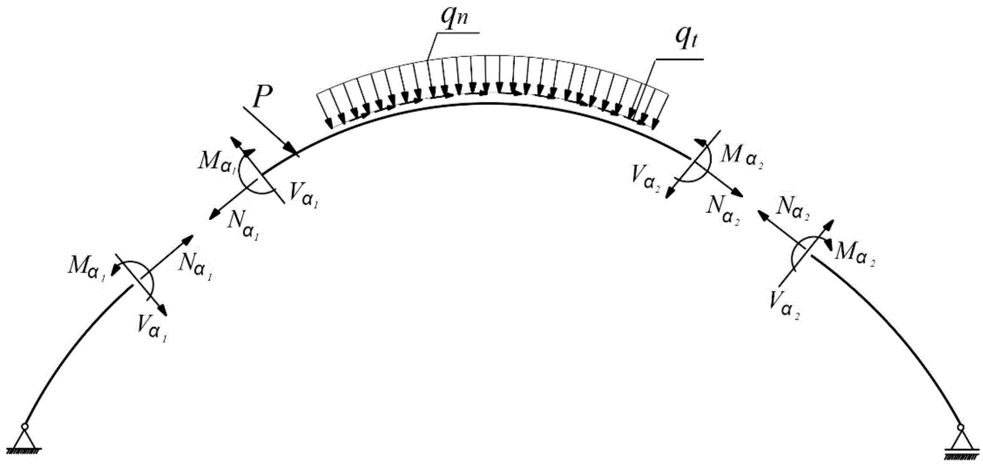


Fig. 2.5.6.a.

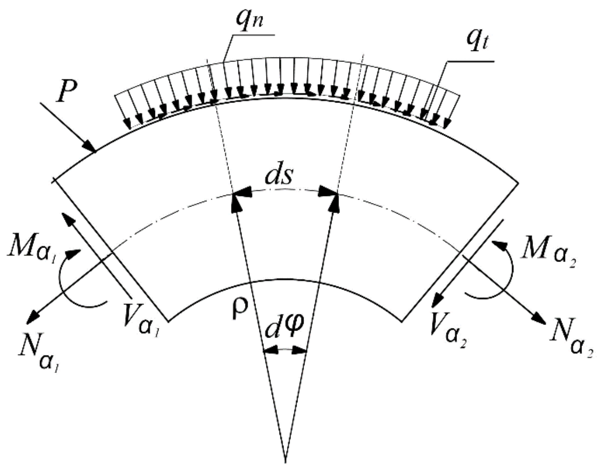


Fig. 2.5.6.b.

As one can see the internal forces (shear and normal) that arise after cutting the bar (section $\alpha_1 - \alpha_1$ and $\alpha_2 - \alpha_2$), which are parallel and perpendicular to the axis of the bar, will change their position along with the curved line of the arch.

Actually, if we know the support reactions and loads we just needs to compose the three equilibrium equations for the left (or right) part of the arch and will find the values of the internal forces.

The equations depend on the adopted coordinate system We can perform calculations either with respect to Cartesian coordinates (x, y) or with respect to polar coordinates (ρ, φ).

2.5.3. Relationships between loads, shear and moment for arch with respect to polar coordinates [1], [4]

Let's take into account the arch member shown in Fig. 2.5.6.a. and its differential element (Fig. 2.5.6.b.). The external loadings have been reduced to the arch central line (Fig. 2.5.7.), hence, there is also the distributed moment $m = q_t \frac{h}{2}$.

Considering V_α, N_α and M_α to be functions of φ, expanding these variables in terms of their differentials, and retaining up to first order terms we have the forces shown in Fig. 2.5.7. One can write three equations of equilibrium for differential arch elements:

$$\sum F_{in} = 0; \quad (2.17)$$

$$-(V_\alpha + dV_\alpha)\cos\frac{d\varphi}{2} + V_\alpha \cos\frac{d\varphi}{2} - N_\alpha \sin\frac{d\varphi}{2} - (N_\alpha + dN_\alpha)\sin\frac{d\varphi}{2} - q_n ds = 0 \quad (2.18)$$

$$-dV_\alpha - 2N_\alpha \frac{ds}{2\varrho} - q_n ds = 0 \quad (2.19)$$

$$\sum F_{it} = 0; \quad (2.20)$$

$$(N_\alpha + dN_\alpha)\cos\frac{d\varphi}{2} - N_\alpha \cos\frac{d\varphi}{2} - V_\alpha \sin\frac{d\varphi}{2} - (V_\alpha + dV_\alpha)\sin\frac{d\varphi}{2} + q_t ds = 0 \quad (2.21)$$

$$dN_\alpha - 2V_\alpha \frac{ds}{2\varrho} + q_t ds = 0 \quad (2.22)$$

$$\sum M_{(c)} = 0; \quad (2.23)$$

$$-M_\alpha + (M_\alpha + dM_\alpha) - V_\alpha \varrho \tan\frac{d\varphi}{2} - (V_\alpha + dV_\alpha)\varrho \tan\frac{d\varphi}{2} - mds = 0 \quad (2.24)$$

$$-dM_\alpha + V_\alpha ds + mds = 0 \quad (2.25)$$

$d\varphi \cong 0$, hence $\sin d\varphi \cong d\varphi$; $\cos d\varphi \cong 1$ $\left(\sin\frac{d\varphi}{2} \cong \frac{d\varphi}{2};\ \cos\frac{d\varphi}{2} \cong 1 \right)$ $ds = \rho d\varphi$

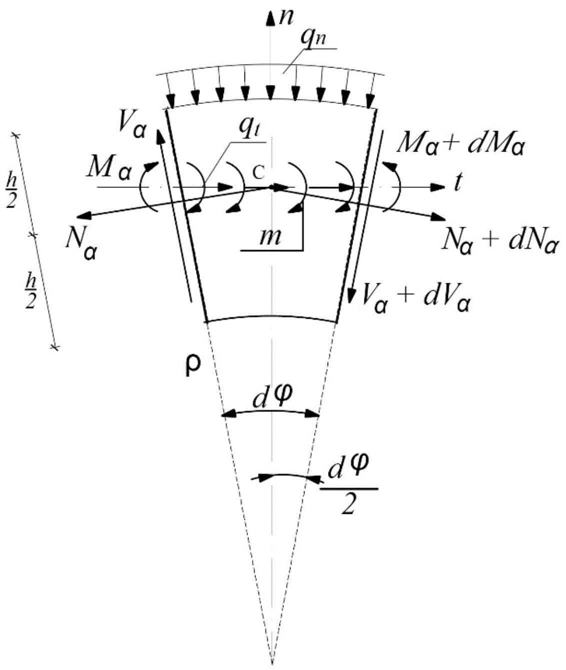


Fig. 2.5.7.

The differential equations of equilibrium have the form:

$$\frac{dV_\alpha}{ds} = -q_n - \frac{N_\alpha}{\varrho} \tag{2.26}$$

$$\frac{dN_\alpha}{ds} = -q_t + \frac{V_\alpha}{\varrho} \tag{2.27}$$

$$\frac{dM_\alpha}{ds} = V_\alpha + m \tag{2.28}$$

As we can see above, there are connections between the distributed loads, shear and moment functions.

2.5.4. Computational problems – arches

In this chapter the members of circular, parabolic and elliptical curvature of the arch will be analyzed.

Computational example 2.5.1.

A semicircular arch is loaded as shown in Fig. 2.5.8.a. Draw internal forces diagrams.

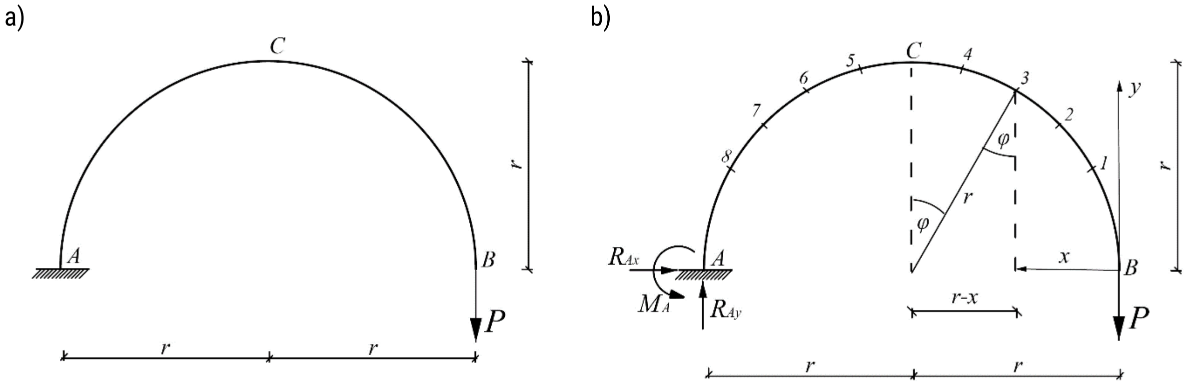


Fig. 2.5.8.

1. Degree of static indeterminacy: n = 0
2. Support reactions (Fig. 2.5.8.b.):

$$\sum F_{iy} = 0; \qquad -P + R_{Ay} = 0 \quad \rightarrow \quad R_{Ay} = P$$

$$\sum F_{ix} = 0; \qquad R_{Ax} = 0$$

$$\sum M_A = 0; \qquad M_A - P \cdot 2r = 0 \quad \rightarrow \quad M_A = 2Pr$$

To determine internal forces, we isolate an arbitrary span such as B–A defined in Fig. 2.5.9.

3. Internal forces equations – span B–A (Fig. 2.5.9.)

In order to evaluate the axial (N_α) force and shear force (V_α), we need to specify the angle φ between the tangential and the horizontal axis Fig. 2.5.9.

Now, we have to determine two components of concentrated P force: longitudinal force component as well as a transverse force component (with respect to the section α) – Fig. 2.5.9.

Enforcing equilibrium for the right span leads to the general solution for the internal forces.

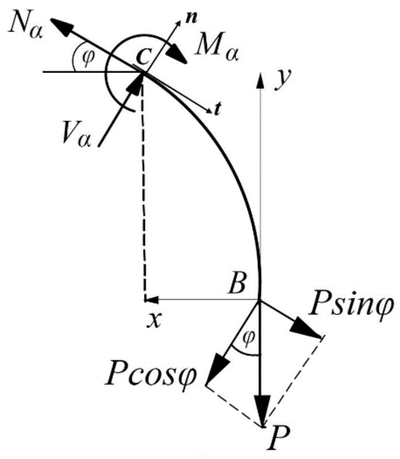


Fig. 2.5.9.

Span B–A $\quad 0 \le x < 2r \quad 0 \le y < r$

$$\sum F_{it} = 0; \qquad N_\alpha = P\sin\varphi$$

$$\sum F_{in} = 0; \qquad V_\alpha = P\cos\varphi$$

$$\sum M_{iC} = 0; \qquad M_\alpha = -Px$$

$$\sin\varphi = \frac{r-x}{r}$$

$$\varphi = arc\ sin\frac{r-x}{r}$$

Table 2.5.1. Internal force magnitudes

Point	x [r]	φ [rad]	V_α [P]	N_α [P]	M_α [Pr]
B	0.00	$\frac{\pi}{2}$	0	1	0
1	0.20	0.927295	0.6	0.8	−0.2
2	0.40	0.6435011	0.8	0.6	−0.4
3	0.60	0.4115168	0.916515139	0.4	−0.6
4	0.80	0.2013579	0.979795897	0.2	−0.8
C	1.00	0	1	0	−1
5	1.20	−0.2013579	0.979795897	−0.2	−1.2
6	1.40	−0.4115168	0.916515139	−0.4	−1.4
7	1.60	−0.6435011	0.8	−0.6	−1.6
8	1.80	−0.927295	0.6	−0.8	−1.8
A	2.00	$-\frac{\pi}{2}$	0	−1	−2

Internal forces diagrams are shown in Fig. 2.5.10.

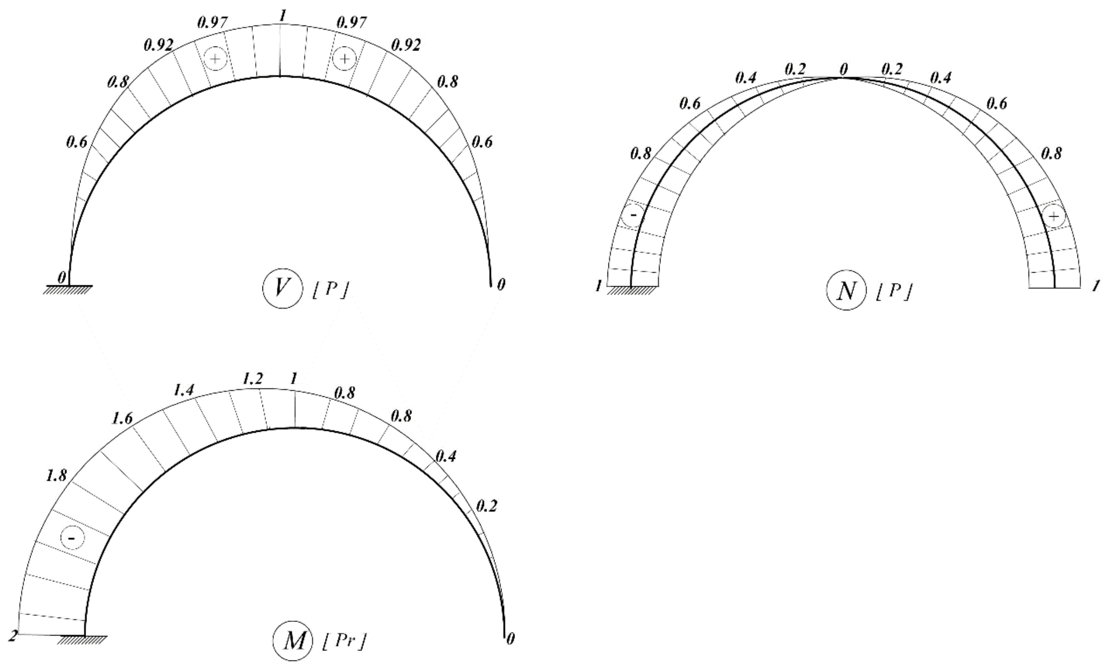


Fig. 2.5.10.

Computational example 2.5.2.

A three-hinged parabolic arch is loaded as shown in Fig. 2.5.11.a. Draw M, V, N diagrams.

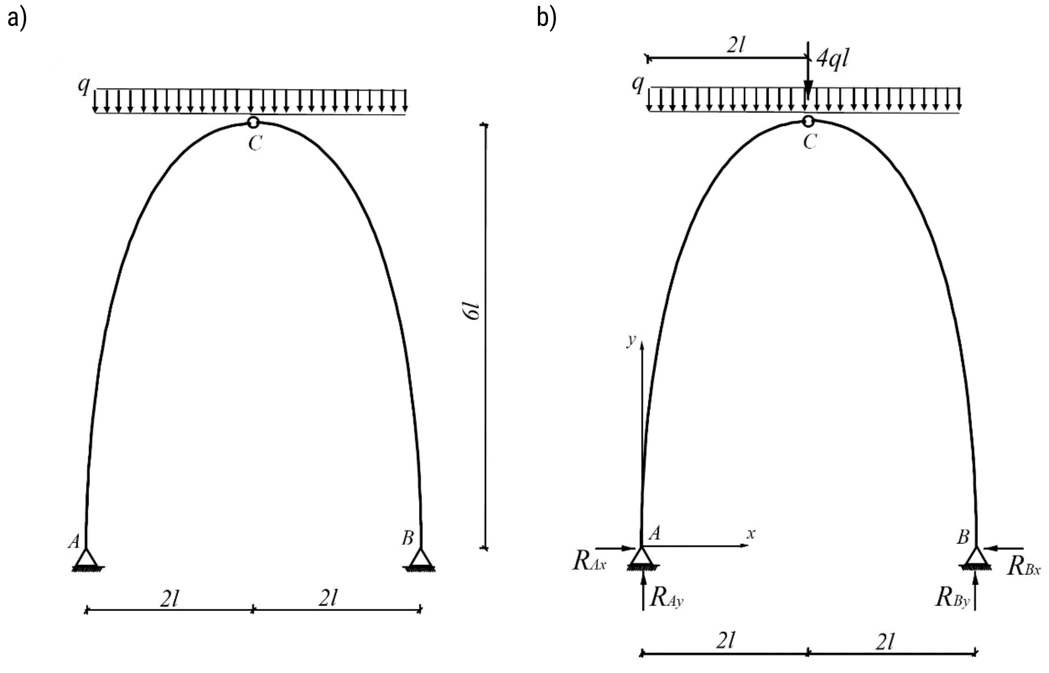


Fig. 2.5.11.

1. Degree of static indeterminacy: n = 4+2 – 3·2 = 0
2. Support reactions (Fig. 2.5.11.b.):

$$\sum M_A = 0; \qquad R_{By} \cdot 4l - 4ql \cdot 2l = 0 \quad \rightarrow \quad R_{By} = 2ql$$
$$\sum F_{iy} = 0; \qquad -4ql + R_{Ay} + R_{By} = 0 \quad \rightarrow \quad R_{Ay} = 2ql$$
$$\sum M_C^P = 0; \qquad R_{By} \cdot 2l - 2ql \cdot l - R_{Bx} \cdot 6l = 0 \quad \rightarrow \quad R_{Bx} = \frac{1}{3}ql$$
$$\sum F_{ix} = 0; \qquad R_{Ax} - R_{Bx} = 0 \quad \rightarrow \quad R_{Ax} = \frac{1}{3}ql$$

3. Internal forces equations (Fig. 2.5.12.)

Normal (axial) force N_α in α section is an algebraic sum of all forces of the longitudinal components located on the left (or right) side of the section. In the case of a curved member, all vertical forces (y direction) should be multiplied by sinφ and horizontal forces (x direction) by cosφ.

Shear (transversal) force V_α in α section is an algebraic sum of all force components normal to the longitudinal direction and located on the left (or right) side of the section. In the case of a curved member, all vertical forces (y direction) should be multiplied by cosφ and horizontal forces (x direction) by sinφ.

Bending moment M_α in α section is an algebraic sum of all moments caused by forces located on the left (or right) side of the section calculated about the center of the section.

Origin of the coordinate axis is located at point A as shown in Fig. 2.5.12.

Taking A as the origin, the equation of the central line for the three-hinged parabolic arch is given by:

$$y = \frac{4(6l)}{(4l)^2} \cdot x \cdot (4l - x) = 6x - \frac{3}{2}\frac{x^2}{l}$$

The slope is evaluated by:

$$\frac{dy}{dx} = tg\varphi = 6 - 3\frac{x}{l}$$

Hence, $\varphi = arctg\left[6 - 3\frac{x}{l}\right] \qquad sin\varphi = \left(6 - 3\frac{x}{l}\right)cos\varphi$

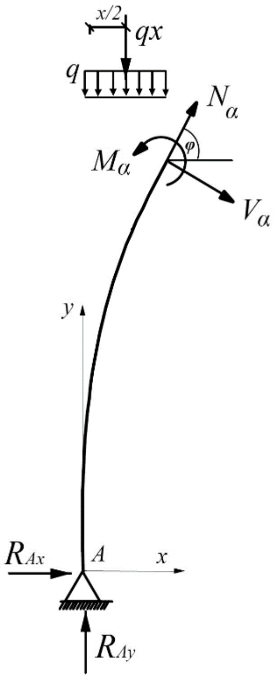


Fig. 2.5.12.

Span A–B $\quad 0 \le x < 4l \quad 0 \le y < 6l$

$$V_\alpha = R_{Ay}cos\varphi - R_{Ax}sin\varphi - qxcos\varphi =$$

$$= 2qlcos\varphi - \frac{ql}{3}sin\varphi - qx\,cos\varphi$$

$$= 2qlcos\varphi - \frac{ql}{3}\left(6 - 3\frac{x}{l}\right)cos\varphi - qx\,cos\varphi = 0$$

$$N_\alpha = -R_{Ay}sin\varphi - R_{Ax}cos\varphi + qxsin\varphi$$

$$= -2qlsin\varphi - \frac{ql}{3}cos\varphi + qx\,sin\varphi$$

$$M_\alpha = R_{Ay}x - R_{Ax}y - qx\frac{x}{2} =$$

$$= 2qlx - \frac{ql}{3}\left(6x - \frac{3}{2}\frac{x^2}{l}\right) - qx\frac{x}{2}=0$$

Table 2.5.2. Internal force magnitudes

Point	x [l]	y [l]	φ [rad]	V_α [ql]	N_α [ql]	M_α [ql^2]
A	0	0	1.405648	0	−2.02759	0
1	0.4	2.16	1.365401	0	−1.63435	0
2	0.8	3.84	1.299849	0	−1.24544	0
3	1.2	5.04	1.176005	0	−0.86667	0
4	1.6	5.76	0.876058	0	−0.52068	0
C	2	6	0	0	−0.33333	0
5	2.4	5.76	−0.87606	0	−0.52068	0
6	2.8	5.04	−1.17601	0	−0.86667	0
7	3.2	3.84	−1.29985	0	−1.24544	0
8	3.6	2.16	−1.3654	0	−1.63435	0
B	4	0	−1.40565	0	−2.02759	0

Internal forces diagrams are shown in Fig. 2.5.13.

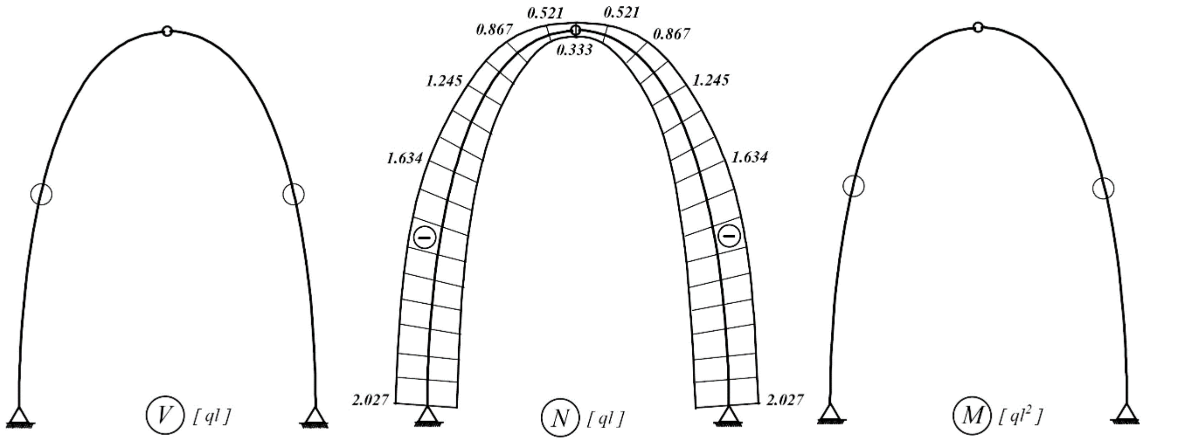


Fig. 2.5.13.

Computational example 2.5.3.

A three-hinged semicircular arch is loaded as shown in Fig. 2.5.14.a. Draw internal forces diagrams.

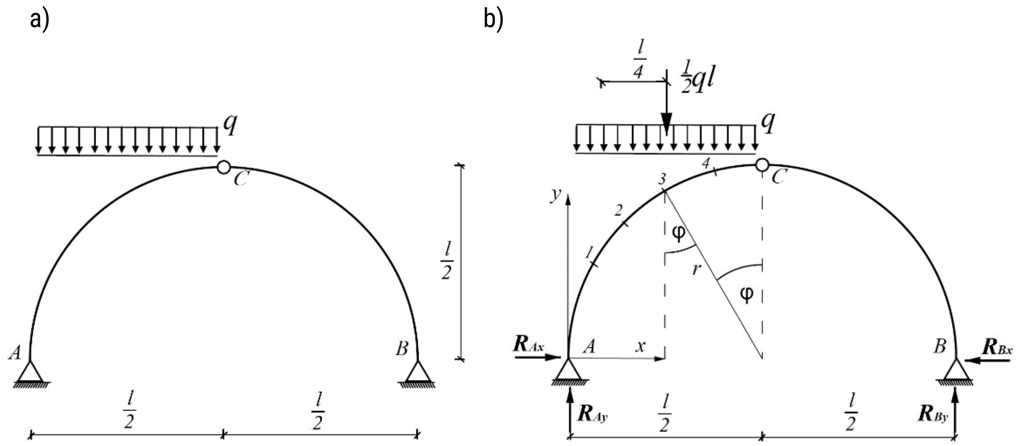


Fig. 2.5.14.

1. Degree of static indeterminacy: n = 4 + 2 – 3·2 = 0
2. Support reactions (Fig. 2.5.14.b.):

$$\sum M_A = 0; \qquad R_{By} \cdot l - \frac{1}{2}ql \cdot \frac{l}{4} = 0 \qquad \rightarrow \qquad R_{By} = \frac{1}{8}ql$$

$$\sum F_{iy} = 0; \qquad -\frac{1}{2}ql + R_{Ay} + R_{By} = 0 \qquad \rightarrow \qquad R_{Ay} = \frac{3}{8}ql$$

$$\sum M_C^P = 0; \qquad R_{By} \cdot \frac{1}{2}l - R_{Bx} \cdot \frac{1}{2}l = 0 \qquad \rightarrow \qquad R_{Bx} = \frac{1}{8}ql$$

$$\sum F_{ix} = 0; \qquad R_{Ax} - R_{Bx} = 0 \qquad \rightarrow \qquad R_{Ax} = \frac{1}{8}ql$$

3. Internal forces equations (Fig. 2.5.15.)

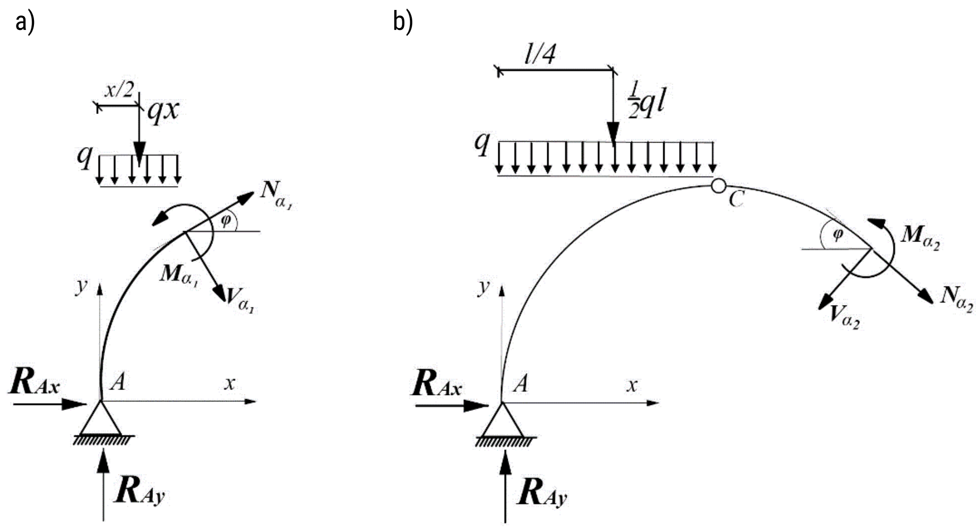


Fig. 2.5.15.

Equation of the central line for semicircular arch presented in Fig. 2.5.14. has a form:

$$y = \sqrt{xl - x^2}$$

$$\frac{dy}{dx} = tg\varphi = \frac{l-2x}{2\sqrt{xl-x^2}} \quad \Rightarrow \quad \varphi = arctg\left[\frac{l-2x}{2\sqrt{xl-x^2}}\right]; \ (x \neq 0 \ \wedge \ x \neq l)$$

$$x{=}0 \ \rightarrow \ \varphi = \frac{\pi}{2} \quad and \quad x{=}l \ \rightarrow \ \varphi = -\frac{\pi}{2}$$

Span A–C $\quad 0 \leq x < \frac{1}{2}l \quad 0 \leq y < \frac{1}{2}l \quad$ (Fig. 2.5.15.a.)

$$V_{\alpha_1} = R_{Ay}cos\varphi - R_{Ax}sin\varphi - qxcos\varphi = \frac{3ql}{8}cos\varphi - \frac{ql}{8}sin\varphi - qx\,cos\varphi$$

$$N_{\alpha_1} = -R_{Ay}sin\varphi - R_{Ax}cos\varphi + qxsin\varphi = -\frac{3ql}{8}sin\varphi - \frac{ql}{8}cos\varphi + qx\,sin\varphi$$

$$M_{\alpha_1} = R_{Ay}x - R_{Ax}y - qx\frac{x}{2} = \frac{3ql}{8}x - \frac{ql}{8}y - qx\frac{x}{2}$$

Span C–B $\quad \frac{1}{2}l \leq x < l \quad \frac{1}{2}l \geq y > 0 \quad$ (Fig. 2.5.15.b.)

$$V_{\alpha_2} = R_{Ay}cos\varphi - R_{Ax}sin\varphi - q\ \frac{l}{2}cos\varphi$$

$$= \frac{3ql}{8}cos\varphi - \frac{ql}{8}sin\varphi - q\frac{l}{2}\ cos\varphi = -\frac{ql}{8}cos\varphi - \frac{ql}{8}sin\varphi$$

$$N_{\alpha_2} = -R_{Ay}sin\varphi - R_{Ax}cos\varphi + q\frac{l}{2}sin\varphi$$

$$= -\frac{3ql}{8}sin\varphi - \frac{ql}{8}cos\varphi + q\frac{l}{2}\ sin\varphi = \frac{ql}{8}sin\varphi - \frac{ql}{8}cos\varphi$$

$$M_{\alpha_2} = R_{Ay}x - R_{Ax}y - q\frac{l}{2}\left(x - \frac{l}{4}\right) = \frac{3ql}{8}x - \frac{ql}{8}y - q\frac{l}{2}\left(x - \frac{l}{4}\right)$$

Table 2.5.3. presents internal force magnitudes.

Internal forces diagrams are shown in Fig. 2.5.16.

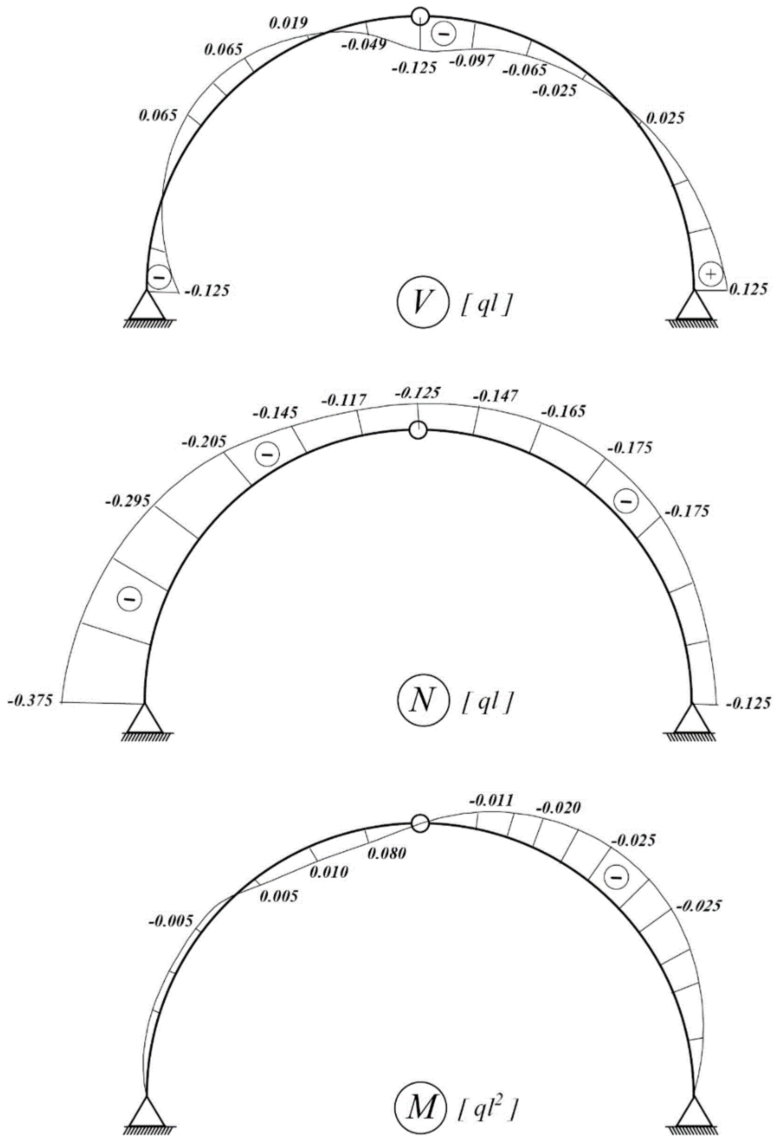


Fig. 2.5.16.

Table 2.5.3. presents internal force magnitudes.

Table 2.5.3. Internal force magnitudes

Point	x [l]	y [l]	φ [rad]	V_α [ql]	N_α [ql]	M_α [ql^2]
A	0	0	1.570796	−0.125	−0.375	0
1	0.1	0.3	0.927295	0.065	−0.295	−0.005
2	0.2	0.4	0.643501	0.065	−0.205	0.005
3	0.3	0.458258	0.411517	0.018738635	−0.14456	0.010218
4	0.4	0.489898	0.201358	−0.0494949	−0.11747	0.008763
C	0.5	0.5	0	−0.125	−0.125	0
5	0.6	0.489898	−0.20136	−0.09747449	−0.14747	−0.01124
6	0.7	0.458258	−0.41152	−0.06456439	−0.16456	−0.01978
7	0.8	0.4	−0.6435	−0.025	−0.175	−0.025
8	0.9	0.3	−0.9273	0.025	−0.175	−0.025
B	1	0	−1.5708	0.125	−0.125	0

Computational example 2.5.4.

An elliptical arch is loaded as shown in Fig. 2.5.17. Draw internal forces diagrams.

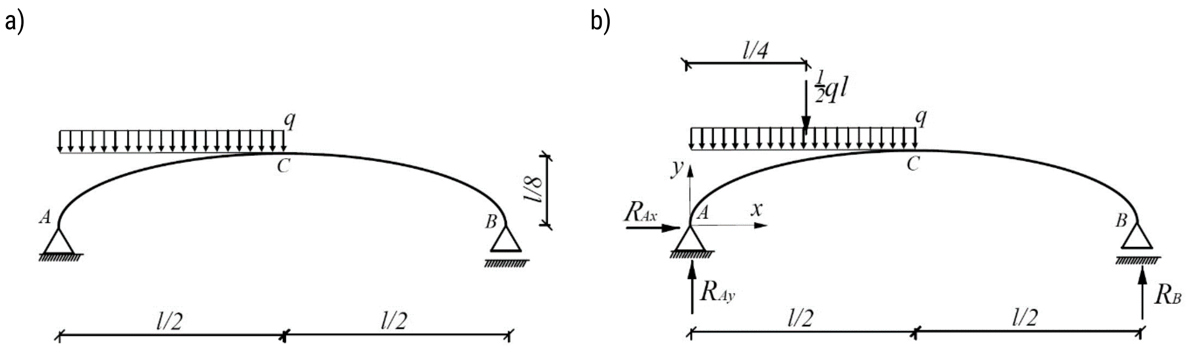


Fig. 2.5.17.

1. Degree of static indeterminacy: n = 0
2. Support reactions (Fig. 2.5.17.b.):

$$\sum M_A = 0; \qquad R_B \cdot l - \frac{1}{2}ql \cdot \frac{l}{4} = 0 \quad \rightarrow \quad R_B = \frac{1}{8}ql$$

$$\sum F_{iy} = 0; \qquad -\frac{1}{2}ql + R_{Ay} + R_B = 0 \quad \rightarrow \quad R_{Ay} = \frac{3}{8}ql$$

$$\sum F_{ix} = 0; \qquad R_{Ax} = 0$$

3. Internal forces equations (Fig. 2.5.18.)

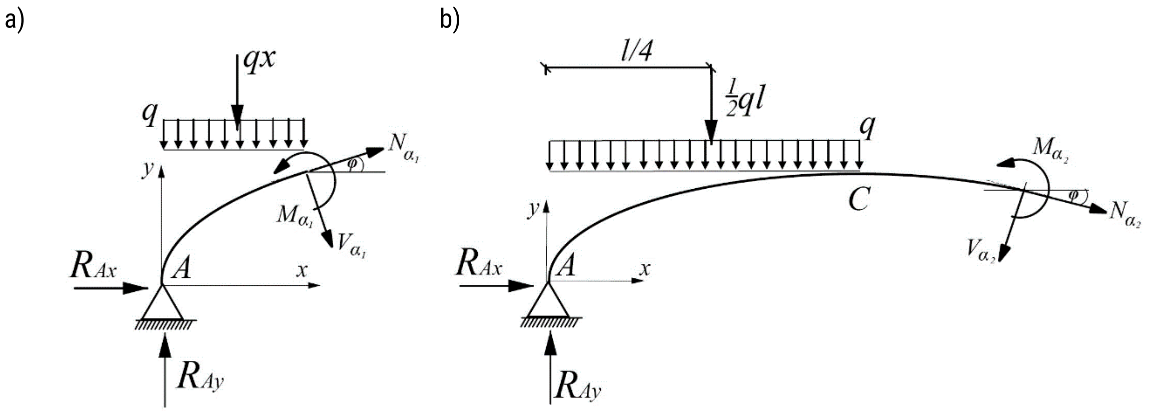


Fig. 2.5.18.

Equation of the central line for the elliptical arch shown in Fig. 2.5.17. has a form:

$$y = \frac{1}{4}\sqrt{xl - x^2}$$

$\frac{dy}{dx} = tg\varphi = \frac{l-2x}{8\sqrt{xl-x^2}} \quad \Rightarrow \quad \varphi = arctg\left[\frac{l-2x}{8\sqrt{xl-x^2}}\right]$; $(x \neq 0 \ \wedge \ x \neq l)$

$x=0 \quad \varphi = \frac{\pi}{2} \quad and \quad x=l \quad \varphi = -\frac{\pi}{2}$

<u>Span A–C</u> $\quad 0 \le x < \frac{1}{2}l \quad 0 \le y < \frac{1}{8}l$

$$V_{\alpha_1} = R_{Ay}cos\varphi - qxcos\varphi = \frac{3ql}{8}cos\varphi - qx\,cos\varphi$$

$$V_{\alpha_1} = 0 \quad \rightarrow \quad x = \frac{3}{8}l$$

$$N_{\alpha_1} = -R_{Ay}sin\varphi + qxsin\varphi = -\frac{3ql}{8}sin\varphi + qx\,sin\varphi$$

$$M_{\alpha_1} = R_{Ay}x - qx\frac{x}{2} = \frac{3ql}{8}x - qx\frac{x}{2}$$

Moment extrema: $M_{\alpha_1}\left(x = \frac{3l}{8}\right) = \frac{3ql}{8}\cdot\frac{3l}{8} - \frac{q}{2}\left(\frac{3l}{8}\right)^2 = \frac{9}{128}ql^2$

<u>Span C–B</u> $\quad \frac{1}{2}l \le x < l \quad \frac{1}{8}l \ge y > 0$

$$V_{\alpha_2} = R_{Ay}cos\varphi - q\,\frac{l}{2}cos\varphi = \frac{3ql}{8}cos\varphi - q\frac{l}{2}\,cos\varphi = -\frac{ql}{8}cos\varphi$$

$$N_{\alpha_2} = -R_{Ay}sin\varphi + q\frac{l}{2}sin\varphi = -\frac{3ql}{8}sin\varphi + q\frac{l}{2}\,sin\varphi = \frac{ql}{8}sin\varphi$$

$$M_{\alpha_2} = R_{Ay}x - q\frac{l}{2}\left(x - \frac{l}{4}\right) = \frac{3ql}{8}x - q\frac{l}{2}\left(x - \frac{l}{4}\right)$$

Table 2.5.4. Internal force magnitudes

Point	x [l]	y [l]	φ [rad]	V_α [ql]	N_α [ql]	M_α [ql^2]
A	0	0	1.570796	0	−0.375	0
1	0.1	0.075	0.927295	0.165	−0.22	0.0325
2	0.2	0.1	0.643501	0.14	−0.105	0.055
3	0.3	0.114564	0.411517	0.068738635	−0.03	0.0675
4	0.4	0.122474	0.201358	−0.0244949	0.005	0.07
C	0.5	0.125	0	−0.125	0	0.0625
5	0.6	0.122474	−0.20136	−0.12247449	−0.025	0.05
6	0.7	0.114564	−0.41152	−0.11456439	−0.05	0.0375
7	0.8	0.1	−0.6435	−0.1	−0.075	0.025
8	0.9	0.075	−0.9273	−0.075	−0.1	0.0125
B	1	0	−1.5708	0	−0.125	0

Internal forces diagrams are shown in Fig. 2.5.19.

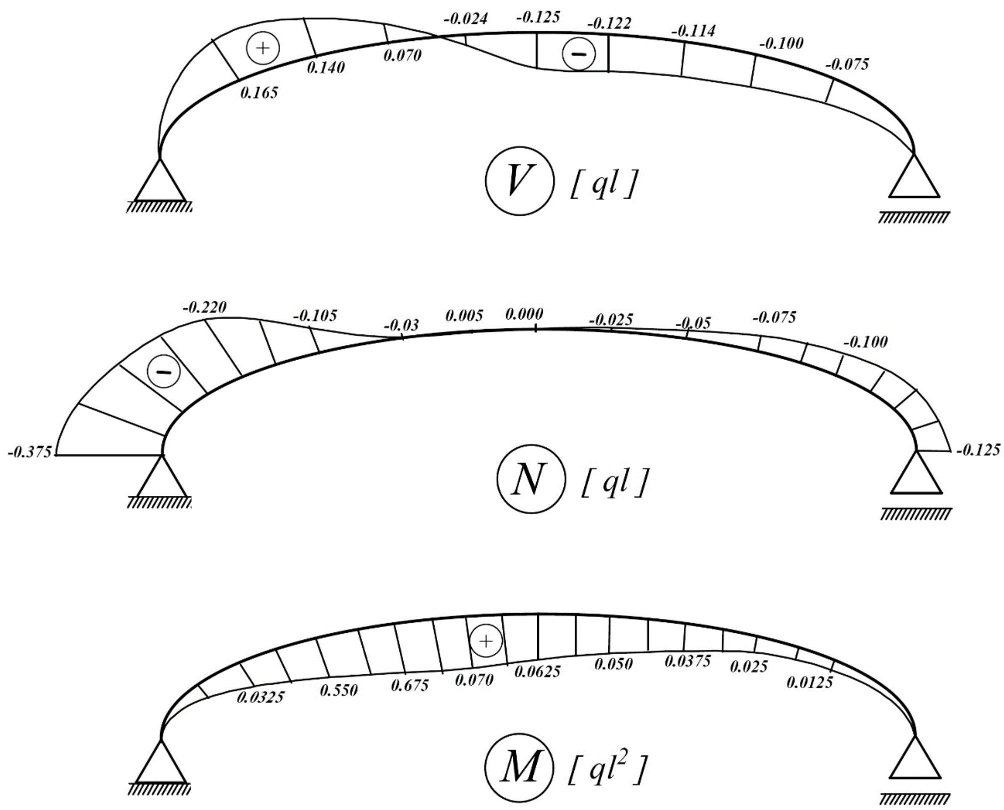


Fig. 2.5.19.

2.6. Statically determinate planar truss structures

2.6.1. Basic information about trusses [1], [3], [5]

The truss is one of the major types of engineering structures. Trusses are used commonly in buildings, towers and bridges. Fig. 2.6.1. presents examples of trusses [1].

We limit our attention to the planar trusses, i.e. all truss members are located in one plane and loads are also applied in this plane.

Truss definition:

Trusses are idealized structures consisting of straight and slender rigid bars (members of a truss), arranged such that its centroidal axis coincides with the line connecting the nodal points. Truss members are connected together with frictionless pin joints and connected only at the ends of the members. All forces (loads and reactions) must be applied at the joints and are transmitted from one member to another through pins. The weights of the members of the truss are also assumed to be applied to the joints.

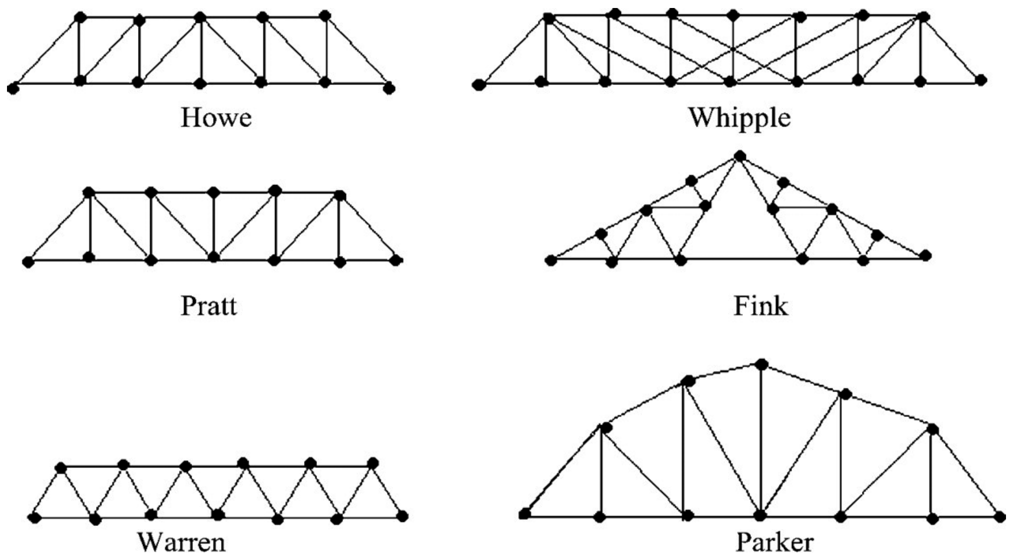


Fig. 2.6.1. Examples of named truss [1].

The consequence of the idealization described is that members of a truss are so-called "two-force members" which carry only a pair of equal magnitude, oppositely directed forces along their length (Fig. 2.6.2.).

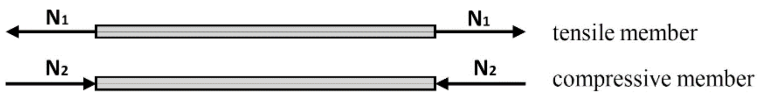


Fig. 2.6.2.

The types of stable trusses:

- simple truss (Fig. 2.6.3.a.),
- compound truss – combination of two or more simple trusses together (Fig. 2.6.3.b.),
- complex truss – one that cannot be classified as being either simple or compound (Fig. 2.6.3.c.).

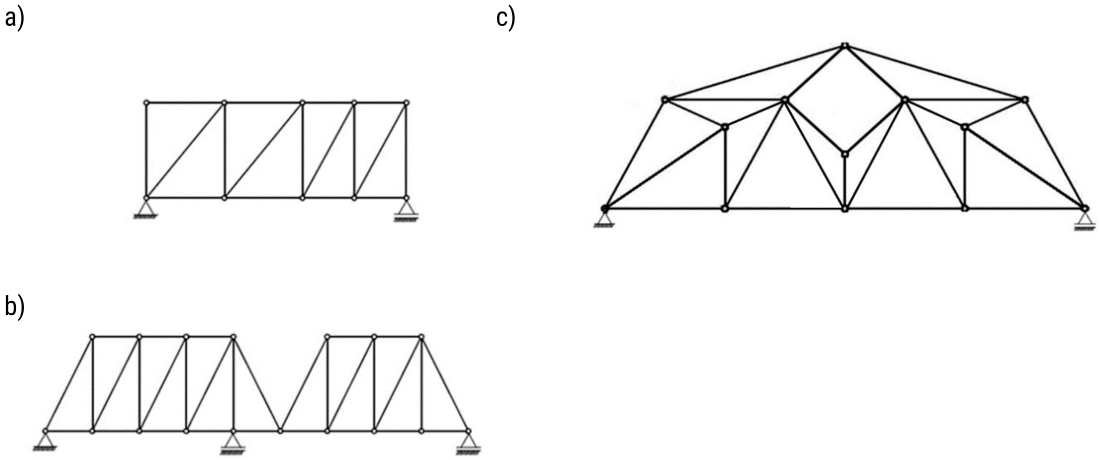


Fig. 2.6.3.

<u>*Kinematic stability and static determinacy of planar truss*</u> [1], [2]

Three bars joined by pins at their ends constitute a rigid structure (Fig. 2.6.4.). The term rigid is used to mean stable and also to mean that deformation of the members due to induced internal strains is negligible.

Four bars pin-jointed to form a rectangle constitute a unstable system (Fig. 2.6.4.).

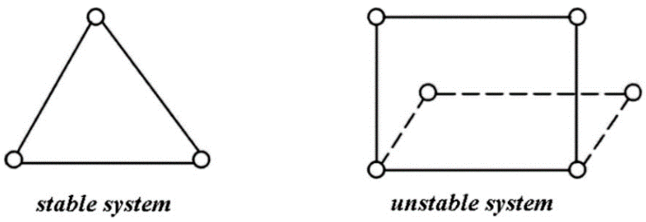


Fig. 2.6.4.

Simple planar truss structures are formed by combining one-dimensional linear members to create a triangular pattern. The structure which consists of a triangular arrangement of members that are pinned at their ends create a rigid structure. In terms of stability, the most simple truss can be constructed in the shape of a triangle using three members.

Each node (pin, joint) of a plane truss is acted upon by a set of coplanar concurrent forces. There are no moments since the pins are frictionless and the lines of action of the forces intersect at the node.

If the entire truss is in equilibrium, a single node is also in equilibrium. For the coplanar concurrent force system there are two equations of equilibrium:

$$\sum F_{ix} = 0 \quad \text{and} \quad \sum F_{\mathrm{iy}} = 0 \tag{2.28}$$

So for j number of nodes (joints) we have $2j$ equations of equilibrium.

Hence, the degree of static indeterminacy can be determined by using the formula:

$$n = r + m - 2j \tag{2.29}$$

where:

n – degree of static indeterminacy,
m – number of truss members,
r – number of support reactions,
j – number of joints.

If
n = 0 – statically determinate truss,
n > 0 – statically indeterminate truss,
n < 0 – unstable truss system – mechanism.

When more members are present than are needed to prevent collapse, the truss is statically indeterminate. A statically indeterminate truss cannot be analyzed using the equations of equilibrium alone.

Sign convention: tensile forces are denoted with + sign (are positive), compressive forces are denoted with – sign (are negative).

There are two basic methods used for truss solving: method of joints and method of sections. Sometimes, we can have statically determinate truss for which both method of joints and method of sections will be not convenient to use. In this case we can use Henneberg's method [2].

2.6.2. Analysis of planar truss [1], [3], [5]

Analysis of planar truss includes:

- determining the support reactions,
- determining the internal forces in each of the members (tensile or compressive).

In some cases we can simplify the analysis by finding "zero-force members".

"Zero-force members"

The simplification is possible when:

- the joint has two non-collinear members with no external force at the joint (Fig. 2.6.5.a.):

$$\Sigma F_{iy_1} = 0 \;\rightarrow\; N_2 = 0 \quad \text{and} \quad \Sigma F_{iy} = 0 \;\rightarrow\; N_1 = 0$$

- in the joint there are two collinear members (N_1 and N_2) and a third member (N_3) (Fig. 2.6.5.b.). We see from the force summation in the y-direction that the force N_3 must be zero and from the force summation in the x-direction that the other two forces must have the same magnitude but opposite sense.

$$\Sigma F_{iy} = 0 \;\rightarrow\; N_3 = 0 \qquad \Sigma F_{ix} = 0 \;\rightarrow\; N_1 = N_2$$

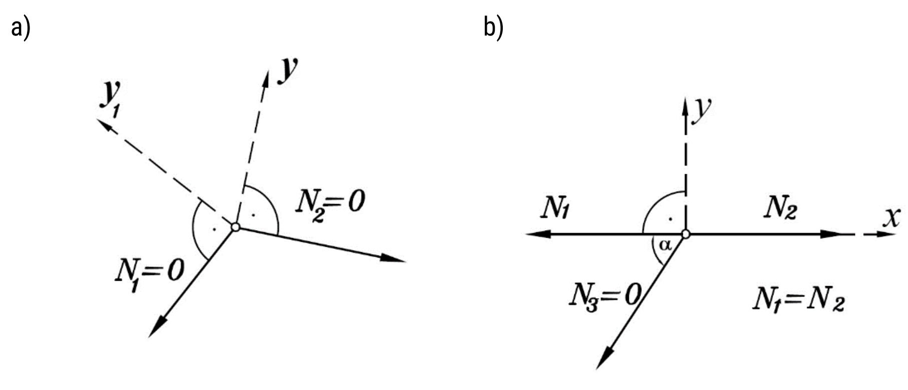


Fig. 2.6.5.

The zero-force members are not useless. Although these members don't carry any loads, some of them would probably carry loads if the loading conditions were changed. They are necessary to maintain the truss in the desired shape and to make the truss stable.

When applying the method of joints or method of sections it is convenient to first determine the reactions from the equations of equilibrium of the total planar structure. Knowing the reactions will make it easier to find the truss members forces.

- *Method of joints*

In the method of joints, we first find a joint with at most 2 members connected (two unknowns) and by using two equations of equilibrium $\sum F_{ix} = 0$ and $\sum F_{\text{iy}} = 0$, the unknown forces can be determined.

Then we work our way through the structure, one joint at a time, picking joints with at most two unknown members. Known reactions can help locate a joint that has only two unknown member forces.

Hence, for a statically determinate truss (Fig. 2.6.6.a.), we calculate the support reactions first from the three global equilibrium equations:

$$\sum M_5 = 0, \;\; \sum F_{\text{iy}} = 0 \;\; \text{and} \;\; \sum F_{\text{ix}} = 0.$$

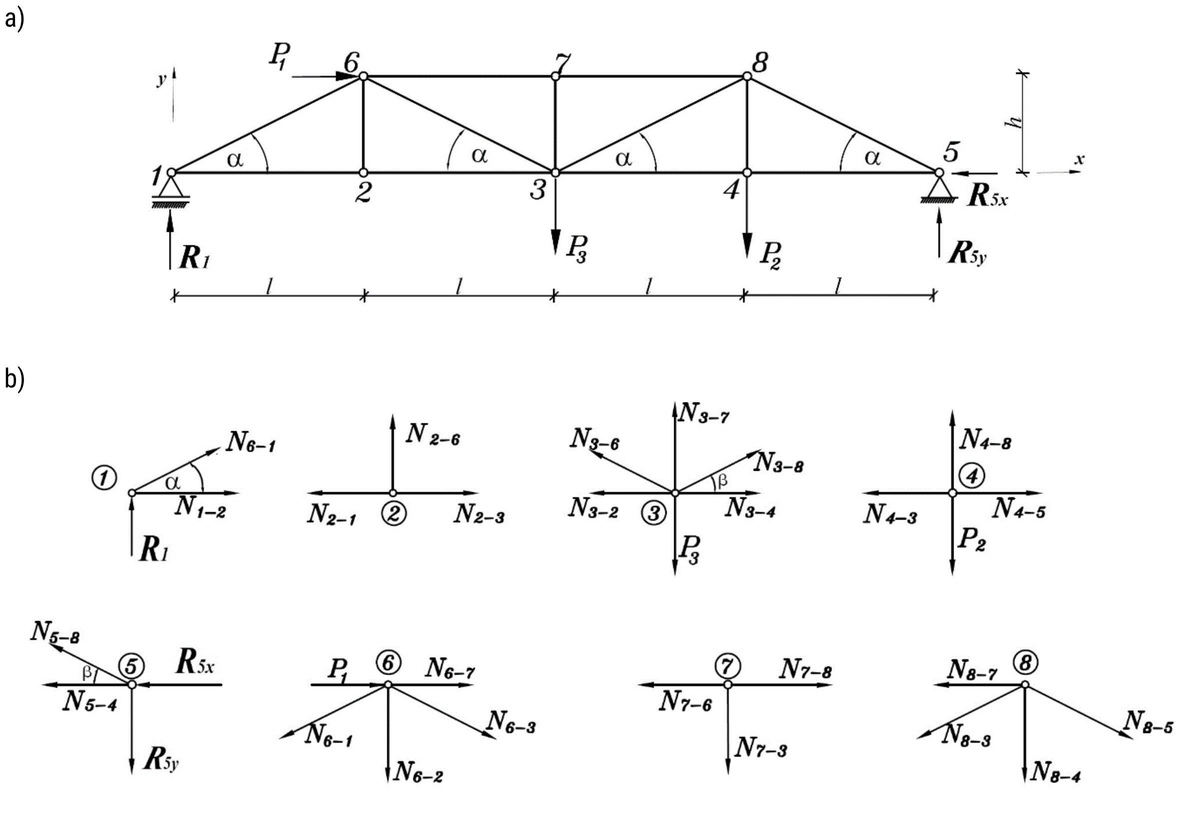


Fig. 2.6.6. Planar truss – method of joints.

We pick joint 1 (Fig. 2.6.6.b.) and for this joint we apply the force-balance equations:

$$\sum F_{ix} = 0 \quad \text{and} \quad \sum F_{\text{iy}} = 0$$

We indicate a tensile member force with an arrow pointing away from the joint. The opposite sense is used for compression.

Next, we can write two similar equations of equilibrium for joint No. 2. After this we can set two equations for the joint No. 6 then No. 7→No. 3→No. 4→No. 8.

After finding the last force (joint No. 8) all truss member forces will be determined. The total number of joints equilibrium equations is equal to sixteen. If we use three global equations of equilibrium to calculate the support reactions, there are only 15 independent equations left to apply to the joints. As one can see, three equations (one for joint No. 8 and two for joint No. 5) haven't been used, so we can use them to verify the results of our calculations. The method of joints is suitable to be used when we need to determine all the member forces.

• *Method of sections* (*Ritter's method*)

This method is suitable to be used if one wants to determine only the force in a particular member. Applying the method of joints might not be convenient because it involves first finding the force in other members.

However, the method of sections can also be used to determine all the member forces in a truss [1]. Let's consider the truss shown in Fig. 2.6.7.a.

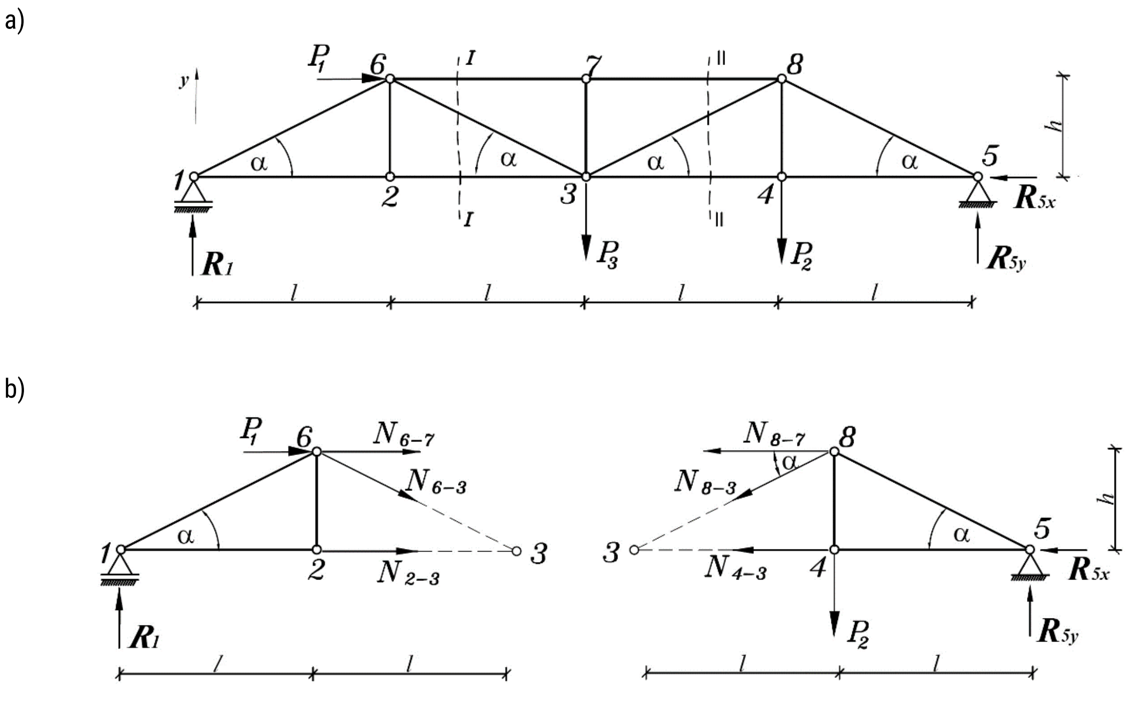


Fig. 2.6.7. Planar truss – method of sections.

Suppose the force in member N_{6-7} is desired.

First we have to determine the reactions from the three global equilibrium equations for the planar truss:

$$\sum M_5 = 0, \quad \sum F_{iy} = 0 \quad \text{and} \quad \sum F_{ix} = 0.$$

With the reactions known, we cut the truss structure into two segments, we isolate either the left or right part, and apply the three global equilibrium equations to the segment.

The cutting plane must cut the particular member whose force is desired, and the other two members that are concurrent. There are only three equilibrium equations of planar loading, and therefore, we can cut only three members (Fig. 2.6.7.a.).

To determine N_{6-7} one can use the vertical cutting plane I–I and consider the left segment shown in Fig. 2.6.7.b.

We can use the moment equilibrium condition with respect to joint No. 3 which is the point of concurrency for members 2–3 and 6–3.

$$\sum M_3 = 0 \qquad -N_{6-7}h - R_1 \cdot 2l - P_1 h = 0 \quad \rightarrow \quad N_{6-7} = -\frac{R_1 2l + P_1 h}{h}$$

If the force in member N_{2-3} is desired, one can use the moment equilibrium condition with respect to joint No. 6 which is the point of concurrency for members 6–7 and 6–3.

$$\sum M_6 = 0 \qquad N_{2-3}h - R_1 l = 0 \quad \rightarrow \quad N_{2-3} = \frac{R_1 l}{h}$$

We can determine force N_{6-3} using the formula:

$$\sum F_{iy} = 0 \qquad -N_{6-3} \cdot \sin\alpha + R_1 = 0 \quad \rightarrow \quad N_{6-3} = \frac{R_1}{\sin\alpha}$$

Similarly, to determine forces in members 7–8, 8–3 and 3–4 we can use the vertical cutting plane II–II and consider the right segment shown in Fig. 2.6.7.c.

- *Henneberg's method* (*truss member replacement*) [2]

In order to determine the member forces, one has to establish the complete set of nodal force equilibrium equations expressed in terms of the member forces. If the truss is statically determinate, the number of equations will be equal to the number of nodal forces unknowns. These unknowns can be found by solving the system of equations. However, finding the solution for a complex truss using the "solve-by-hand" method can be both inconvenient and time consuming.

In the case of statically determinate complex truss (Fig. 2.6.8.) for which both method of joints and method of sections are inconvenient to use, we can use Henneberg's method.

In this method we replace one truss member by its axial force X (Fig. 2.6.8.b.c.d.). This will make the structure unstable. Therefore, we add one "extra" (***z***) member to ensure truss stability. Obviously, the magnitude of force in the "extra" member is equal zero. Using this condition and the superposition method we can calculate the force in the removed bar.

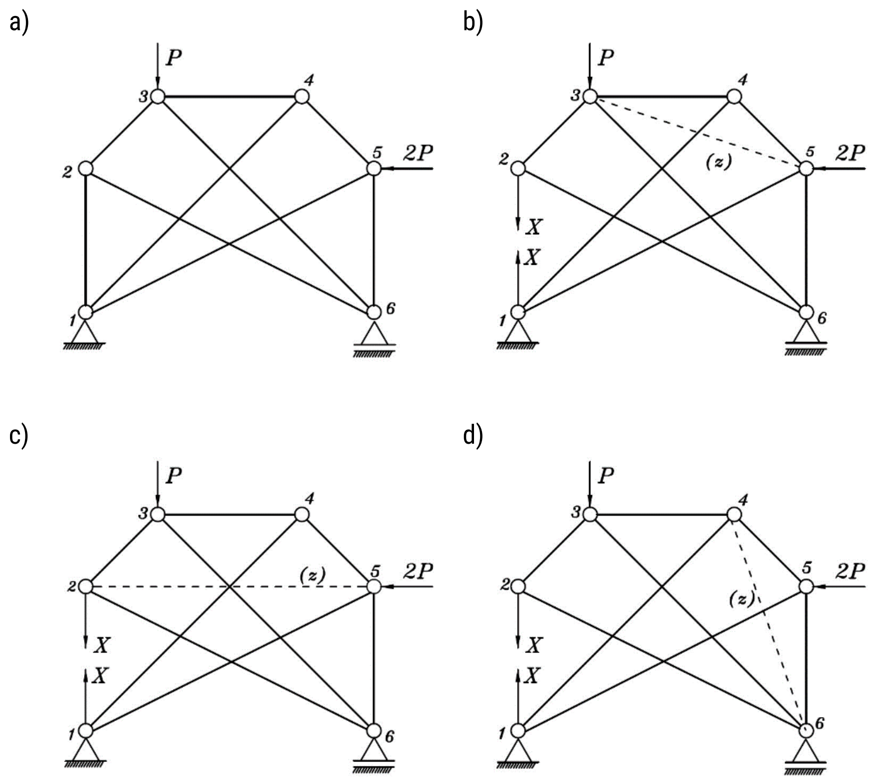


Fig. 2.6.8. Complex truss – Henneberg's method

The truss modified in this way can be first solved (using both method of joints and method of sections) for a given loading and then solved for the unit force X = 1.
Hence, we have:

$$N^i = N^i_{X=1} X + N^i_P \quad (2.30)$$

where:
$N^i_{X=1}$ – "i" member force determined in the modified truss under the loading X = 1,
N^i_P – "i" member force determined in the modified truss under the external loading P,
X – removed member force.

The "extra" (z) member doesn't exist, so the magnitude of N^Z is equal zero.
Thus,

$$N^z = N^z_{X=1} X + N^z_P = 0 \quad (2.31)$$

$$X = -\frac{N^z_P}{N^z_{X=1}} \quad (2.32)$$

If one knows the X force, the forces in the remaining truss members can be determined using the formula:

$$N^i = N^i_{X=1} X + N^i_P \quad (2.33)$$

2.6.3. Computational problems – trusses

Computational example 2.6.1.

Determine the member forces of planar truss by the method of joints. The truss and loading are defined by Fig. 2.6.9.

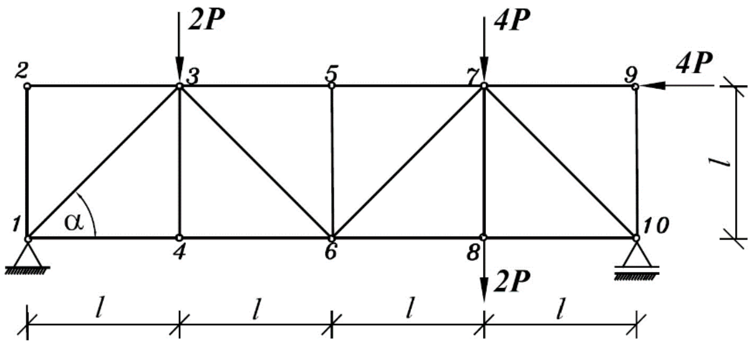


Fig. 2.6.9.

Solution:

The degree of static indeterminacy:
$n = r + m - 2j = 3 + 17 - 2 \cdot 10 = 0$

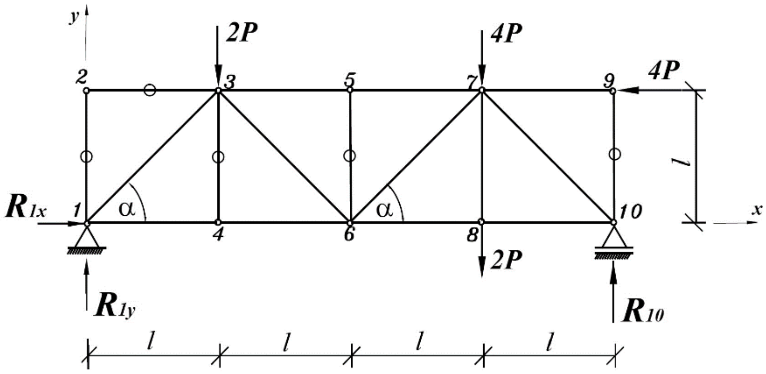


Fig. 2.6.10.

Support reactions (Fig. 2.6.10.) can be determined by using the three global equilibrium equations for the coplanar forces system.

$\Sigma M_1 = 0 \;\rightarrow\; -2P \cdot l - 2P \cdot 3l - 4P \cdot 3l + R_{10} \cdot 4l + 4P \cdot l = 0 \;\rightarrow\; R_{10} = 4P$

$\Sigma F_{ix} = 0 \;\rightarrow\; R_{1x} - 4P = 0 \;\rightarrow\; R_{1x} = 4P$

$\Sigma F_{iy} = 0 \;\rightarrow\; -2P + R_{1y} - 2P - 4P + R_{10} = 0 \;\rightarrow\; R_{1y} = 4P$

"Zero-force" members are shown in Fig. 2.6.10.

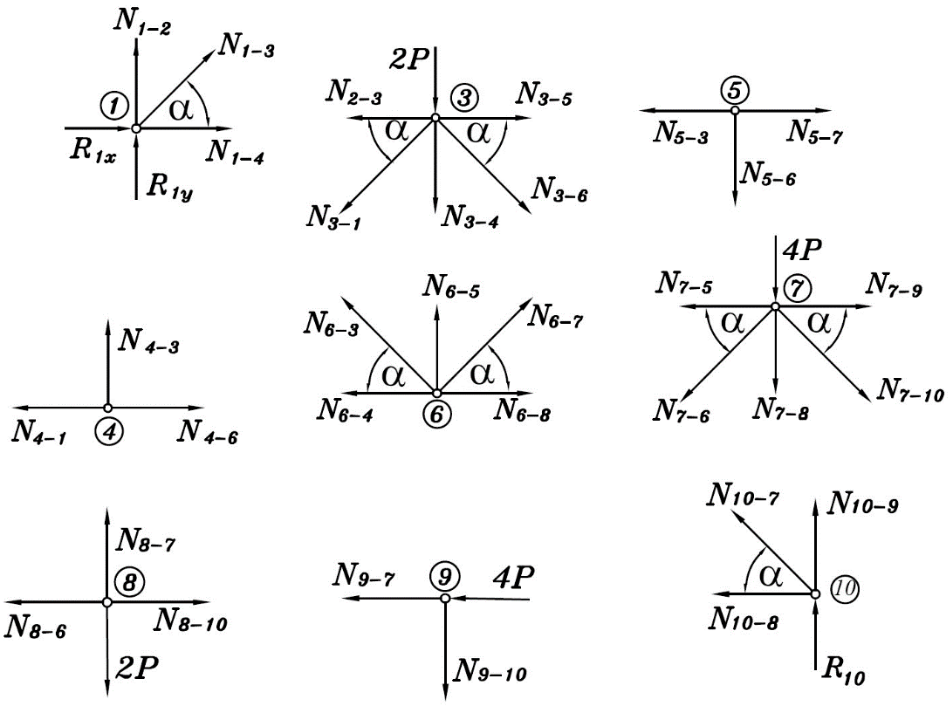


Fig. 2.6.11.

<u>Joint 1</u> (Fig. 2.6.11.)

$N_{1-2} = 0 \qquad sin\alpha = cos\alpha = \frac{\sqrt{2}}{2}$

$\Sigma F_{iy} = 0 \qquad N_{1-3}sin\alpha + R_{1y} + N_{1-2} = 0 \qquad \rightarrow \ N_{1-3} = -4\sqrt{2}\,P$

$\Sigma F_{ix} = 0 \qquad N_{1-3}cos\alpha + N_{1-4} + R_{1x} = 0 \qquad \rightarrow \quad N_{1-4} = 0$

<u>Joint 4</u>

$\Sigma F_{ix} = 0 \qquad -N_{4-1} + N_{4-6} = 0 \qquad \rightarrow \quad N_{4-6} = 0$

<u>Joint 3</u> (Fig. 2.6.11.)

$sin\alpha = cos\alpha = \frac{\sqrt{2}}{2}$

$\Sigma F_{iy} = 0 \qquad -N_{3-1}sin\alpha - 2P - N_{3-6}sin\alpha - N_{3-4} = 0 \qquad \rightarrow \ N_{3-6} = 2\sqrt{2}\,P$

$\Sigma F_{ix} = 0 \qquad -N_{2-3} - N_{3-1}cos\alpha + N_{3-5} + N_{3-6}cos\alpha = 0 \qquad \rightarrow \quad N_{3-5} = -6\mathrm{P}$

<u>Joint 5</u> (Fig. 2.6.11.)

$\Sigma F_{ix} = 0 \qquad -N_{5-3} + N_{5-7} = 0 \qquad \rightarrow \quad N_{5-7} = -6\mathrm{P}$

<u>Joint 6</u> (Fig. 2.6.11.)

$sin\alpha = cos\alpha = \frac{\sqrt{2}}{2}$

$\Sigma F_{iy} = 0 \qquad N_{6-3}sin\alpha + N_{6-7}sin\alpha - N_{6-5} = 0 \qquad \rightarrow \ N_{6-7} = -\,2\sqrt{2}\,P$

$\Sigma F_{ix} = 0 \qquad -N_{6-4} - N_{6-3}cos\alpha + N_{6-8} + N_{6-7}cos\alpha = 0 \qquad \rightarrow \quad N_{6-8} = 4\mathrm{P}$

<u>Joint 8</u> (Fig. 2.6.11.)

$\Sigma F_{iy} = 0 \qquad N_{8-7} - 2P = 0 \qquad \rightarrow \ N_{8-7} = 2\,P$

$\Sigma F_{ix} = 0 \qquad -N_{8-6} + N_{8-10} = 0 \qquad \rightarrow \quad N_{8-10} = 4\mathrm{P}$

Joint 7 (Fig. 2.6.11.)

$sin\alpha = cos\alpha = \frac{\sqrt{2}}{2}$

$\Sigma F_{iy} = 0 \quad -N_{7-6}sin\alpha - 4P - N_{7-10}sin\alpha - N_{7-8} = 0 \quad \rightarrow \; N_{7-10} = -4\sqrt{2}\,P$

$\Sigma F_{ix} = 0 \quad -N_{7-5} - N_{7-6}cos\alpha + N_{7-9} + N_{7-10}cos\alpha = 0 \quad \rightarrow \; N_{7-9} = -4P$

Verification:

Joint 9 (Fig. 2.6.11.)

$\Sigma F_{ix} = -N_{9-7} - 4P = 4P - 4P = 0$

$\Sigma F_{iy} = -N_{9-10} = 0$

Joint 10 (Fig. 2.6.11.)

$sin\alpha = cos\alpha = \frac{\sqrt{2}}{2}$

$\Sigma F_{ix} = -N_{10-8} - N_{10-7}cos\alpha = -4P + 4\sqrt{2}\,P\frac{\sqrt{2}}{2} = 0$

$\Sigma F_{iy} = R_{10} + N_{10-9} + N_{10-7}sin\alpha = 4P - 4\sqrt{2}P\frac{\sqrt{2}}{2} = 0$

Computational example 2.6.2.

Determine the member forces 5–7, 6–7, 6–8. The truss and loading are defined by Fig. 2.6.12.

The degree of static indeterminacy:
$n = r + m - 2j = 3 + 17 - 2 \cdot 10 = 0$

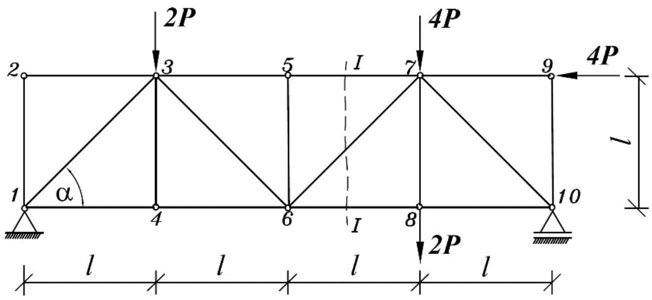


Fig. 2.6.12.

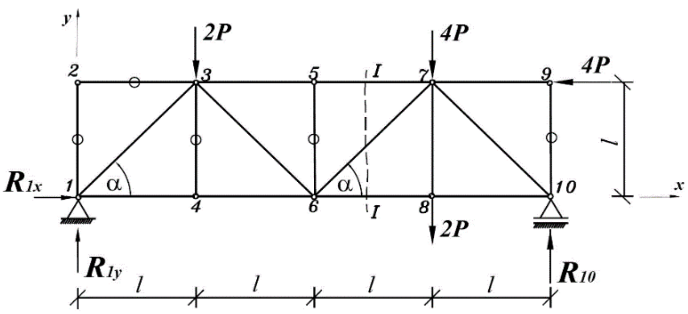


Fig. 2.6.13.

Support reactions (Fig. 2.6.13.) can be determined by using the three global equilibrium equations for the planar structures.

$\Sigma M_1 = 0 \quad -2P \cdot l - 2P \cdot 3l - 4P \cdot 3l + R_{10} \cdot 4l + 4P \cdot l = 0 \rightarrow R_{10} = 4P$

$\Sigma F_{ix} = 0 \quad R_{1x} - 4P = 0 \rightarrow R_{1x} = 4P$

$\Sigma F_{iy} = 0 \quad -2P + R_{1y} - 2P - 4P + R_{10} = 0 \rightarrow R_{1y} = 4P$

Zero-force members are shown in Fig. 2.6.13.

To determine forces of members 5–7, 6–7, 6–8 we can use the vertical cutting plane I–I shown in Fig. 2.6.14.

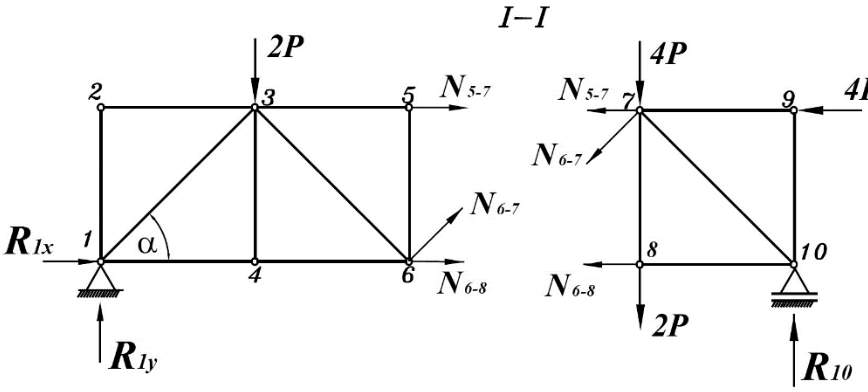


Fig. 2.6.14.

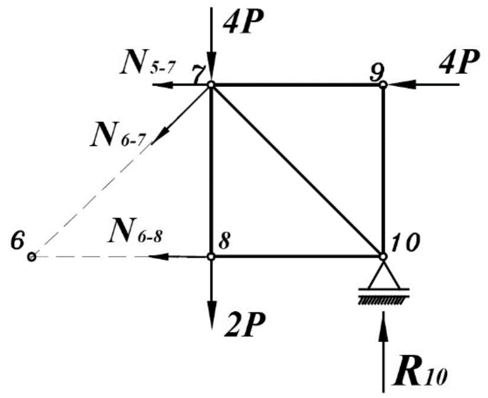


Fig. 2.6.15.

Equilibrium equations for the right segment of truss (Fig. 2.6.15.):

$\sum M_7 = 0 \rightarrow -N_{6-8} \cdot l - R_{10} \cdot l = 0 \rightarrow N_{6-8} = 4P$

$\sum M_6 = 0 \rightarrow N_{5-7} \cdot l + R_{10} \cdot 2l + 4P \cdot l - 4P \cdot l - 2P \cdot l = 0 \rightarrow N_{5-7} = -6P$

$\sum F_{iy} = 0 \rightarrow R_{10} - N_{6-7} \cdot sin\alpha - 4P - 2P = 0 \rightarrow N_{6-7} = -2\sqrt{2}P$

Computational example 2.6.3.

Determine the member forces 2–3, 9–8, 8–2, 7–8, 7–3, 3–4 of planar truss. The truss and loading are defined by Fig. 2.6.16.

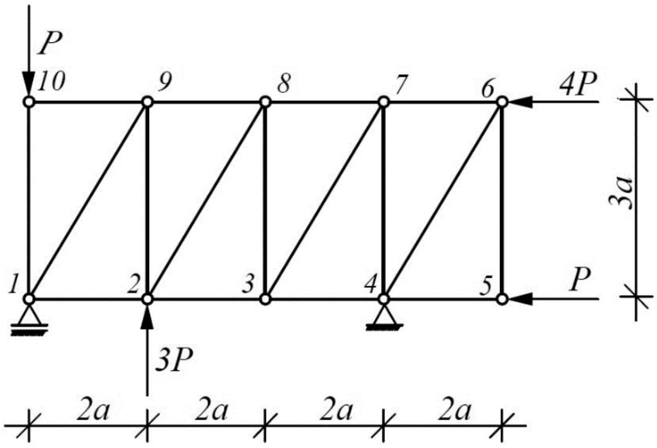


Fig. 2.6.16.

Solution:

The degree of static indeterminacy:
$n = r + m - 2j = 3 + 17 - 2 \cdot 10 = 0$

„Zero-force" members are presented in Fig. 2.6.17.

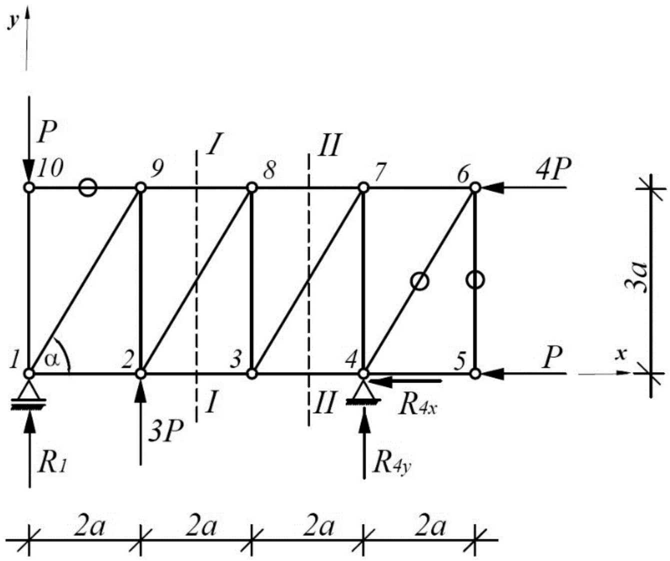


Fig. 2.6.17.

Support reactions (Fig. 2.6.17.) can be determined by using the three global equilibrium equations for the planar structures:

$\sum M_1 = 0 \;\rightarrow\; 3P \cdot 2a + R_{4y} \cdot 6a + 4P \cdot 3a = 0 \;\rightarrow\; R_{4y} = -3P$

$\sum F_{iy} = 0 \;\rightarrow\; R_1 + 3P + R_{4y} - \mathrm{P} = 0 \;\rightarrow\; R_1 = P$

$\sum F_{ix} = 0 \;\rightarrow\; -R_{4x} - 4P - \mathrm{P} = 0 \;\rightarrow\; R_{4x} = -5P$

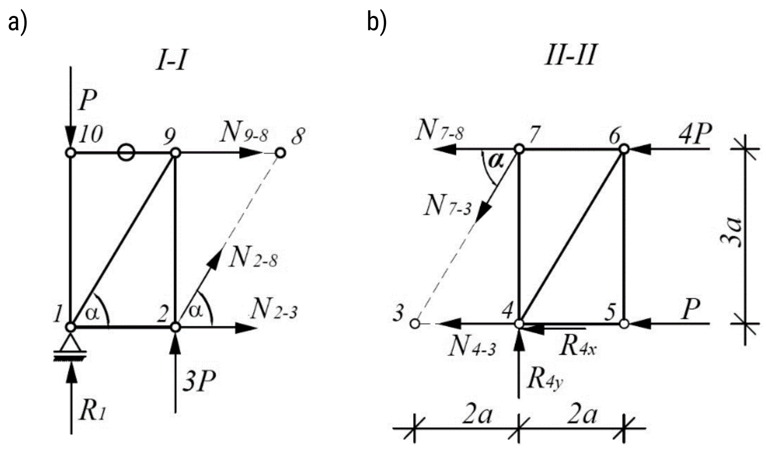


Fig. 2.6.18.

To determine forces of members 2–3, 9–8, 2–8 we can use the vertical cutting plane I–I shown in Fig. 2.6.17. and consider the left truss segment (Fig. 2.6.18.a.).

Three equations of equilibrium have the form:

$$sin\alpha = \frac{3}{\sqrt{13}}$$

$$\sum M_2 = 0 \;\rightarrow\; -N_{9-8} \cdot 3a + P \cdot 2a - R_1 \cdot 2a = 0 \;\rightarrow\; N_{9-8} = 0$$

$$\sum M_8 = 0 \;\rightarrow\; N_{2-3} \cdot 3a - R_1 \cdot 4a - 3P \cdot 2a + P \cdot 4a = 0 \;\rightarrow\; N_{2-3} = 2P$$

$$\sum F_{iy} = 0 \;\rightarrow\; R_1 + N_{2-8} \cdot sin\alpha + 3P - P = 0 \;\rightarrow\; N_{2-8} = -\sqrt{13}P$$

To determine forces of members 7–8, 7–3, 3–4 we will use the vertical cutting plane II–II shown in Fig. 2.6.17. and will consider the right truss segment (Fig. 2.6.18.b.).

Three equations of equilibrium have the form:

$$\sum M_3 = 0 \;\rightarrow\; N_{7-8} \cdot 3a + 4P \cdot 3a + R_{4y} \cdot 2a = 0 \;\rightarrow\; N_{7-8} = -2P$$

$$\sum M_7 = 0 \;\rightarrow\; -N_{4-3} \cdot 3a - R_{4x} \cdot 3a - P \cdot 3a = 0 \;\rightarrow\; N_{4-3} = 4P$$

$$\sum F_{iy} = 0 \;\rightarrow\; R_{4y} - N_{7-3} \cdot sin\alpha = 0 \;\rightarrow\; N_{7-3} = -\sqrt{13}P$$

Computational example 2.6.4.

Determine the member forces 3–5, 3–6, 4–6, 7–5, 7–6, 6–8 of planar truss. The truss and loading are defined by Fig. 2.6.19.a.

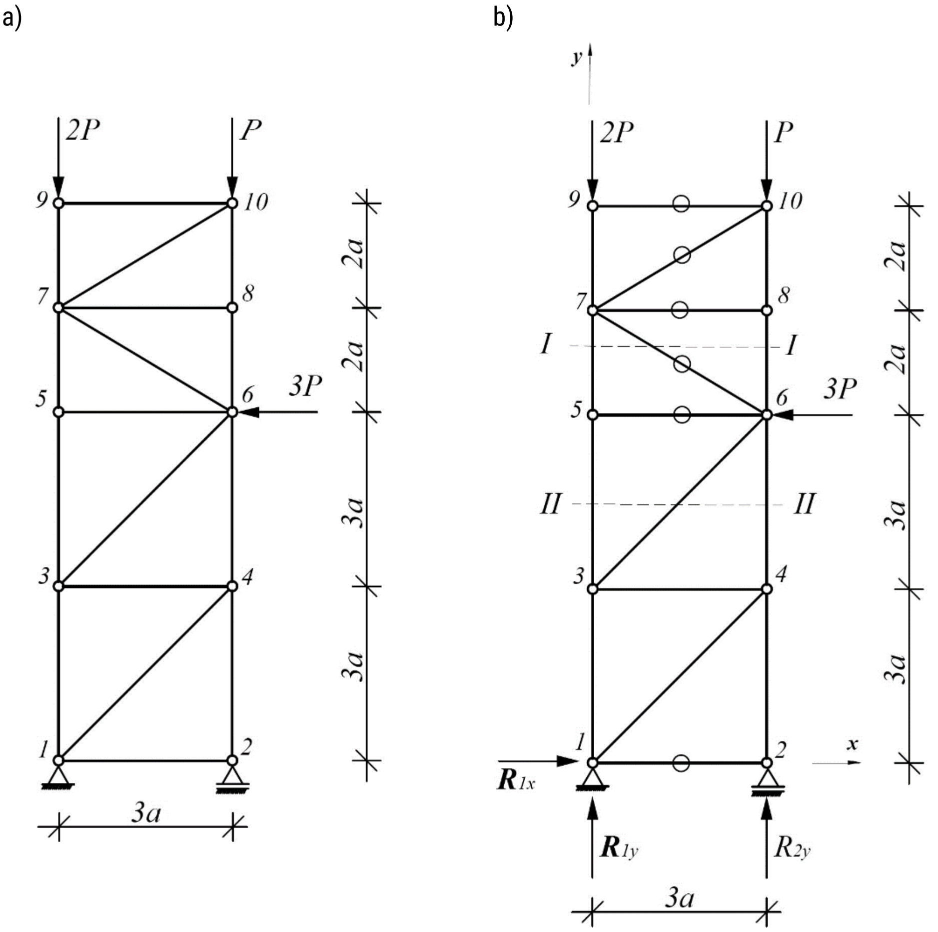


Fig. 2.6.19.

Solution:

The degree of static indeterminacy:

$n = r + m - 2j = 3 + 17 - 2 \cdot 10 = 0$

„Zero-force" members are shown in Fig. 2.6.19.

Support reactions (Fig. 2.6.19.b.):

$\sum M_1 = 0 \;\rightarrow\; 3P \cdot 6a + R_{2y} \cdot 3a - P \cdot 3a = 0 \;\rightarrow\; R_{2y} = -5P$

$\sum F_{iy} = 0 \;\rightarrow\; R_{1y} - 2P + R_{2y} - \mathrm{P} = 0 \;\rightarrow\; R_1 = 8P$

$\sum F_{ix} = 0 \;\rightarrow\; R_{1x} - 3P = 0 \;\rightarrow\; R_{1x} = 3P$

To determine forces of members 7–5, 7–6, 6–8, we will use the horizontal cutting plane I–I shown in Fig. 2.6.19.b. and will consider the upper truss segment (Fig. 2.6.20.a.).

$cos\beta = \dfrac{3\sqrt{13}}{13}$

In this case three equations of equilibrium have the form:

$\sum F_{ix} = 0 \rightarrow \; N_{6-7} \cdot cos\beta = 0 \rightarrow \; N_{6-7} = 0$

$\sum M_7 = 0 \rightarrow \; -N_{6-8} \cdot 3a - P \cdot 3a = 0 \rightarrow \; N_{6-8} = -P$

$\sum M_6 = 0 \rightarrow \; 2P \cdot 3a + N_{5-7} \cdot 3a = 0 \;\rightarrow\; N_{5-7} = -2P$

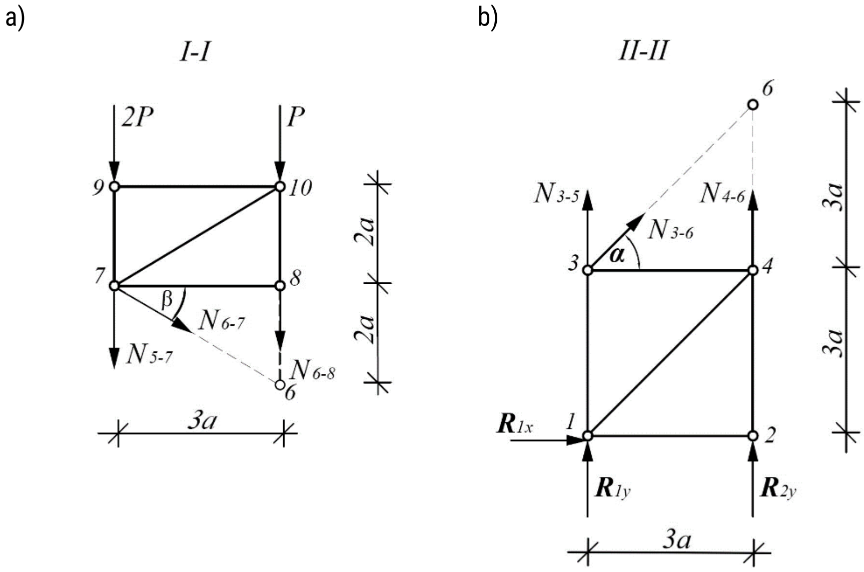


Fig. 2.6.20.

To determine forces of members 3–5, 3–6, 4–6, we will use the horizontal cutting plane II–II shown in Fig. 2.6.19.b. and will consider the bottom truss segment (Fig. 2.6.20.b.).

Equilibrium equations for the bottom segment of truss:

$$cos\alpha = \frac{\sqrt{2}}{2}$$

$$\sum F_{ix} = 0 \rightarrow \quad R_{1x} + N_{6-3} \cdot cos\alpha = 0 \rightarrow \quad N_{6-3} = \frac{-3P}{cos\alpha} = -3\sqrt{2}P$$

$$\sum M_6 = 0 \rightarrow \quad -N_{3-5} \cdot 3a - R_{1y} \cdot 3a + R_{1x} \cdot 6a = 0 \rightarrow \quad N_{3-5} = -2P$$

$$\sum M_3 = 0 \rightarrow \quad N_{4-6} \cdot 3a + R_{2y} \cdot 3a + R_{1x} \cdot 3a = 0 \rightarrow \quad N_{4-6} = 2P$$

Computational example 2.6.5.

Determine the member forces 8–7, 3–4, 13–7, 13–4 of K-type truss using the method of sections. The truss and loading are defined by Fig. 2.6.21.

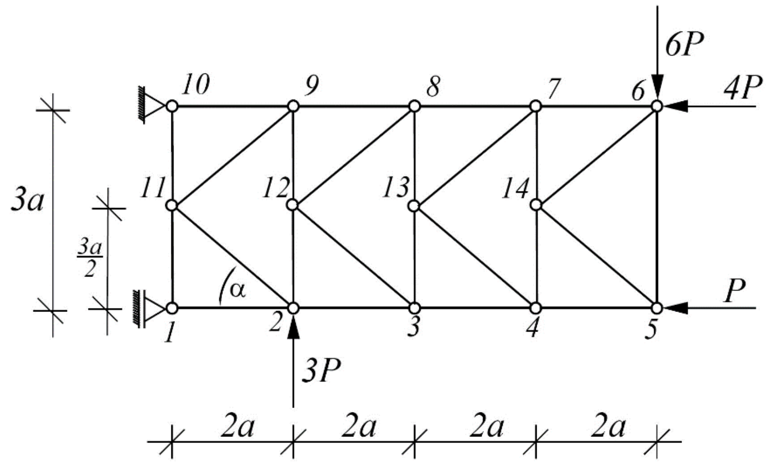


Fig. 2.6.21.

Solution:

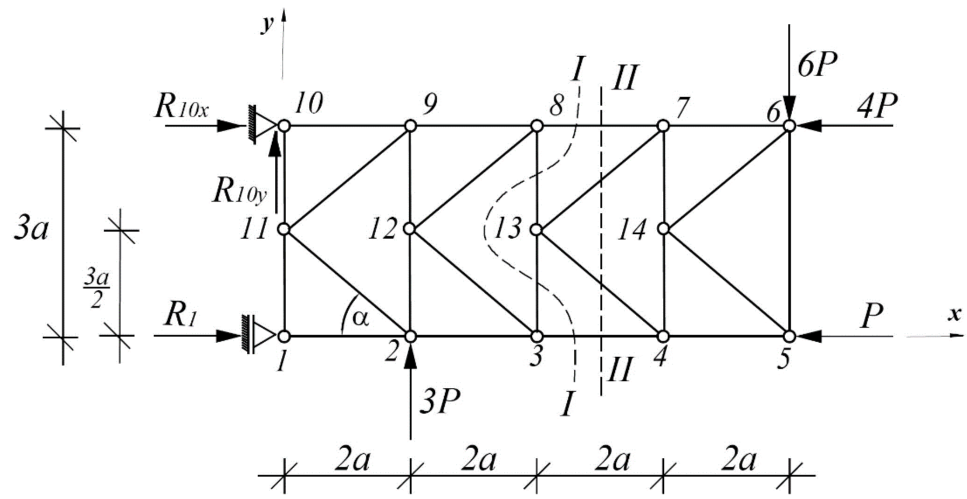


Fig. 2.6.22.

The degree of static indeterminacy:
$n = r + m - 2j = 3 + 25 - 2 \cdot 14 = 0$

We do not have to determine the support reactions if we consider the right segment of truss.

A vertical section such as I–I cuts four truss members and does not lead to a solution. There are no vertical cutting planes that involve only three unknown forces. For this type of truss, we have to also take into account plane II–II to get the solution.

To determine forces of members 8–7 and 3–4 we can use the vertical cutting plane I–I shown in Fig. 2.6.22. and consider the right truss segment (Fig. 2.6.23.a.).

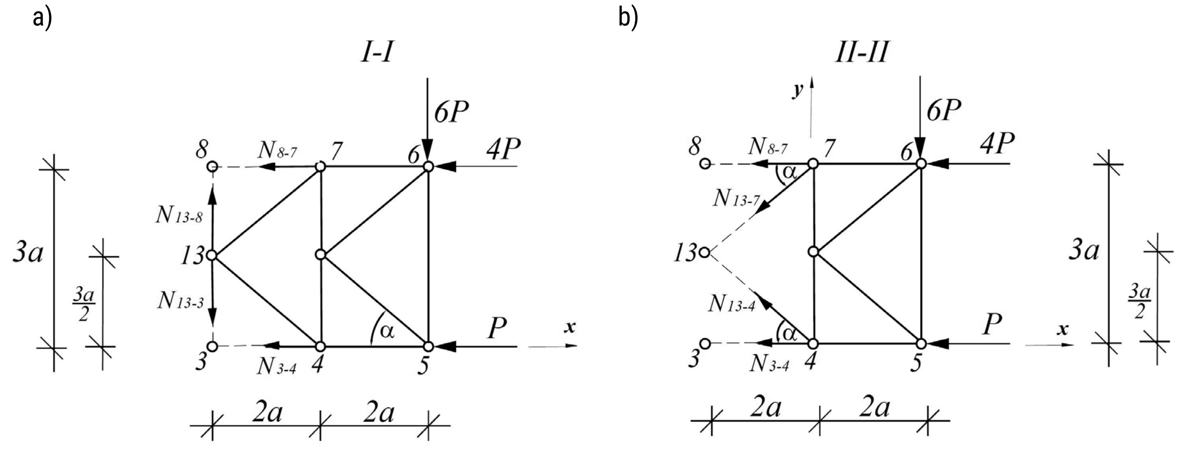


Fig. 2.6.23.

Equilibrium equations have the form:

$$\sum M_3 = 0 \;\rightarrow\; N_{8-7} \cdot 3a + 4P \cdot 3a - 6P \cdot 4a = 0 \;\rightarrow\; N_{8-7} = 4P$$
$$\sum M_8 = 0 \;\rightarrow\; -N_{3-4} \cdot 3a - P \cdot 3a - 6P \cdot 4a = 0 \;\rightarrow\; N_{3-4} = -9P$$

To determine forces of members 13–7 and 13–4 we can use the vertical cutting plane II–II shown in Fig. 2.6.22. and consider the equilibrium of the right truss segment (Fig. 2.6.23.b.).

Summation of forces in the *x* and *y* direction gives:

$$\sum F_{iy} = 0 \ \rightarrow \ -6P - N_{13-7} \cdot sin\alpha + N_{13-4} \cdot sin\alpha = 0$$

$$\sum F_{ix} = 0 \ \rightarrow \ -4P - P - N_{13-7} \cdot cos\alpha - N_{13-4} \cdot cos\alpha - N_{8-7} - N_{3-4} = \\ = -N_{13-7} \cdot cos\alpha - N_{13-4} \cdot cos\alpha = 0$$

$sin\alpha = \frac{3}{5} \qquad cos\alpha = \frac{4}{5}$

Hence:

$N_{13-7} = -5P$

$N_{13-4} = 5P$

2.6.4. Review problems – trusses

Problem 1.

Determine the member forces 3–5, 3–4, 2–4, 5–6, 6–B, B–8, 6–7, 6–8, 9–11, 10–11, 10–C for the compound truss shown in Fig. 2.6.24.

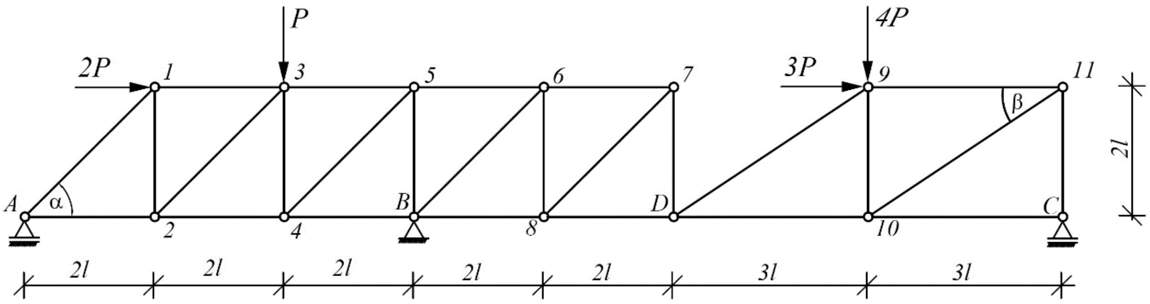


Fig. 2.6.24.

Problem 2.

Determine the member forces of planar truss by the method of joints. The truss and loading are defined by Fig. 2.6.25.

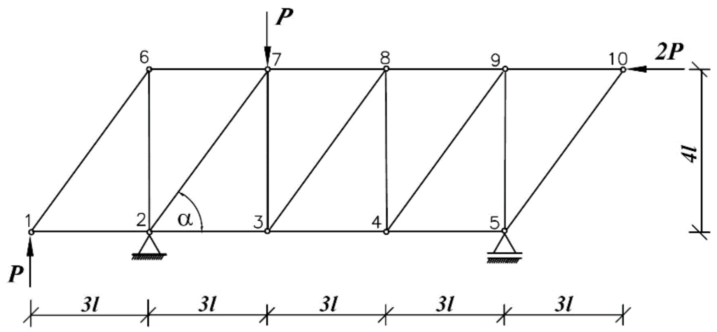


Fig. 2.6.25.

Problem 3.

Determine the member forces 9–8, 2–8, 2–3, 7–6, 7–4 and 3–4 of planar truss. The truss and loading are defined by Fig. 2.6.26.

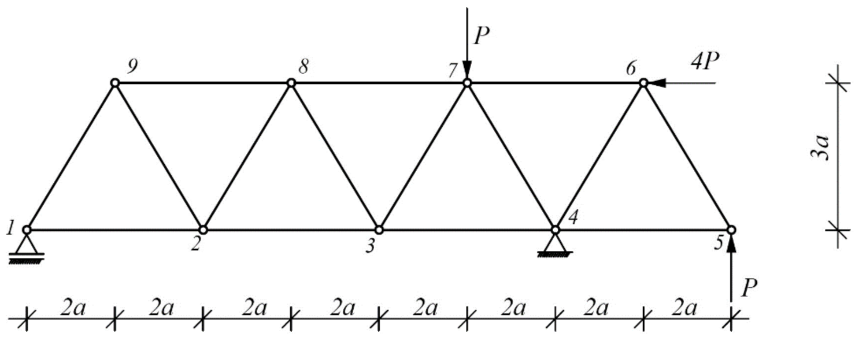
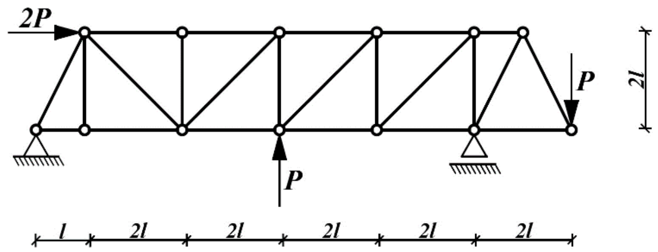


Fig. 2.6.26.

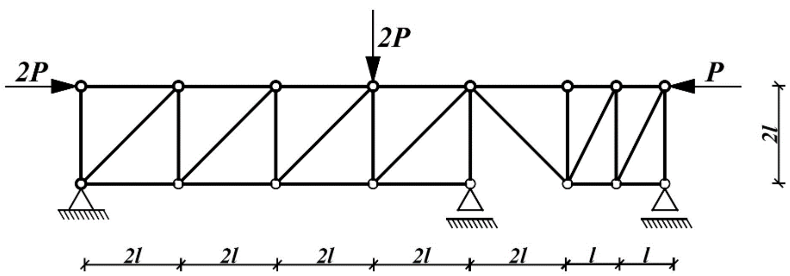
Problem 4.

Determine the member forces of planar truss presented below.

a)



b)



c)

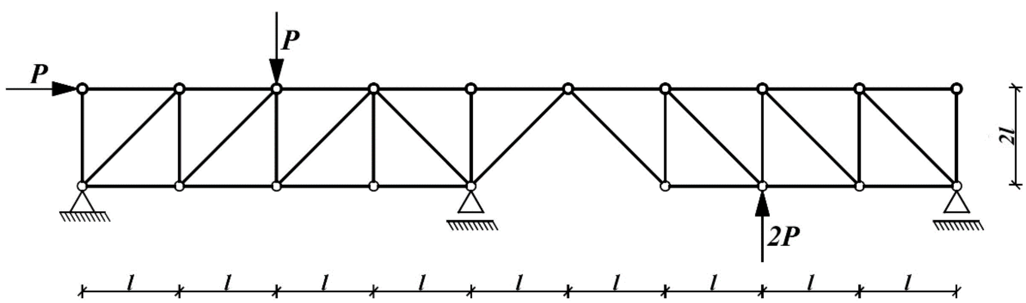


Fig. 2.6.27.

d)

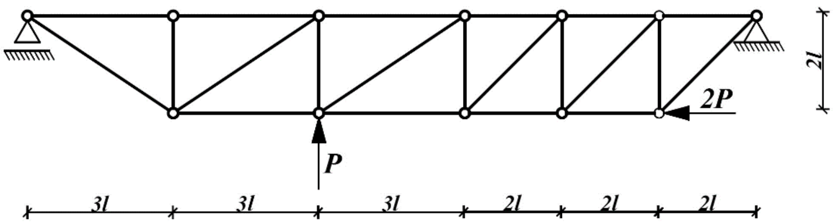


Fig. 2.6.27.

3. Review problem solutions

3.1. Review problem solutions – beams

Problem 1.

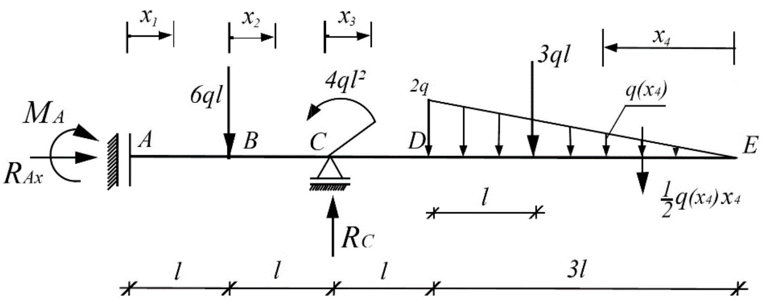


Fig. 3.1.1.

1. n = 0
2. Reactions:

$\sum F_{iy} = 0;$ $\quad -6ql + R_C - 3ql = 0$

$\sum F_{ix} = 0;$ $\quad R_{Ax} = 0$

$\sum M_{(C)} = 0;$ $\quad -M_A + 6ql \cdot l + 4ql^2 - 3ql \cdot 2l = 0$

$R_{Ax} = 0 \qquad R_C = 9ql \qquad M_A = 4ql^2$

3. Internal forces equations:

Segment A–B $\quad 0 \le x_1 < l$

$$V(x_1) = 0$$
$$M(x_1) = 4ql^2$$
$$N(x_1) = 0$$

Segment B–C $\quad 0 \le x_2 < l$

$$V(x_2) = -6ql$$
$$M(x_2) = -6ql \cdot x_2 + 4ql^2$$
$$M(x_2 = 0) = 4ql^2$$
$$M(x_2 = l) = -6ql \cdot l + 4ql^2 = -2ql^2$$
$$N(x_2) = 0$$

Segment C–D $\quad 0 \le x_3 < l$

$$V(x_3) = -6ql + 9ql = 3ql$$
$$M(x_3) = 4ql^2 - 6ql \cdot (l + x_3) + 9ql \cdot x_3 - 4ql^2$$
$$M(x_3 = 0) = 4ql^2 - 6ql \cdot l - 4ql^2 = -6ql^2$$
$$M(x_3 = l) = 4ql^2 - 6ql \cdot (l + l) + 9ql \cdot l - 4ql^2 = -3ql^2$$
$$N(x_3) = 0$$

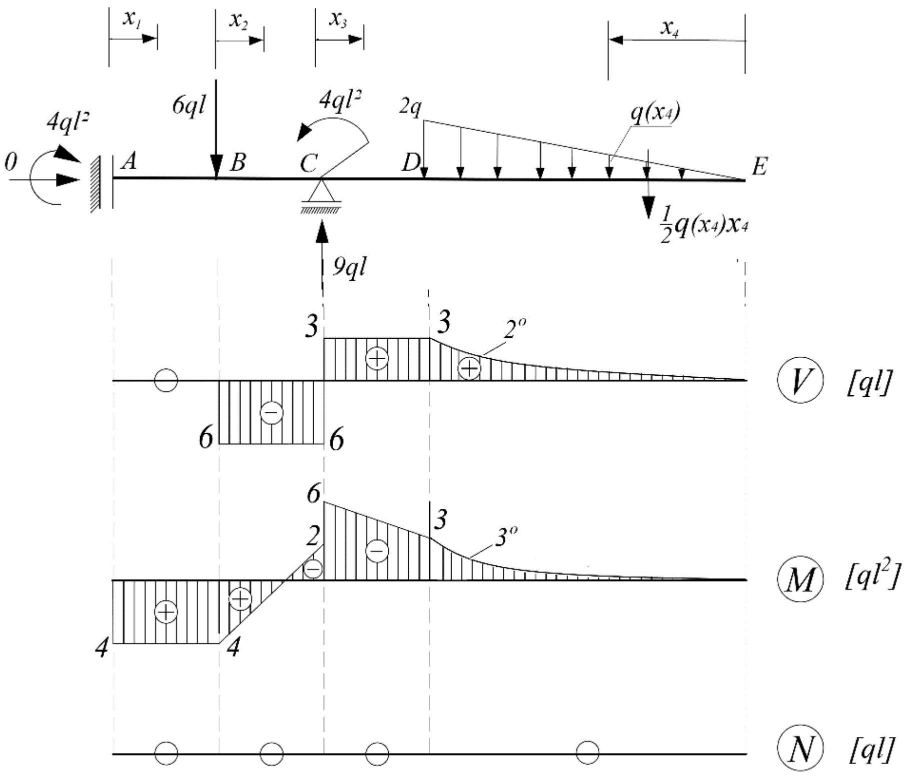


Fig. 3.1.2.

<u>Segment E–D</u> $\quad 0 \leq x_4 < 3l$

$$\frac{q(x_4)}{x_4} = \frac{2q}{3l} \quad \rightarrow \quad q(x_4) = \frac{2qx_4}{3l}$$

$$V(x_4) = \frac{1}{2}q(x_4) \cdot x_4 = \frac{1}{2} \cdot \frac{2qx_4}{3l} \cdot x_4 = \frac{q{x_4}^2}{3l}$$

$$V(x_4 = 0) = 0$$

$$V\left(x_4 = \frac{3}{2}l\right) = \frac{3}{4}ql$$

$$V(x_4 = 3l) = 3ql$$

$$M(x_4) = -\frac{1}{2}q(x_4) \cdot x_4 \cdot \frac{x_4}{3} = -\frac{1}{2} \cdot \frac{2qx_4}{3l} \cdot x_4 \cdot \frac{x_4}{3} = -q \cdot \frac{(x_4)^3}{9l}$$

$$M(x_4 = 0) = 0$$

$$M\left(x_4 = \frac{3}{2}l\right) = -\frac{3}{8}ql^2$$

$$M(x_4 = 3l) = -q \cdot \frac{(3l)^3}{9l} = -3ql^2$$

$$N(x_4) = 0$$

Problem 2.

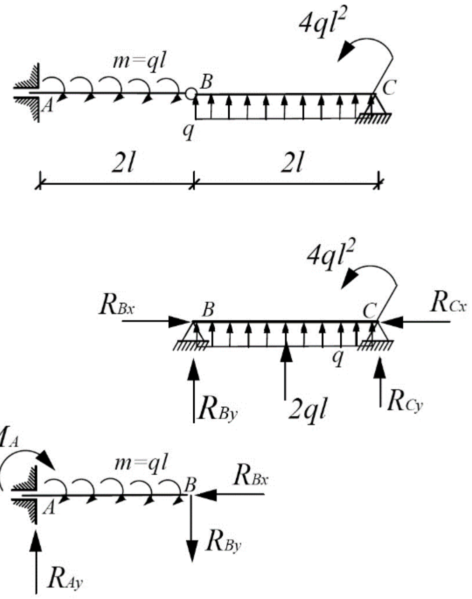


Fig. 3.1.3.

1. $n = r + 2s - 3t = 4 + 2 - 3 \cdot 2 = 0$

2. Reactions:

$\sum F_{ix} = 0; \qquad R_{Bx} = R_{Cx} = 0$

– right beam:

$\sum M_{(C)} = 0; \quad -R_{By} \cdot 2l - 2ql \cdot l + 4ql^2 = 0 \qquad R_{By} = ql$

$\sum F_{iy} = 0; \qquad R_{By} + R_{Cy} + 2ql = 0 \qquad R_{Cy} = -3ql$

– left beam:

$\sum M_{(A)} = 0; \quad -M_A - m \cdot 2l - R_{By} \cdot 2l = -M_A - ql \cdot 2l - R_{By} \cdot 2l = 0 \qquad M_A = -4ql^2$

$\sum F_{iy} = 0; \qquad -R_{By} + R_{Ay} = 0 \qquad R_{Ay} = ql$

3. Internal forces equations:

<u>Segment B–A</u> $\quad 0 \le x_1 < 2l$

$V(x_1) = ql$

$M(x_1) = -m \cdot x_1 - ql \cdot x_1 = -ql \cdot x_1 - ql \cdot x_1 = -2qlx_1$

$M(x_1 = 0) = 0$

$M(x_1 = 2l) = -4ql^2$

<u>Segment B–C</u> $\quad 0 \le x_2 < 2l$

$V(x_2) = ql + qx_2$

$V(x_2 = 0) = ql$

$V(x_2 = 2l) = 3ql$

$M(x_2) = ql \cdot x_2 + qx_2 \cdot \frac{x_2}{2} = ql \cdot x_2 + q \cdot \frac{(x_2)^2}{2}$

$M(x_2 = 0) = 0$

$M(x_2 = 2l) = 4ql^2$

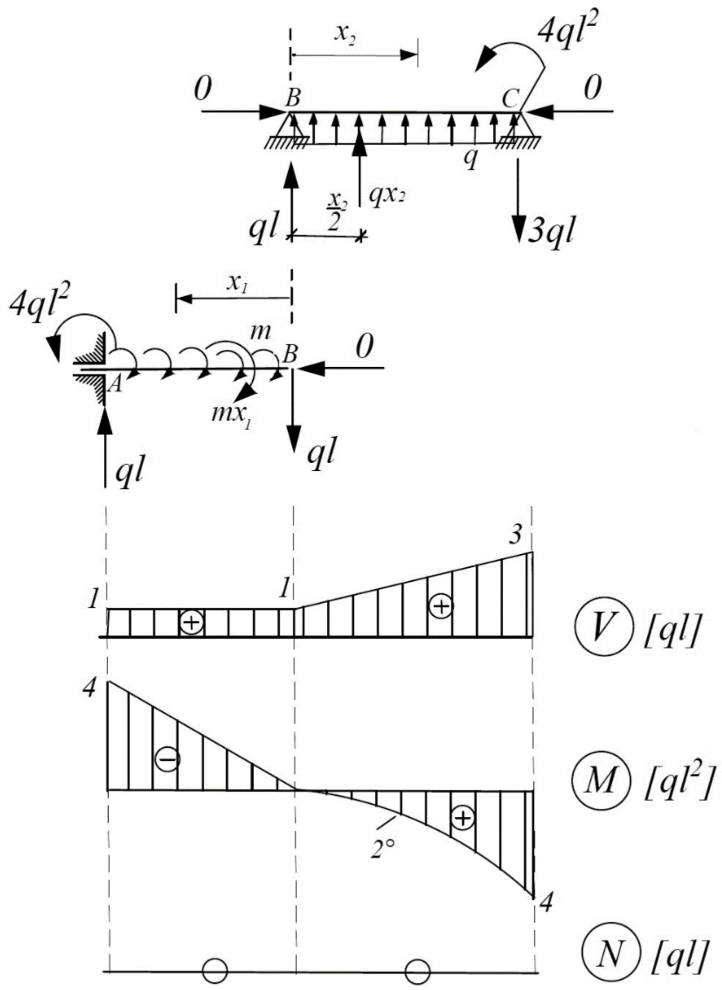


Fig. 3.1.4.

Problem 3.

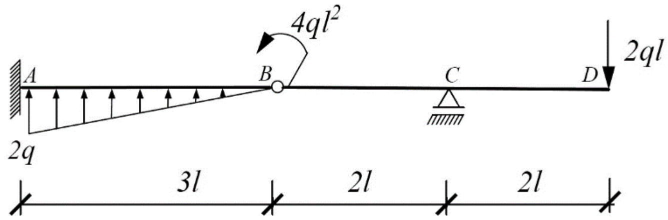


Fig. 3.1.5.

1. $n = r + 2s - 3t = 4 + 2 - 3 \cdot 2 = 0,$

2. Reactions (Fig. 3.1.6.):
   $R_{Bx} = R_{Ax} = 0$

– right beam:

$\sum M_{(C)} = 0; \quad -R_{By} \cdot 2l - 2ql \cdot 2l + 4ql^2 = 0 \qquad R_{By} = 0\ ql$

$\sum F_{iy} = 0; \qquad R_{By} + R_C - 2ql = 0 \qquad R_C = 2ql$

– left beam:

$\sum M_{(A)} = 0; \quad -M_A + 3ql \cdot l - R_{By} \cdot 3l = 0 \qquad M_A = 3\ ql^2$

$\sum F_{iy} = 0; \qquad -R_{By} + 3ql + R_{Ay} = 0 \qquad R_{Ay} = -3ql$

3. Internal forces equations:

<u>Segment B–A</u> $0 \le x_1 < 3l$

$$\frac{q(x_1)}{x_1} = \frac{2q}{3l} \quad \rightarrow \quad q(x_1) = \frac{2qx_1}{3l}$$

$$V(x_1) = -\frac{1}{2}q(x_1) \cdot x_1 = -\frac{1}{2} \cdot \frac{2qx_1}{3l} \cdot x_1 = -\frac{q{x_1}^2}{3l}$$

$$V(x_1 = 0) = 0$$

$$V\left(x_1 = \frac{3}{2}l\right) = -\frac{3}{4}ql$$

$$V(x_1 = 3l) = -3ql$$

$$M(x_1) = \frac{1}{2}q(x_1) \cdot x_1 \cdot \frac{x_1}{3} = \frac{1}{2} \cdot \frac{2qx_1}{3l} \cdot x_1 \cdot \frac{x_1}{3} = q \cdot \frac{(x_1)^3}{9l}$$

$$M(x_1 = 0) = 0$$

$$M\left(x_1 = \frac{3}{2}l\right) = \frac{3}{8}ql^2$$

$$M(x_1 = 3l) = q \cdot \frac{(3l)^3}{9l} = 3ql^2$$

<u>Segment B–C</u> $0 \le x_2 < 2l$

$$V(x_2) = 0$$

$$M(x_2) = -4ql^2$$

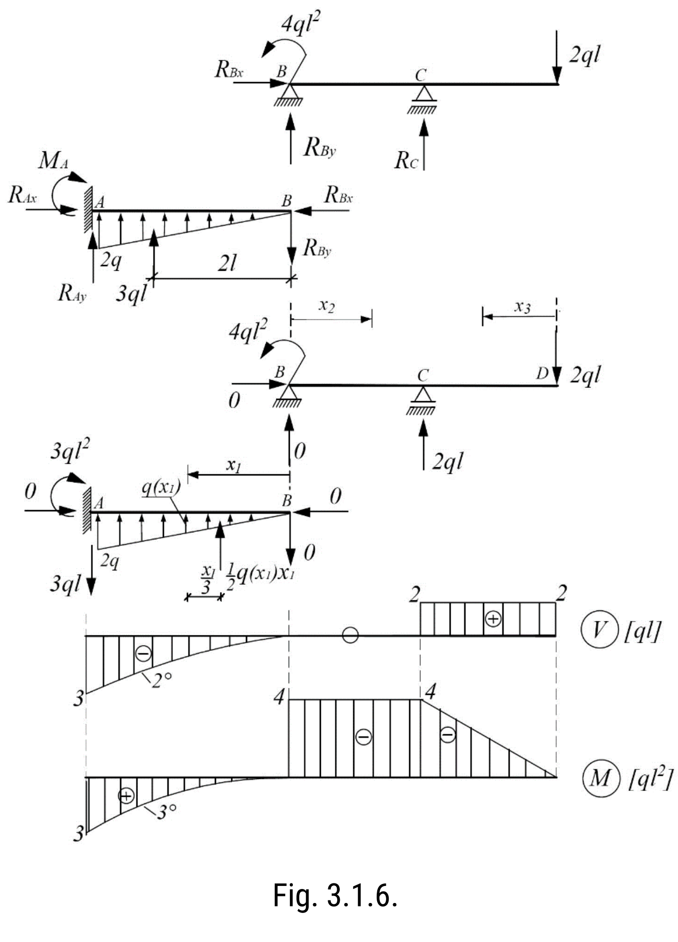


Fig. 3.1.6.

Segment D–C $\quad 0 \leq x_3 < 2l$

$V(x_3) = 2ql$

$M(x_3) = -2ql \cdot x_3$

$M(x_3 = 0) = 0$

$M(x_3 = 2l) = -4ql^2$

There are no horizontal loads, hence $N(x) = 0$.

Problem 4.

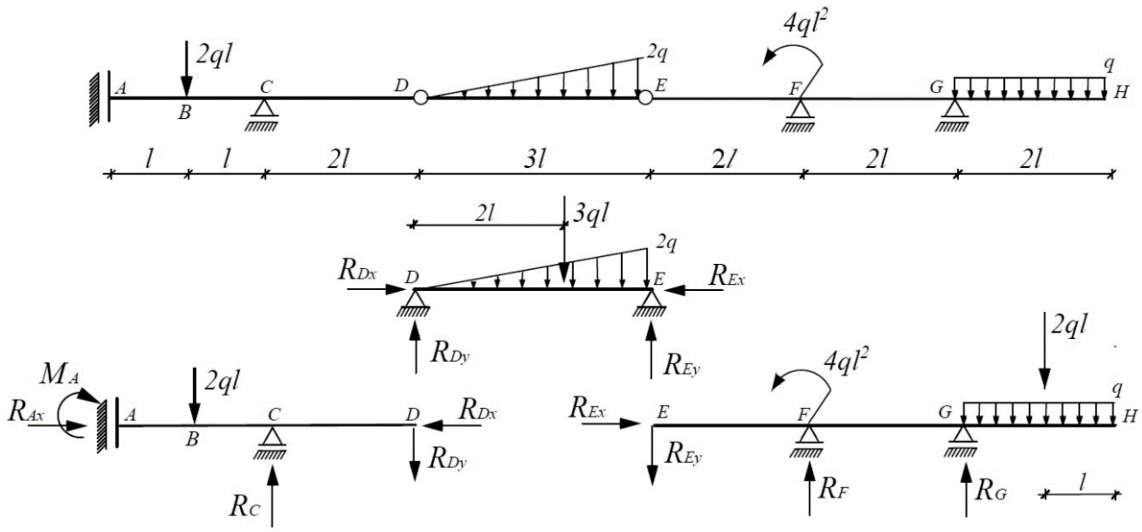


Fig. 3.1.7.

1. $n = r + 2s - 3t = 5 + 4 - 3 \cdot 3 = 0$

2. Reactions:

$R_{Ax} = R_{Dx} = R_{Ex} = 0$

– middle beam:

$\sum M_{(D)} = 0; \quad R_{Ey} \cdot 3l - 3ql \cdot 2l = 0 \qquad R_{Ey} = 2\,ql$

$\sum F_{iy} = 0; \quad R_{Ey} + R_{Dy} - 3ql = 0 \qquad R_{Dy} = ql$

– right beam:

$\sum M_{(F)} = 0; \quad R_G \cdot 2l - 2ql \cdot 3l + R_{Ey} \cdot 2l + 4ql^2 = 0 \qquad R_G = -\,ql$

$\sum F_{iy} = 0; \quad R_G + R_F - R_{Ey} - 2ql = 0 \qquad R_F = 5ql$

– left beam:

$\sum F_{iy} = 0; \quad -R_{Dy} - 2ql + R_C = 0 \qquad R_C = 3ql$

$\sum M_{(A)} = 0; \quad -M_A - 2ql \cdot l + R_C \cdot 2l - R_{Dy} \cdot 4l = 0 \qquad M_A = 0$

3. Internal forces equations:

Segment A–B $\quad 0 \leq x_1 < l$

$V(x_1) = 0$

$M(x_1) = 0$

Segment B–C $\quad 0 \leq x_2 < l$

$$V(x_2) = -2ql$$
$$M(x_2) = -2ql \cdot x_2$$
$$M(x_2 = 0) = 0$$
$$M(x_2 = l) = -2ql^2$$

Segment D–C $\quad 0 \leq x_3 < 2l$

$$V(x_3) = ql$$
$$M(x_3) = -ql \cdot x_3$$
$$M(x_3 = 0) = 0$$
$$M(x_3 = 2l) = -2ql^2$$

Segment D–E $\quad 0 \leq x_4 < 3l$

$$\frac{q(x_4)}{x_4} = \frac{2q}{3l} \quad \rightarrow \quad q(x_4) = \frac{2qx_4}{3l}$$
$$V(x_4) = ql - \frac{1}{2} q(x_4) \cdot x_4 = ql - \frac{1}{2} \cdot \frac{2qx_4}{3l} \cdot x_4 = ql - \frac{q{x_4}^2}{3l}$$
$$V(x_4 = 0) = ql$$
$$V\left(x_4 = \frac{3}{2}l\right) = \frac{1}{4}ql$$
$$V(x_4 = 3l) = -2ql$$
$$V(x_4) = 0 \quad \rightarrow \quad ql - \frac{1}{2} \cdot \frac{2qx_4}{3l} \cdot x_4 = 0 \quad \rightarrow \quad 3ql^2 - q{x_4}^2 = 0 \quad \rightarrow x_4 = \sqrt{3}l$$
$$M(x_4) = qlx_4 - \frac{1}{2} q(x_4) \cdot x_4 \cdot \frac{x_4}{3} = qlx_4 - \frac{1}{2} \cdot \frac{2qx_4}{3l} \cdot x_4 \cdot \frac{x_4}{3} = qlx_4 - q \cdot \frac{(x_4)^3}{9l}$$
$$M(x_4 = 0) = 0$$
$$M(x_4 = 3l) = 0$$
$$M\left(x_4 = \sqrt{3}l\right) = ql \cdot \sqrt{3}l - q \cdot \frac{\left(\sqrt{3}l\right)^3}{9l} = \frac{2}{3}\sqrt{3}ql^2$$

Segment E–F $\quad 0 \leq x_5 < 2l$

$$V(x_5) = -2ql$$
$$M(x_5) = -2ql \cdot x_5$$
$$M(x_5 = 0) = 0$$
$$M(x_5 = 2l) = -4ql^2$$

Segment F–G $\quad 0 \leq x_6 < 2l$

$$V(x_6) = -2ql + 5ql = 3ql$$
$$M(x_6) = -4ql^2 - 2ql \cdot (2l + x_6) + 5ql \cdot x_6$$
$$M(x_6 = 0) = -4ql^2 - 2ql \cdot 2l = -8ql^2$$
$$M(x_6 = 2l) = -4ql^2 - 2ql \cdot (2l + 2l) + 5ql \cdot 2l = -2ql^2$$

Segment H–G $0 \leq x_7 < 2l$

$$V(x_7) = qx_7$$
$$V(x_7 = 0) = 0$$
$$V(x_7 = 2l) = 2ql$$
$$M(x_7) = -qx_7 \cdot \frac{x_7}{2} = -q \cdot \frac{(x_7)^2}{2}$$
$$M(x_7 = 0) = 0$$
$$M(x_7 = 2l) = -2ql^2$$

There are no horizontal loads, hence $N(x) = 0$.

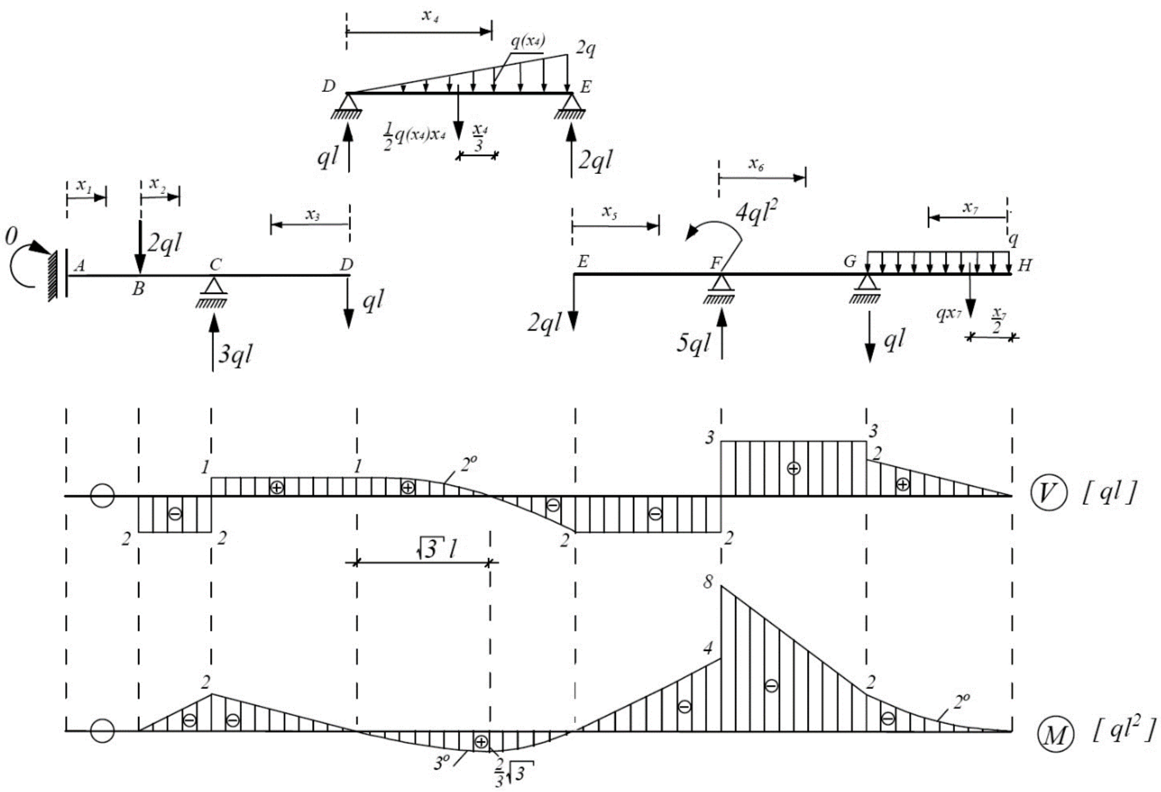


Fig. 3.1.8.

Problem 5.

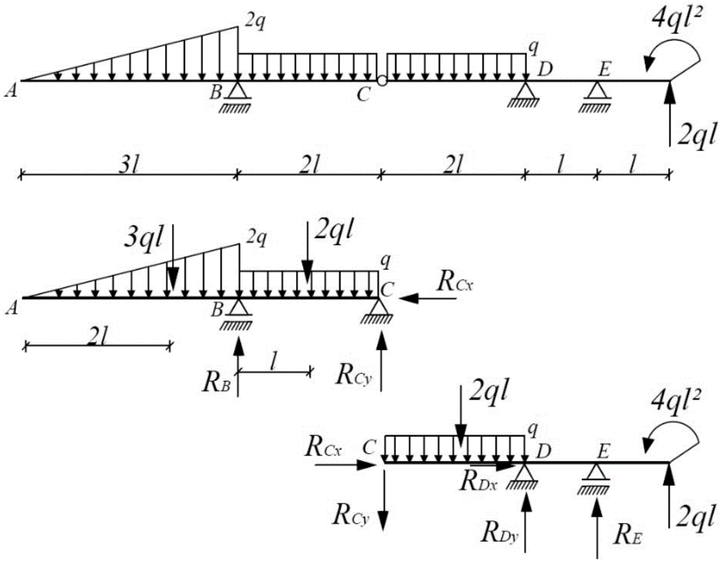


Fig. 3.1.9.

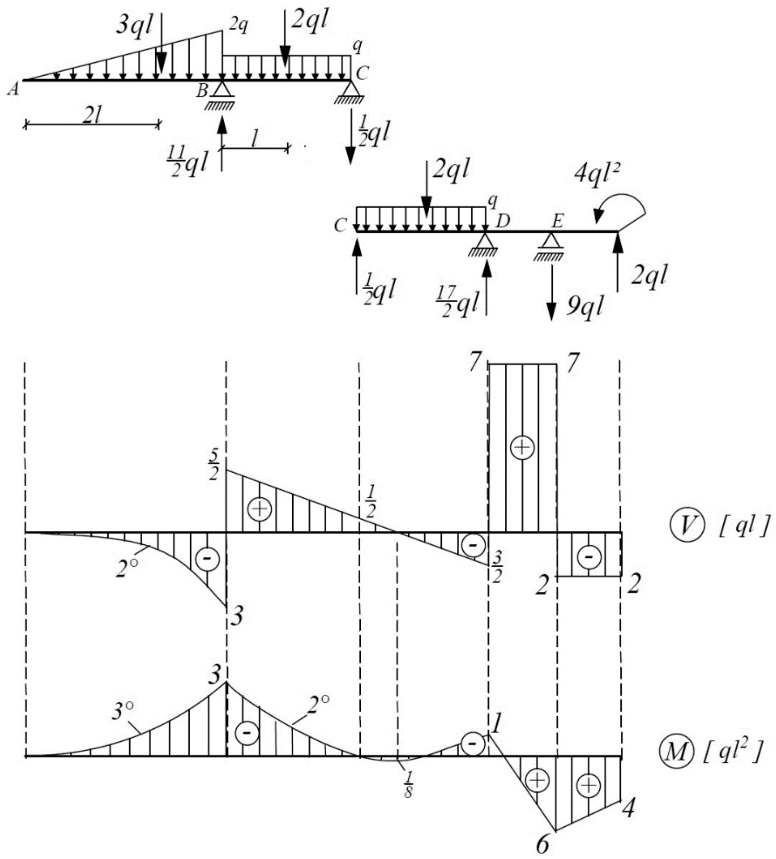


Fig. 3.1.10.

There are no horizontal loads, hence $N(x) = 0$.

Problem 6.

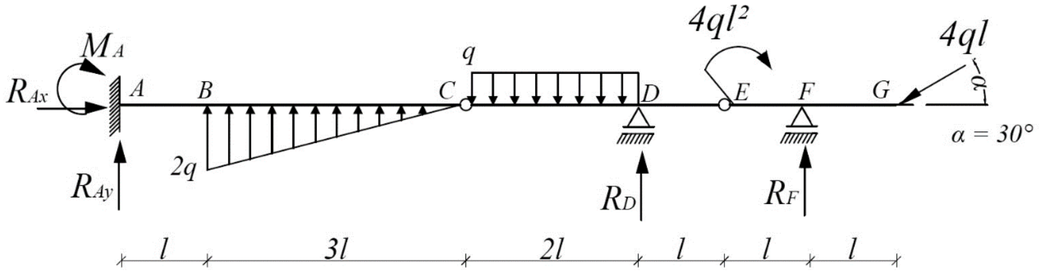


Fig. 3.1.11.

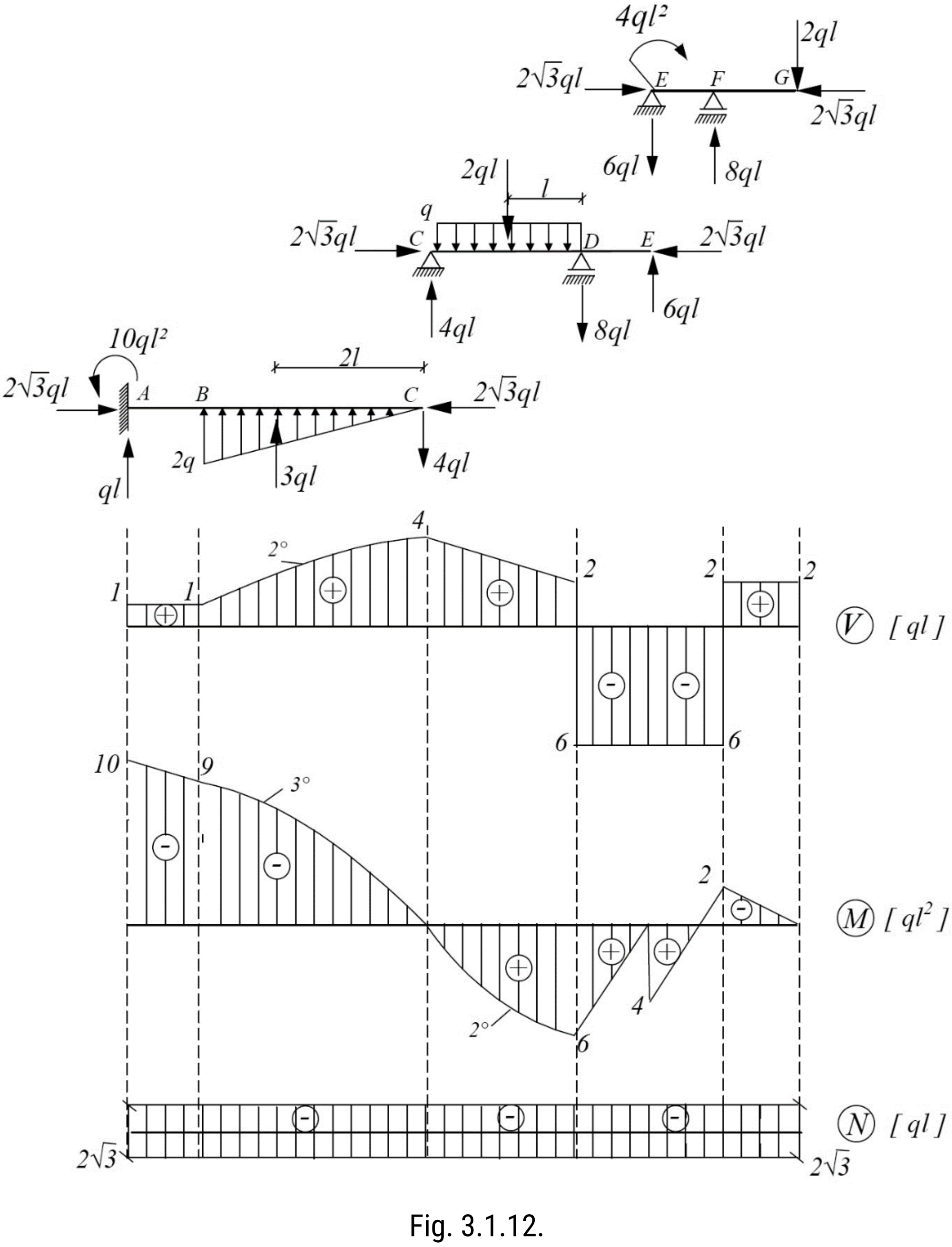


Fig. 3.1.12.

3.2. Review problem solutions – plane frames

Problem 1.

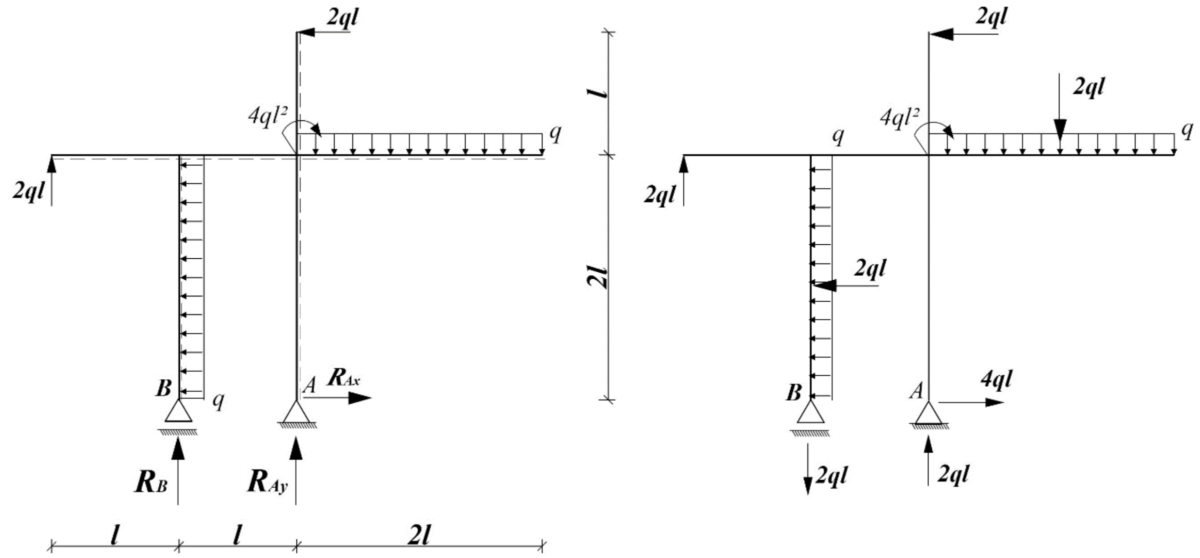


Fig. 3.2.1.

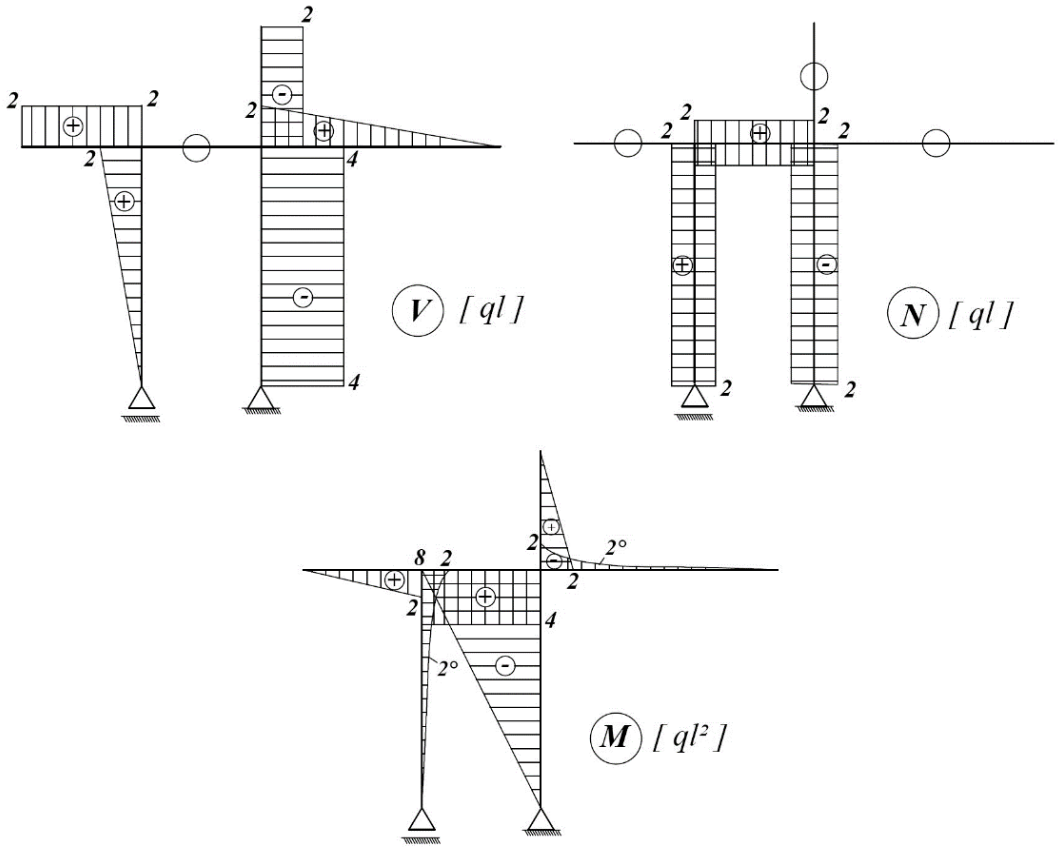


Fig. 3.2.2.

Problem 2.

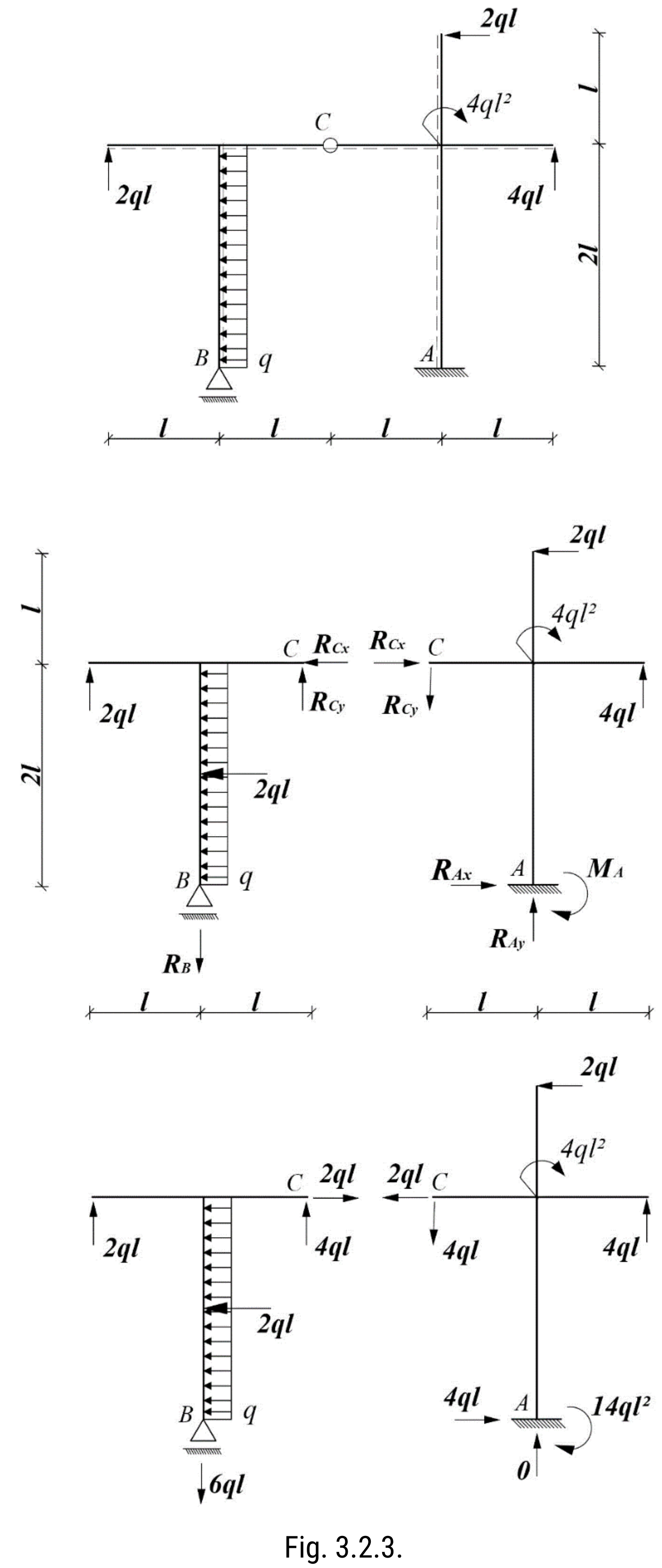


Fig. 3.2.3.

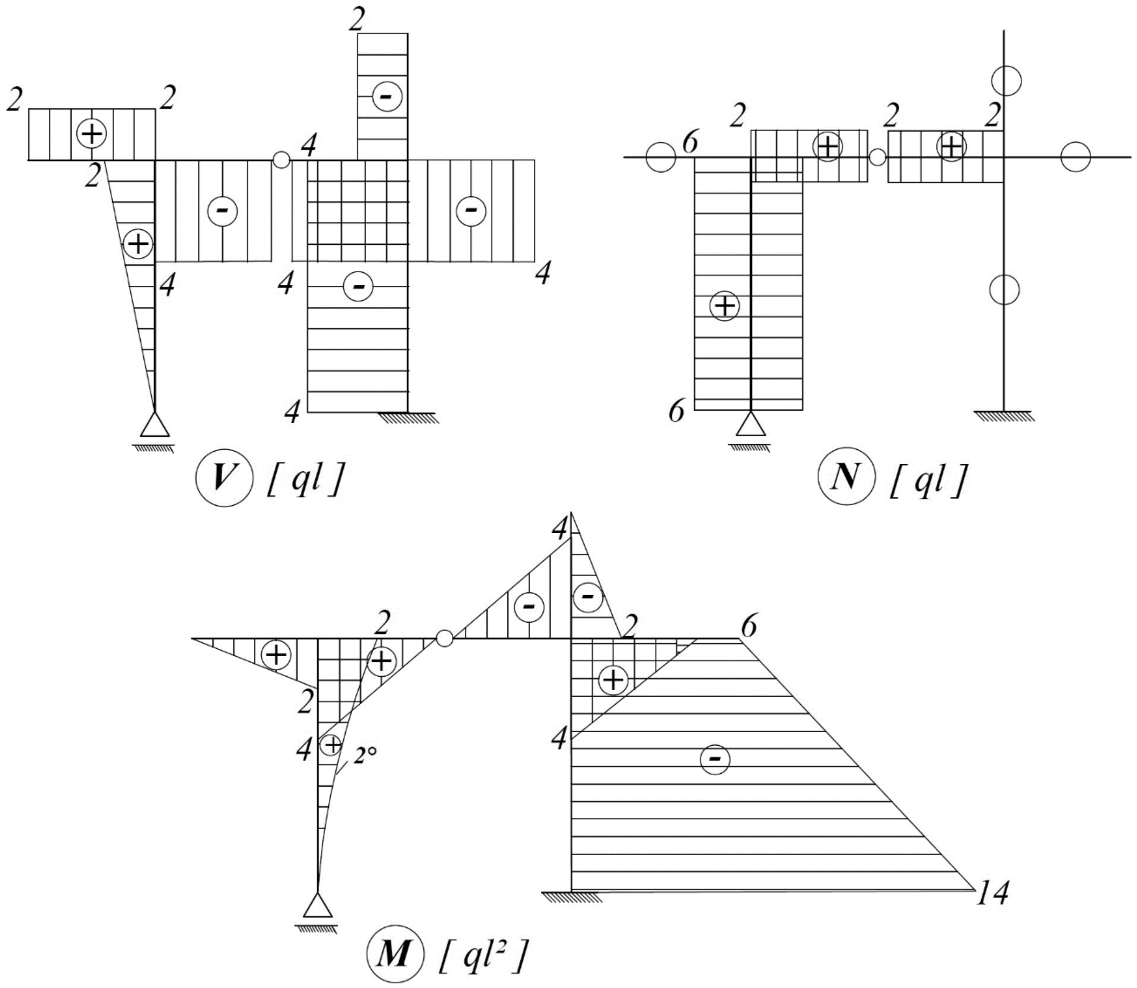


Fig. 3.2.4.

<u>*Problem* 3.</u>

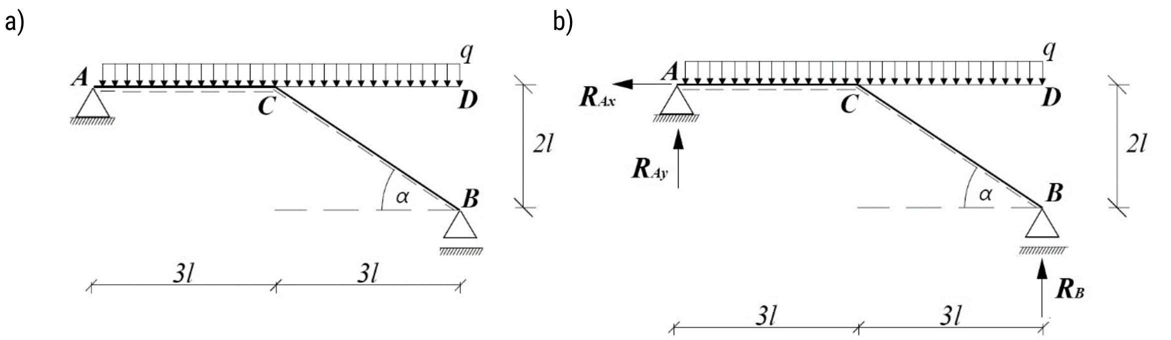


Fig. 3.2.5.

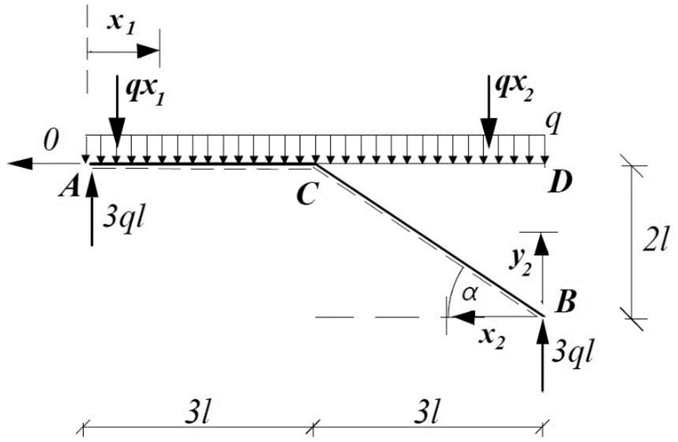


Fig. 3.2.6.

Segment A–C $\quad 0 \leq x_1 < 3l$

$$V(x_1) = 3ql - qx_1$$
$$V(x_1 = 0) = 3ql$$
$$V(x_1 = 3l) = 0$$
$$M(x_1) = 3ql \cdot x_1 - q \cdot \frac{(x_1)^2}{2}$$
$$M(x_1 = 0) = 0$$
$$M(x_1 = 3l) = \frac{9}{2}ql^2$$
$$N(x_1) = 0$$

Segment B–C $\quad 0 \leq x_2 < 3l, \quad 0 \leq y_2 < 2l$

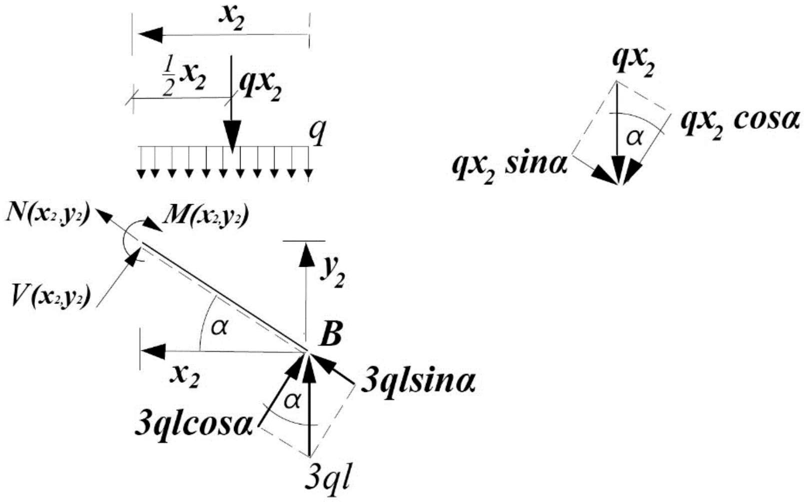


Fig. 3.2.7.

$$cos\alpha = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}, \quad sin\alpha = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$V(x_2, y_2) = -3qlcos\alpha + qx_2cos\alpha = (-3ql + qx_2)\frac{3\sqrt{13}}{13}$$

$$V(x_2 = 0, \quad y_2 = 0) = \frac{-9\sqrt{13}}{13}\, ql$$

$$V(x_2 = 3l, \quad y_2 = 2l) = 0$$

$$M(x_2, y_2) = 3ql \cdot x_2 - qx_2 \cdot \frac{x_2}{2} = 3ql \cdot x_2 - q \cdot \frac{(x_2)^2}{2}$$

$$M(x_2 = 0, \quad y_2 = 0) = 0$$

$$M(x_2 = 3l, \quad y_2 = 2l) = \frac{9}{2}ql^2$$

$$N(x_2, y_2) = -3qlsin\alpha + qx_2sin\alpha = (-3ql + qx_2) \cdot \frac{2\sqrt{13}}{13}$$

$$N(x_2 = 0, \quad y_2 = 0) = \frac{-6\sqrt{13}}{13}\, ql$$

$$N(x_2 = 3l, \quad y_2 = 2l) = 0$$

Internal forces diagrams:

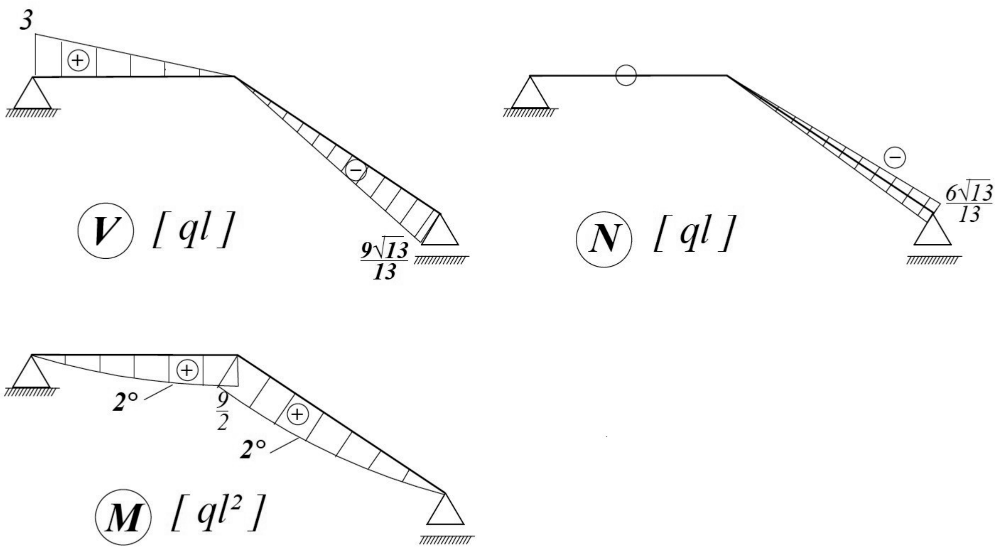


Fig. 3.2.8.

<u>*Problem 4.*</u>

1. $n = r + 2s - 3t = 3 + 6 - 3 \cdot 3 = 0$
2. Reactions (Fig. 3.2.9.):

$\sum M_{(A)} = 0; \quad R_{By} \cdot 2l - 2ql \cdot l - 2ql \cdot l - 2ql \cdot 3l + 4ql^2 = 0 \quad R_{By} = 3ql$

$\sum F_{iy} = 0; \quad R_{By} + 2ql - 2ql - R_A = 0 \quad R_A = 3ql$

$\sum F_{ix} = 0; \quad -R_{Bx} + 2ql = 0 \quad R_{Bx} = 2ql$

– right frame (Fig. 3.2.10.):

$$\sum M_{(C)} = 0; \quad R_{By} \cdot l - R_{Bx} \cdot 4l - 2ql \cdot 2l + N \cdot 2l = 0 \qquad N = \frac{9}{2}ql$$

$$\sum F_{iy} = 0; \quad R_{By} - R_{Cy} - 2ql = 0 \qquad R_{Cy} = ql$$

$$\sum F_{ix} = 0; \quad -R_{cx} - R_{Bx} + N = 0 \qquad R_{Cx} = \frac{5}{2}ql$$

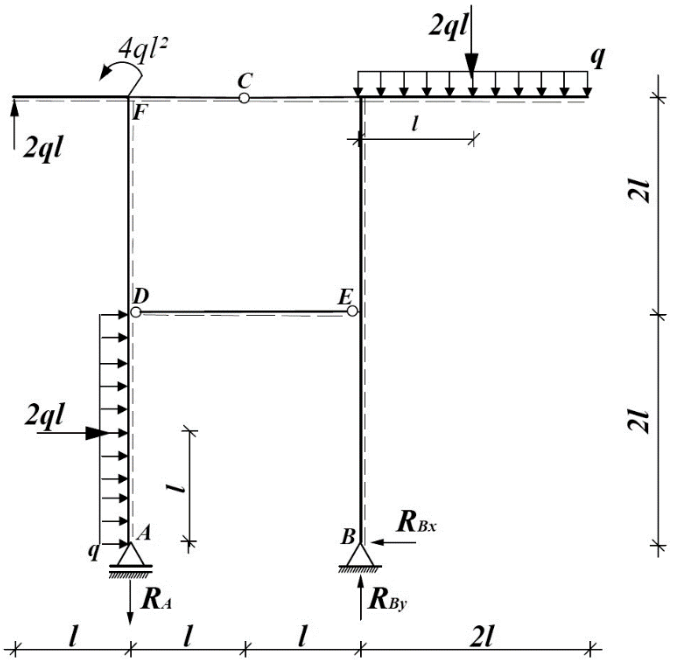


Fig. 3.2.9.

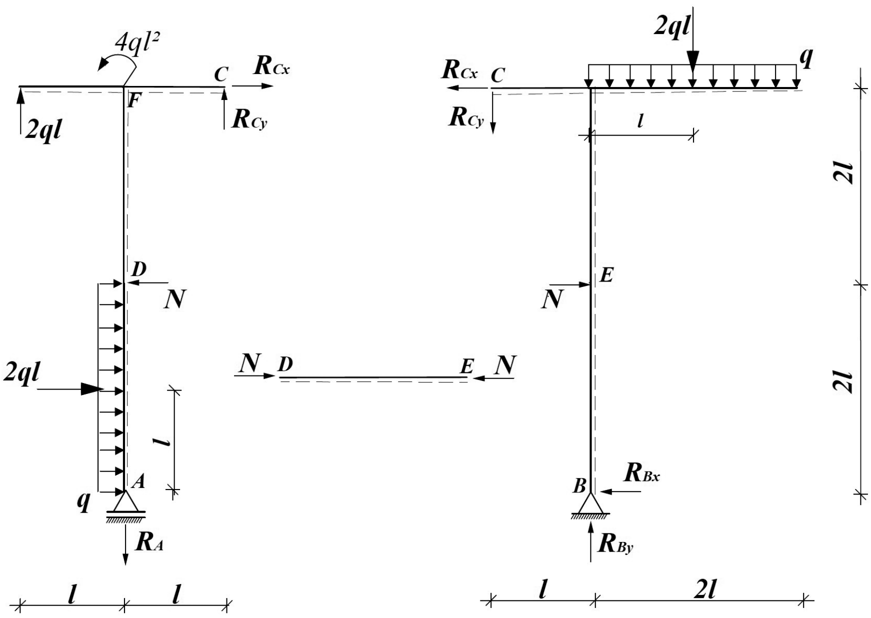


Fig. 3.2.10.

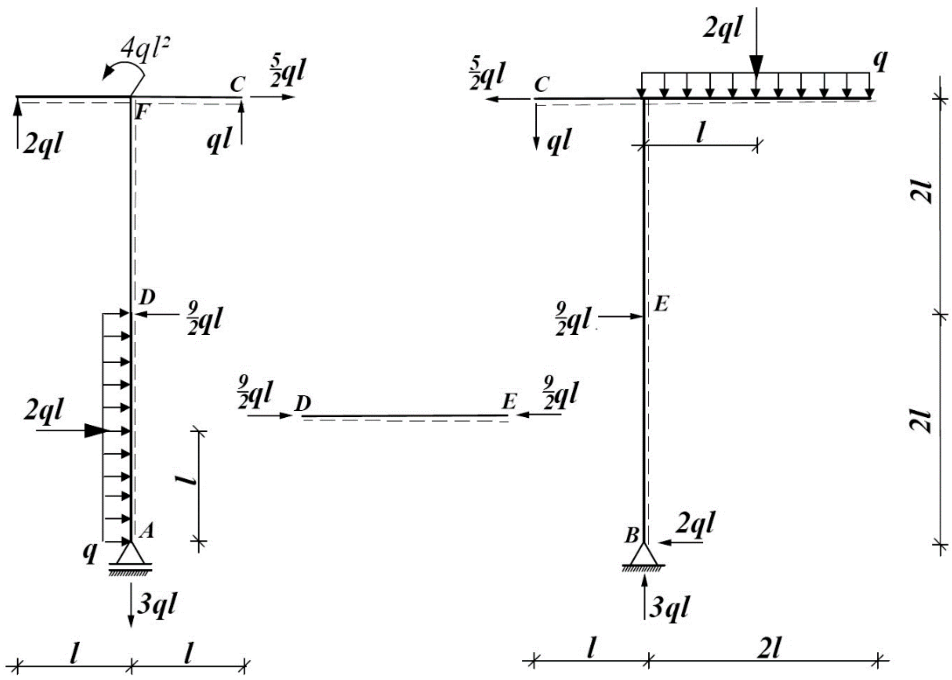


Fig. 3.2.11.

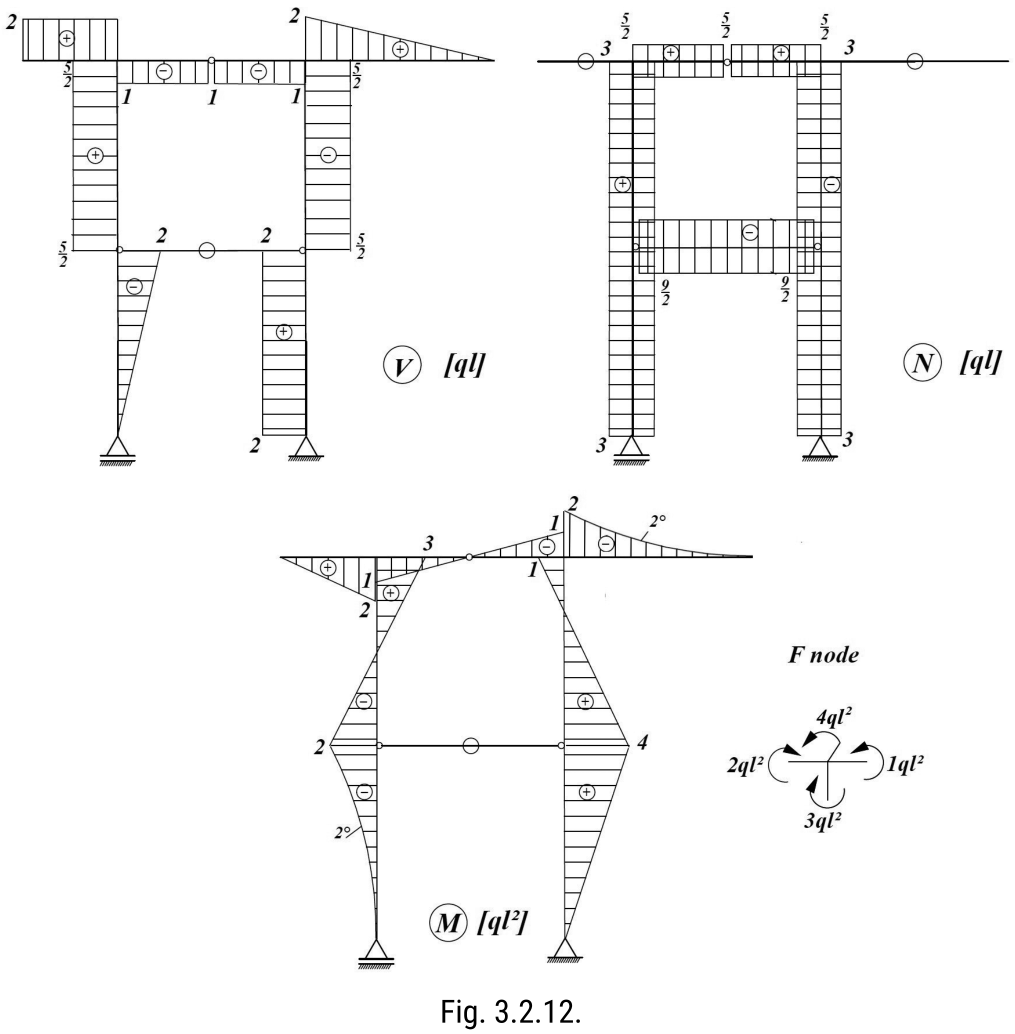


Fig. 3.2.12.

Problem 5.

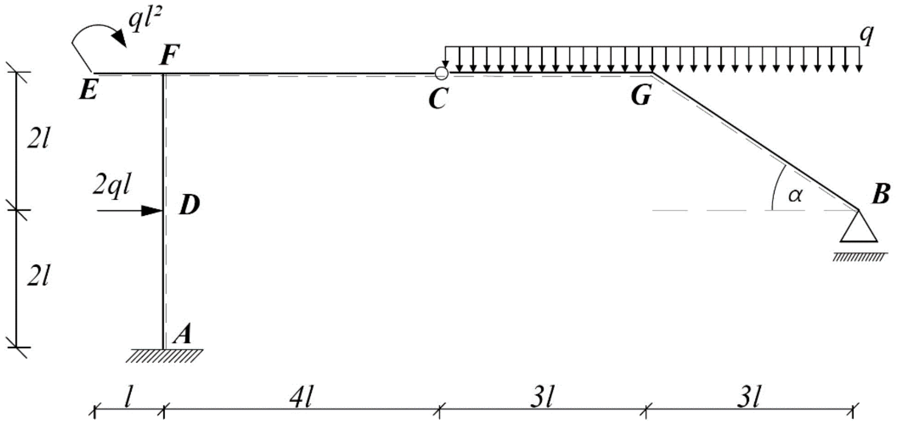


Fig. 3.2.13.a.

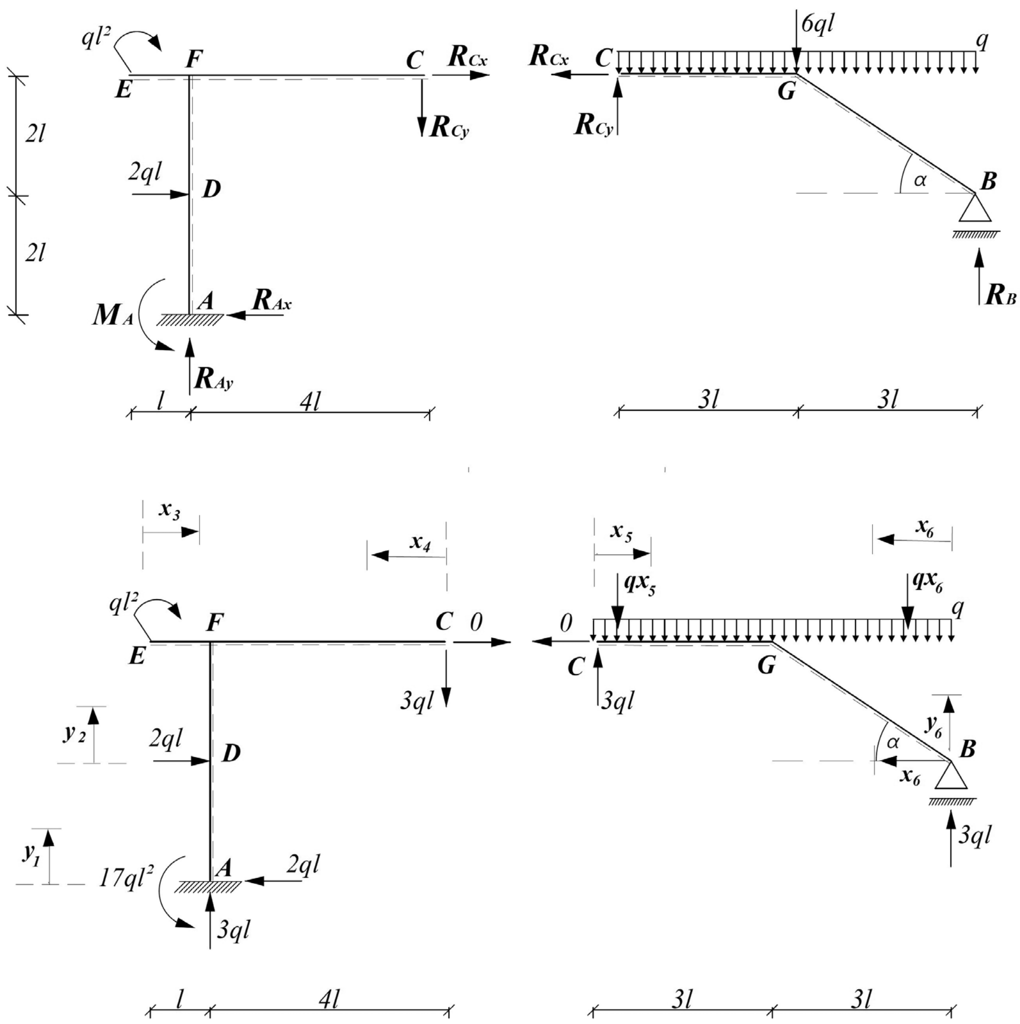


Fig. 3.2.13.b.

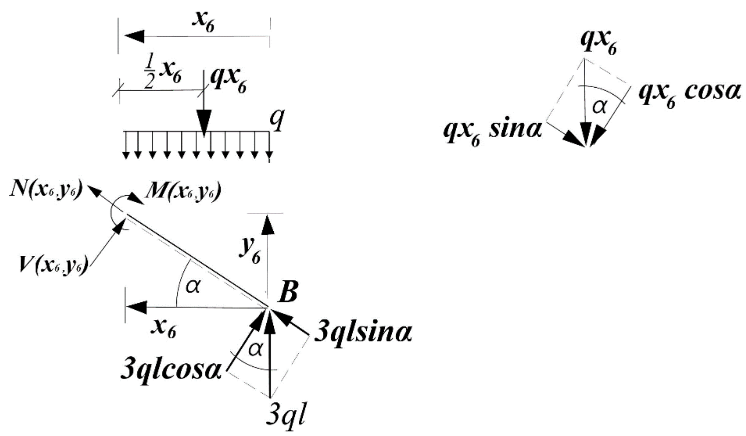


Fig. 3.2.13.c.

<u>Segment B–G</u> $\quad 0 \le x_6 < 3l, \quad 0 \le y_6 < 2l$

The system of forces in segment B–G is presented in Fig. 3.2.13.c.

$$cos\alpha = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}, \qquad sin\alpha = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$V(x_6, y_6) = -3qlcos\alpha + qx_6 cos\alpha = -3ql\frac{3\sqrt{13}}{13} + qx_6\frac{3\sqrt{13}}{13}$$

$$V(x_6 = 0, \quad y_6 = 0) = -9\frac{\sqrt{13}}{13}ql$$

$$V(x_6 = 3l, \quad y_6 = 2l) = 0$$

$$M(x_6, y_6) = 3ql \cdot x_6 - \frac{q(x_6)^2}{2}$$

$$M(x_6 = 0, \quad y_6 = 0) = 0$$

$$M(x_6 = 3l, \quad y_6 = 2l) = \frac{9}{2}\,ql^2$$

$$N(x_6, y_6) = -3qlsin\alpha + qx_6 sin\alpha = -3ql\frac{2\sqrt{13}}{13} + qx_6\frac{2\sqrt{13}}{13}$$

$$N(x_1 = 0, \quad y_1 = 0) = -6\frac{\sqrt{13}}{13}ql$$

$$N(x_1 = 3l, \quad y_1 = 2l) = 0$$

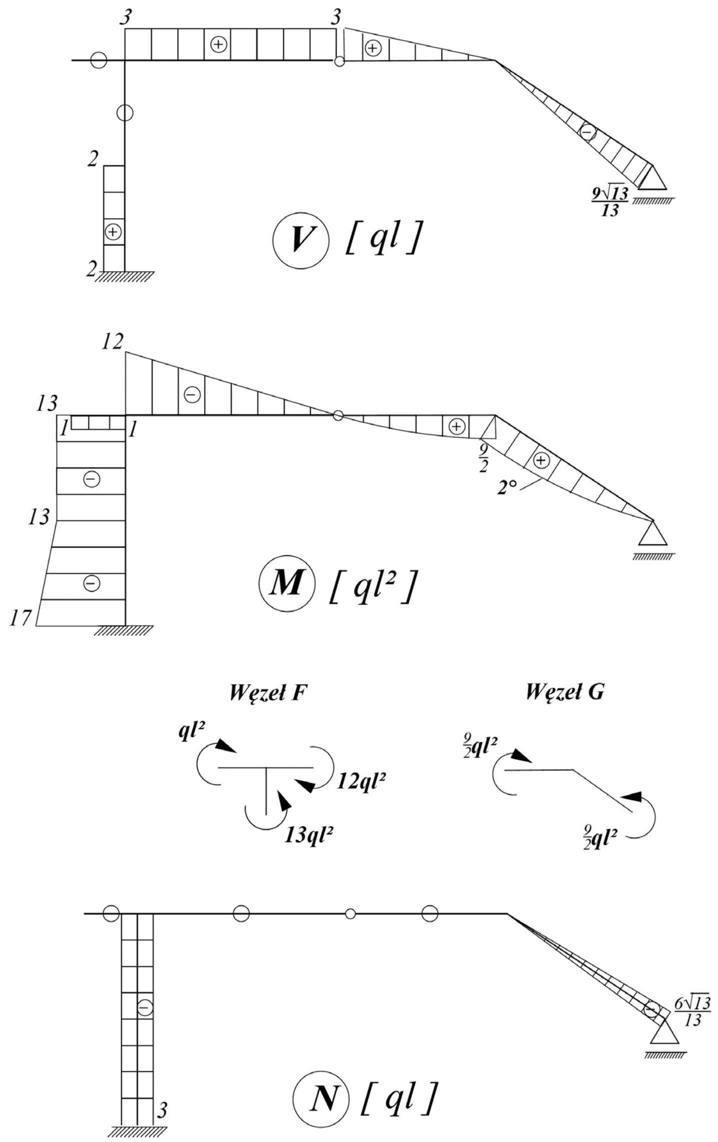

3
3
2
2
9√13
13
V [ql]
12
13
1
1
9
2
2°
13
17
M [ql²]
Węzeł F
ql²
12ql²
13ql²
Węzeł G
9/2ql²
9/2ql²
6√13
13
3
N [ql]

Fig. 3.2.14.

3.3. Review problem solutions – trusses

Problem 1.

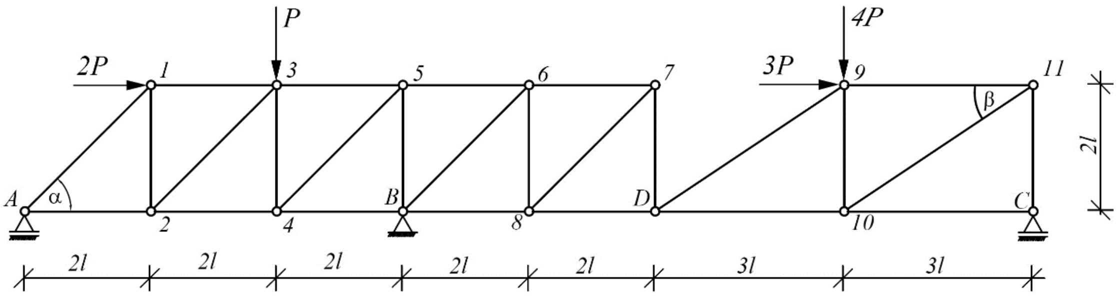
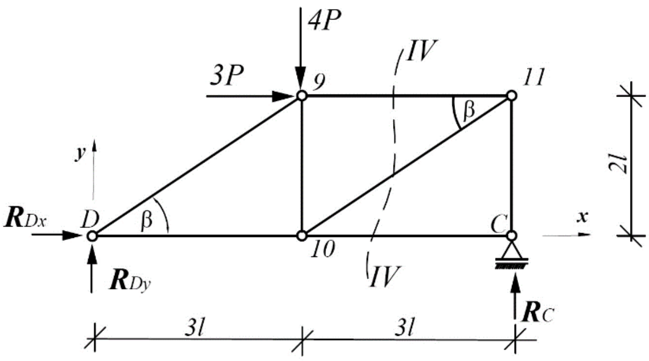


Fig. 3.3.1.

The degree of static indeterminacy:
$n = r + m - 2j = 4 + 26 - 2 \cdot 15 = 0$

a)



b)

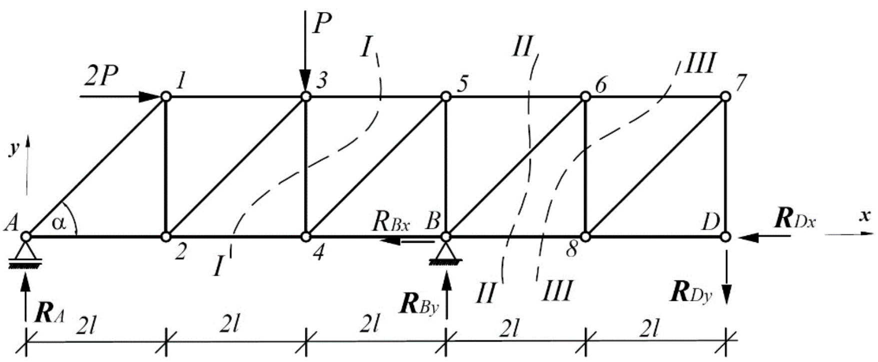


Fig. 3.3.2.

The compound truss shown in Fig. 3.3.1. has been separated into two single truss (Fig. 3.3.2.). The support reactions have to be first determined for the right truss (Fig. 3.3.2.a.):

$\sum F_{ix}^{R} = 0 \quad \rightarrow \quad R_{Dx} + 3\mathrm{P} = 0 \quad \rightarrow \quad R_{Dx} = -3\mathrm{P}$

$\sum M_{D}^{R} = 0 \quad \rightarrow \quad R_{C} \cdot 6l - 4P \cdot 3l - 3P \cdot 2l = 0 \quad \rightarrow \quad R_{C} = 3P$

$\sum F_{iy}^{R} = 0 \quad \rightarrow \quad R_{Dy} + R_{c} - 4P = 0 \rightarrow R_{Dy} = P$

Then we can determine support reactions for the left truss (Fig. 3.3.2.b.):

$\sum M_A^L = 0 \quad \rightarrow \quad R_{By} \cdot 6l - P \cdot 4l - 2P \cdot 2l - R_{Dy} \cdot 10l = 0 \ \rightarrow \ R_{By} = 3P$

$\sum F_{iy}^L = 0 \quad \rightarrow \quad R_A + R_{By} - P - R_{Dy} = 0 \ \rightarrow \ R_A = -P$

$\sum F_{ix}^L = 0 \quad \rightarrow \quad -R_{Bx} + 2P - R_{Dx} = 0 \quad \rightarrow \ R_{Bx} = 5\mathrm{P}$

To determine forces of members 3–5, 3–4, 2–4 we can use the vertical cutting plane I–I shown in Fig. 3.3.2.b. and consider the equilibrium of the left truss segment (Fig. 3.3.3.a.).

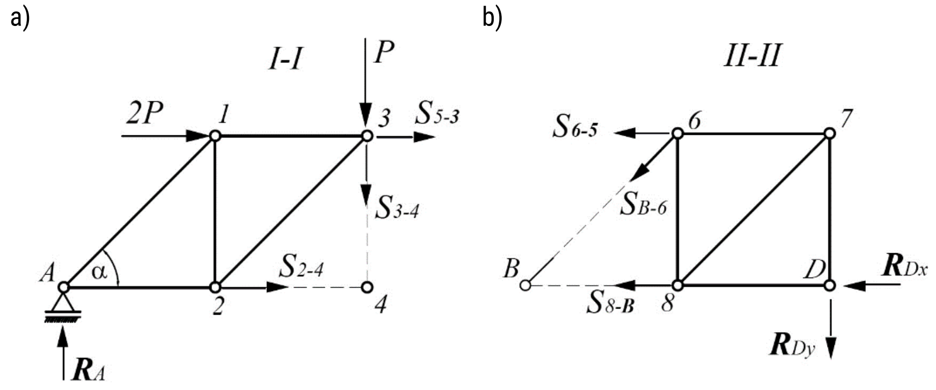


Fig. 3.3.3.

$\sum M_4 = -R_A \cdot 4l - 2P \cdot 2l - S_{3-5} \cdot 2l = 0 \ \rightarrow \quad S_{3-5} = 0$

$\sum M_3 = -R_A \cdot 4l + S_{2-4} \cdot 2l = 0 \ \rightarrow \quad S_{2-4} = -2P$

$\sum F_{iy} = 0 \ \rightarrow \ R_A - P - S_{3-4} = 0 \ \rightarrow \ S_{3-4} = -2P$

To determine forces of members B–6, 6–5 and 8–B we can use the vertical cutting plane II–II and consider the right truss segment (Fig. 3.3.3.b.).

$sin\alpha = \dfrac{\sqrt{2}}{2}$

$\sum F_{iy} = 0 \ \rightarrow \ -S_{B-6} sin\alpha - R_{Dy} = 0 \quad \rightarrow \ S_{B-6} = -\sqrt{2}P$

$\sum M_B = 0 \ \rightarrow \ -R_{Dy} \cdot 4l + S_{6-5} \cdot 2l = 0 \ \rightarrow \ S_{6-5} = 2P$

$\sum M_6 = 0 \ \rightarrow \ -R_{Dy} \cdot 2l - R_{Dx} \cdot 2l - S_{8-B} \cdot 2l = 0 \ \rightarrow \ S_{8-B} = 2P$

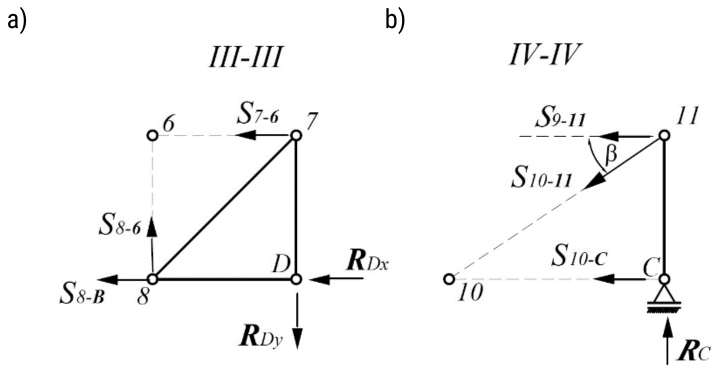


Fig. 3.3.4.

Cutting plane III–III (Fig. 3.3.4.a.)

$\sum F_{iy} = 0 \;\rightarrow\; S_{8-6} - R_{Dy} = 0 \;\rightarrow\; S_{8-6} = P$

$\sum M_8 = 0 \;\rightarrow\; -R_{Dy} \cdot 2l + S_{7-6} \cdot 2l = 0 \;\rightarrow\; S_{7-6} = P$

$\sum M_6 = 0 \;\rightarrow\; -R_{Dy} \cdot 2l - R_{Dx} \cdot 2l - S_{8-B} \cdot 2l = 0 \;\rightarrow\; S_{8-B} = 2P$

Cutting plane IV– IV (Fig. 3.3.4.b.)

$$sin\beta = \frac{2\sqrt{13}}{13}$$

$\sum F_{iy} = 0 \;\rightarrow\; -S_{10-11} sin\beta + R_C = 0 \;\rightarrow\; S_{10-11} = \frac{3\sqrt{13}}{2} P$

$\sum M_{10} = 0 \;\rightarrow\; R_C \cdot 3l + S_{9-11} \cdot 2l = 0 \;\rightarrow\; S_{9-11} = -\frac{9}{2} P$

$\sum M_{11} = 0 \;\rightarrow\; S_{10-C} = 0$

<u>Problem 2.</u>

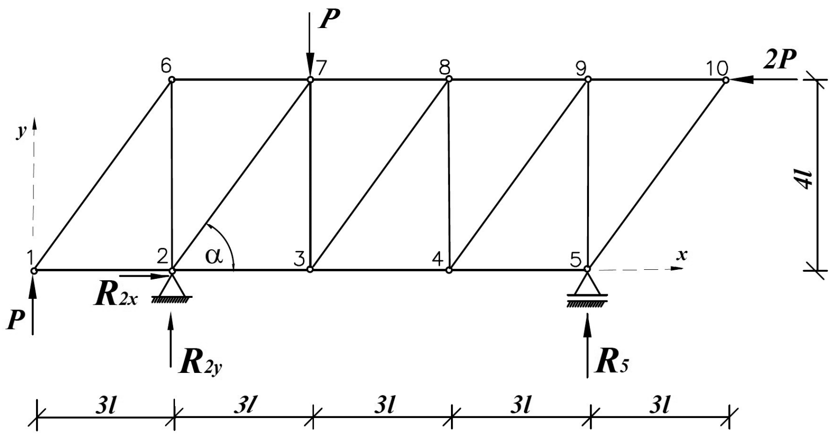


Fig. 3.3.5.

The degree of static indeterminacy:
$n = r + m - 2j = 3 + 17 - 2 \cdot 10 = 0$

Support reactions (Fig. 3.3.5.) can be determined by using the three global equilibrium equations for the planar structures.

$\Sigma M_2 = 0 \;\rightarrow\; -P \cdot 3l - P \cdot 3l + R_5 \cdot 9l + 2P \cdot 4l = 0 \;\rightarrow\; R_5 = -\frac{2}{9} P$

$\Sigma F_{ix} = 0 \;\rightarrow\; R_{2x} - 2P = 0 \;\rightarrow\; R_{2x} = 2P$

$\Sigma F_{iy} = P + R_{2y} - P + R_5 = 0 \;\rightarrow\; R_{2y} = \frac{2}{9} P$

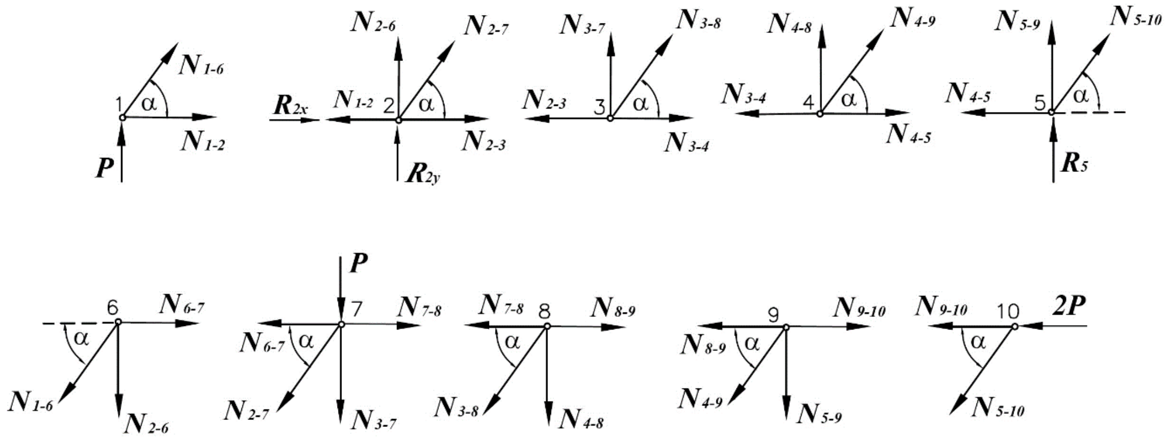


Fig. 3.3.6.

<u>Joint 1</u> (Fig. 3.3.6.)

$sin\alpha = \frac{4}{5} \qquad cos\alpha = \frac{3}{5}$

$\Sigma F_{iy} = 0 \quad N_{1-6}sin\alpha + P = 0 \quad \rightarrow N_{1-6} = -\frac{P}{sin\alpha} = -\frac{5}{4}P$

$\Sigma F_{ix} = 0 \quad N_{1-6}cos\alpha + N_{1-2} = 0 \quad \rightarrow N_{1-2} = -N_{1-6}cos\alpha = \frac{5}{4}P \cdot \frac{3}{5} = \frac{3}{4}P$

<u>Joint 6</u> (Fig. 3.3.6.)

$\Sigma F_{iy} = 0 \rightarrow -N_{2-6} - N_{1-6}sin\alpha = 0 \rightarrow -N_{2-6} + \frac{5}{4}P\frac{4}{5} = 0 \rightarrow N_{2-6} = P$

$\Sigma F_{ix} = 0 \rightarrow N_{6-7} - N_{1-6}cos\alpha = 0 \rightarrow N_{6-7} + \frac{5}{4}P\frac{3}{5} = 0 \rightarrow N_{6-7} = -\frac{3}{4}P$

<u>Joint 2</u> (Fig. 3.3.6.)

$\Sigma F_{iy} = 0 \rightarrow R_{2y} + N_{2-6} + N_{2-7}sin\alpha = 0 \rightarrow N_{2-7} = -\frac{55}{36}P$

$\Sigma F_{ix} = 0 \rightarrow -N_{1-2} + N_{2-3} + R_{2x} + N_{2-7}cos\alpha = 0 \rightarrow N_{2-3} = -\frac{1}{3}P$

<u>Joint 7</u> (Fig. 3.3.6.)

$\Sigma F_{iy} = 0 \rightarrow -P - N_{3-7} - N_{2-7}sin\alpha = 0 \rightarrow N_{3-7} = \frac{2}{9}P$

$\Sigma F_{ix} = 0 \rightarrow -N_{6-7} + N_{7-8} - N_{2-7}cos\alpha = 0 \rightarrow N_{7-8} = -\frac{5}{3}P$

<u>Joint 3</u> (Fig. 3.3.6.)

$\Sigma F_{iy} = 0 \rightarrow N_{3-8}sin\alpha + N_{3-7} = 0 \rightarrow N_{3-8} = -\frac{5}{18}P$

$\Sigma F_{ix} = 0 \rightarrow -N_{2-3} + N_{3-8}cos\alpha + N_{3-4} = 0 \rightarrow N_{3-4} = -\frac{1}{6}P$

Joint 8 (Fig. 3.3.6.)

$$\Sigma F_{iy} = 0 \rightarrow -N_{3-8}sin\alpha - N_{4-8} = 0 \rightarrow N_{4-8} = \frac{2}{9}P$$

$$\Sigma F_{ix} = 0 \rightarrow -N_{7-8} + N_{8-9} - N_{3-8}cos\alpha = 0 \rightarrow N_{8-9} = -\frac{11}{6}P$$

Joint 4 (Fig. 3.3.6.)

$$\Sigma F_{iy} = 0 \rightarrow N_{4-9}sin\alpha + N_{4-8} = 0 \rightarrow N_{4-9} = -\frac{5}{18}P$$

$$\Sigma F_{ix} = 0 \rightarrow -N_{3-4} + N_{4-9}cos\alpha + N_{4-5} = 0 \rightarrow N_{4-5} = 0$$

Joint 9 (Fig. 3.3.6.)

$$\Sigma F_{iy} = -N_{4-9}sin\alpha - N_{5-9} = 0 \rightarrow N_{5-9} = \frac{2}{9}P$$

$$\Sigma F_{ix} = -N_{8-9} - N_{4-9}cos\alpha + N_{9-10} = 0 \rightarrow N_{9-10} = -2P$$

Verification:

Joint 10 (Fig. 3.3.6.)

$$\Sigma F_{ix} = -N_{9-10} - 2P = 2P - 2P = 0$$

$$\Sigma F_{iy} = - N_{5-10} = 0$$

Joint 5 (Fig. 3.3.6.)

$$\Sigma F_{ix} = -N_{4-5} + N_{5-10}cos\alpha = 0 + 0cos\alpha = 0$$

$$\Sigma F_{iy} = R_5 + N_{5-9} + N_{5-10}sin\alpha = -\frac{2}{9}P + \frac{2}{9}P + 0sin\alpha = 0$$

Problem 3.

The degree of static indeterminacy $n = r + m - 2j = 3 + 15 - 2 \cdot 9 = 0$

Support reactions (Fig. 3.3.7.) can be determined by using the three global equilibrium equations for the planar structures.

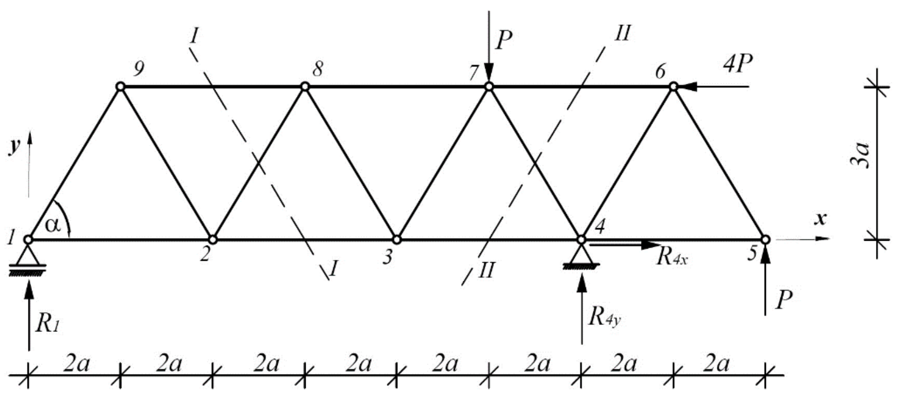


Fig. 3.3.7.

$$\sum M_1 = 0 \rightarrow R_{4y} \cdot 12a + P \cdot 16a + 4P \cdot 3a - P \cdot 10a = 0 \rightarrow R_{4y} = -\frac{18}{12}P = -\frac{3}{2}P$$

$$\sum F_{iy} = 0 \rightarrow R_1 + R_{4y} + P - P = 0 \rightarrow R_1 = \frac{3}{2}P$$

$\sum F_{ix} = 0 \;\rightarrow\; R_{4x} - 4P = 0 \;\rightarrow\; R_{4x} = 4P$

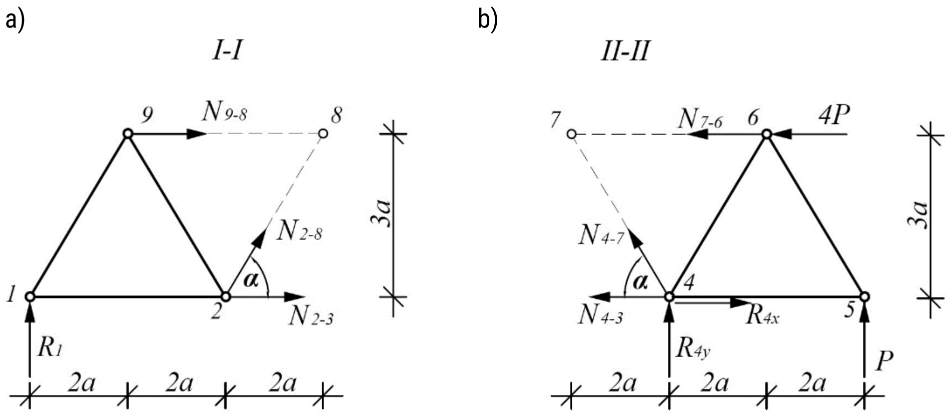


Fig. 3.3.8.

Cutting plane I–I (Fig. 3.3.8.a.)

$$sin\alpha = \frac{3\sqrt{13}}{13}$$

$$\sum F_{iy} = 0 \;\rightarrow\; R_1 + N_{2-8} \cdot sin\alpha = 0 \;\rightarrow\; N_{2-8} = -\frac{3}{2}P \cdot \frac{\sqrt{13}}{3} = -\frac{\sqrt{13}}{2}P$$

$$\sum M_2 = 0 \;\rightarrow\; -N_{9-8} \cdot 3a - R_1 \cdot 4a = 0 \;\rightarrow\; N_{9-8} = -2P$$

$$\sum M_8 = 0 \;\rightarrow\; -R_1 \cdot 6a + N_{2-3} \cdot 3a = 0 \;\rightarrow\; N_{2-3} = 3P$$

Cutting plane II–II (Fig. 3.3.8.b.)

$$\sum M_7 = 0 \;\rightarrow\; R_{4y} \cdot 2a + R_{4x} \cdot 3a + P \cdot 6a - N_{4-3} \cdot 3a = 0$$

$$-\frac{3}{2}P \cdot 2a + 4P \cdot 3a + P \cdot 6a = N_{4-3} \cdot 3a \;\rightarrow\; N_{4-3} = 5P$$

$$\sum M_4 = 0 \;\rightarrow\; 4P \cdot 3a + N_{7-6} \cdot 3a + P \cdot 4a = 0 \;\rightarrow\; N_{7-6} = -\frac{16}{3}P$$

$$\sum F_{iy} = 0 \;\rightarrow\; R_{4y} + P + N_{4-7} \cdot sin\alpha = 0 \;\rightarrow\; -\frac{3}{2}P + P + N_{4-7} \cdot \frac{3}{\sqrt{13}} = 0$$

$$N_{4-7} = \frac{\sqrt{13}}{6}P$$

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