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Coproducts in Categories without Uniqueness of cod and dom

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Summary. The paper introduces coproducts in categories without uniqueness of **cod** and **dom**. It is proven that set-theoretical disjoint union is the coproduct in the category Ens [9].

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Keywords: coproducts; disjoined union

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The notation and terminology used in this paper have been introduced in the following articles: [10], [7], [6], [1], [11], [2], [3], [8], [4], [12], [14], [13], and [5].

From now on I denotes a set and E denotes a non empty set.

Let I be a non empty set, A be a many sorted set indexed by I, and i be an element of I. Let us observe that $\mathrm{coprod}(i, A)$ is relation-like and function-like.

Let C be a non empty category structure, o be an object of C, I be a set, and f be an objects family of I and C. A morphisms family of f and o is a many sorted set indexed by I and is defined by

(Def. 1) Let us consider an element i. Suppose $i \in I$. Then there exists an object o_1 of C such that

(i) $o_1 = f(i)$, and

(ii) $it(i)$ is a morphism from o_1 to o.

Let I be a non empty set. Let us note that a morphisms family of f and o can equivalently be formulated as follows:

(Def. 2) Let us consider an element i of I. Then $it(i)$ is a morphism from $f(i)$ to o.

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Let M be a morphisms family of f and o and i be an element of I. Note that the functor $M(i)$ yields a morphism from $f(i)$ to o. Let C be a functional non empty category structure. Let I be a set. Let us note that every morphisms family of f and o is function yielding.

Now we state the proposition:

(1) Let us consider a non empty category structure C, an object o of C, and an objects family f of $\emptyset$ and C. Then $\emptyset$ is a morphisms family of f and o.

Let C be a non empty category structure, I be a set, A be an objects family of I and C, B be an object of C, and P be a morphisms family of A and B. We say that P is feasible if and only if

(Def. 3) Let us consider a set i. Suppose $i \in I$. Then there exists an object o of C such that

(i) $o = A(i)$, and

(ii) $P(i) \in \langle o, B \rangle$.

Let I be a non empty set. Let us observe that P is feasible if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let us consider an element i of I. Then $P(i) \in \langle A(i), B \rangle$.

Let C be a category and I be a set. We say that P is coprojection morphisms if and only if

(Def. 5) Let us consider an object X of C and a morphisms family F of A and X. Suppose F is feasible. Then there exists a morphism f from B to X such that

(i) $f \in \langle B, X \rangle$, and

(ii) for every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to B such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$, and

(iii) for every morphism f_1 from B to X such that for every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to B such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f_1 \cdot P_i$ holds $f = f_1$.

Let I be a non empty set. Let us note that P is coprojection morphisms if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let us consider an object X of C and a morphisms family F of A and X. Suppose F is feasible. Then there exists a morphism f from B to X such that

(i) $f \in \langle B, X \rangle$, and

(ii) for every element i of I, $F(i) = f \cdot P(i)$, and

(iii) for every morphism f_1 from B to X such that for every element i of I, $F(i) = f_1 \cdot P(i)$ holds $f = f_1$.

Let A be an objects family of $\emptyset$ and C. Note that every morphisms family of A and B is feasible.

Now we state the propositions:

(2) Let us consider a category C, an objects family A of $\emptyset$ and C, and an object B of C. Suppose B is initial. Then there exists a morphisms family P of A and B such that P is empty and coprojection morphisms. The theorem is a consequence of (1).

(3) Let us consider an objects family A of I and $\mathrm{Ens}_{\{\emptyset\}}$ and an object o of $\mathrm{Ens}_{\{\emptyset\}}$. Then $I \longmapsto \emptyset$ is a morphisms family of A and o.

(4) Let us consider an objects family A of I and $\mathrm{Ens}_{\{\emptyset\}}$, an object o of $\mathrm{Ens}_{\{\emptyset\}}$, and a morphisms family P of A and o. If $P = I \longmapsto \emptyset$, then P is feasible and coprojection morphisms. Proof: P is feasible by [11, (7)]. Reconsider $f = \emptyset$ as a morphism from o to Y. For every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to o such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$ by [11, (7)]. □

Let C be a category. We say that C has coproducts if and only if

(Def. 7) Let us consider a set I and an objects family A of I and C. Then there exists an object B of C and there exists a morphisms family P of A and B such that P is feasible and coprojection morphisms.

Note that $\mathrm{Ens}_{\{\emptyset\}}$ has coproducts and there exists a category which is strict and has products and coproducts.

Let C be a category, I be a set, A be an objects family of I and C, and B be an object of C. We say that B is A-category coproduct-like if and only if

(Def. 8) There exists a morphisms family P of A and B such that P is feasible and coprojection morphisms.

Let C be a category with coproducts. Let us observe that there exists an object of C which is A-category coproduct-like.

Let C be a category and A be an objects family of $\emptyset$ and C. Note that every object of C which is A-category coproduct-like is also initial.

Now we state the propositions:

(5) Let us consider a category C, an objects family A of $\emptyset$ and C, and an object B of C. If B is initial, then B is A-category coproduct-like. The theorem is a consequence of (2).

(6) Let us consider a category C, an objects family A of I and C, and objects C_1, C_2 of C. Suppose

(i) C_1 is A-category coproduct-like, and

(ii) C_2 is A-category coproduct-like.

Then C_1,C_2 are iso.

From now on A denotes an objects family of I and Ens_E.

Let us consider I, E, and A. Assume $\bigcup \mathrm{coprod}(A) \in E$. The functor $\coprod A$ yielding an object of Ens_E is defined by the term

(Def. 9) $\bigcup \mathrm{coprod}(A)$.

The functor $\mathrm{Coprod}(A)$ yielding a many sorted set indexed by I is defined by

(Def. 10) Let us consider an element i. Suppose $i \in I$. Then there exists a function F from $A(i)$ into $\bigcup \mathrm{coprod}(A)$ such that

(i) $it(i) = F$, and

(ii) for every element x such that $x \in A(i)$ holds $F(x) = \langle x, i \rangle$.

Let us observe that $\mathrm{Coprod}(A)$ is function yielding.

Assume $\bigcup \mathrm{coprod}(A) \in E$. The functor $\coprod A$ yielding a morphisms family of A and $\coprod A$ is defined by the term

(Def. 11) $\mathrm{Coprod}(A)$.

Now we state the propositions:

(7) If $\bigcup \mathrm{coprod}(A) = \emptyset$, then $\mathrm{Coprod}(A)$ is empty yielding.

(8) If $\bigcup \mathrm{coprod}(A) = \emptyset$, then A is empty yielding.

(9) If $\bigcup \mathrm{coprod}(A) \in E$ and $\bigcup \mathrm{coprod}(A) = \emptyset$, then $\coprod A = I \longmapsto \emptyset$. The theorem is a consequence of (7).

(10) If $\bigcup \mathrm{coprod}(A) \in E$, then $\coprod A$ is feasible and coprojection morphisms. The theorem is a consequence of (7) and (8).

(11) If $\bigcup \mathrm{coprod}(A) \in E$, then $\coprod A$ is A-category coproduct-like. The theorem is a consequence of (10).

(12) If for every I and A, $\bigcup \mathrm{coprod}(A) \in E$, then Ens_E has coproducts. The theorem is a consequence of (10).

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Formulation of Cell Petri Nets

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Summary. Based on the Petri net definitions and theorems already formalized in the Mizar article [10], in this article we were able to formalize the definition of Cell Petri nets. It is based on [**?**]. Colored Petri net is already have been defined in [9]. In addition the conditions of the firing-rule and ColoredSet to this definition, that defines the Cell Petri nets extended to CPNT.i further. Although it was synthesis of two Petri nets in [9], it is synthesis from the family of Colored Petri nets (?? Colored-PT-net-Family of I) of finite number of pieces. That is, extension to a CPNT family is performed by defining the output arc from the transition of a certain Colored Petri nets to Place of a certain another Colored Petri nets (definition of the neighborhood). Finally, activation of Colored Petri nets was formalized.

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Keywords: Petri net; system modelling

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The notation and terminology used in this paper have been introduced in the following articles: [**?**], [17], [18], [4], [19], [5], [2], [**?**], [15], [**?**], [10], [9], [1], [**?**], [7], [13], [8], [16], [**?**], [**?**], [12], [6], [14], [3], and [11].

1. Preliminaries

Let I be a non empty set and C_1 be a many sorted set indexed by I. We say that C_1 is colored-pt-net-family-like if and only if

(Def. 1) Let us consider an element i of I. Then $C_1(i)$ is a colored place/transition net.

Note that there exists a many sorted set indexed by I which is colored-pt-net-family-like.

A colored place/transition net family of I is a colored-pt-net-family-like many sorted set indexed by I. Let C_1 be a colored place/transition net family of I

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and i be an element of I. One can check that the functor $C_1(i)$ yields a colored place/transition net. Let C_2 be a colored place/transition net family of I. We say that C_2 is disjoint valued if and only if

(Def. 2) Let us consider elements i, j of I. Suppose $i \neq j$. Then

(i) the carrier of $C_2(i)$ misses the carrier of $C_2(j)$, and

(ii) the carrier' of $C_2(i)$ misses the carrier' of $C_2(j)$.

Now we state the propositions:

(1) Let us consider a set I and many sorted sets F, D, R indexed by I. Suppose

(i) for every element i such that $i \in I$ there exists a function f such that $f = F(i)$ and $\operatorname{dom} f = D(i)$ and $\operatorname{rng} f = R(i)$, and

(ii) for every elements i, j and for every functions f, g such that $i, j \in I$ and $i \neq j$ and $f = F(i)$ and $g = F(j)$ holds $\operatorname{dom} f$ misses $\operatorname{dom} g$.

Then there exists a function G such that

(iii) $G = \bigcup \operatorname{rng} F$, and

(iv) $\operatorname{dom} G = \bigcup \operatorname{rng} D$, and

(v) $\operatorname{rng} G = \bigcup \operatorname{rng} R$, and

(vi) for every elements i, x and for every function f such that $i \in I$ and $f = F(i)$ and $x \in \operatorname{dom} f$ holds $G(x) = f(x)$.

PROOF: For every element z such that $z \in \bigcup \operatorname{rng} F$ there exist elements x, y, i such that $z = \langle x, y \rangle$ and $z \in F(i)$ and $i \in I$. For every element z such that $z \in \bigcup \operatorname{rng} F$ there exist elements x, y such that $z = \langle x, y \rangle$. Reconsider $G = \bigcup \operatorname{rng} F$ as a binary relation. G is a function. For every element x, $x \in \operatorname{dom} G$ iff $x \in \bigcup \operatorname{rng} D$ by [4, (3)]. For every element x, $x \in \operatorname{rng} G$ iff $x \in \bigcup \operatorname{rng} R$ by [4, (3)]. For every elements i, x and for every function f such that $i \in I$ and $f = F(i)$ and $x \in \operatorname{dom} f$ holds $G(x) = f(x)$ by [4, (1), (3)]. □

(2) Let us consider a set I and many sorted sets Y, Z indexed by I. Suppose elements i, j. If $i, j \in I$ and $i \neq j$, then $Y(i) \cap Z(j) = \emptyset$. Then $\bigcup (Y \setminus Z) = \bigcup Y \setminus \bigcup Z$. PROOF: Set $X = Y \setminus Z$. For every element x, $x \in \bigcup \operatorname{rng} X$ iff $x \in \bigcup \operatorname{rng} Y \setminus \bigcup \operatorname{rng} Z$ by [4, (3)]. □

(3) Let us consider a set I and many sorted sets X, Y, Z indexed by I. Suppose

(i) $X \subseteq Y \setminus Z$, and

(ii) for every elements i, j such that $i, j \in I$ and $i \neq j$ holds $Y(i) \cap Z(j) = \emptyset$.

Then $\bigcup X \subseteq \bigcup Y \setminus \bigcup Z$. The theorem is a consequence of (2).

2. Synthesis of CPNT and I

Let I be a non trivial set. The functor XorDelta I yielding a non empty set is defined by the term

(Def. 3) $\{\langle i, j\rangle$, where i, j are elements of $I : i \neq j\}$.

Now we state the proposition:

(4) Let us consider a non trivial finite set I and a colored place/transition net family C_2 of I. Then $\bigcup\{$(the carrier of $C_2(j))^{\mathrm{Outbds}(C_2(i))}$, where i, j are elements of $I : i \neq j\}$ is not empty.

Let I be a non trivial finite set and C_2 be a colored place/transition net family of I. A connecting mapping of C_2 is a many sorted set indexed by XorDelta I and is defined by

(Def. 4) (i) rng $it \subseteq \bigcup\{$(the carrier of $C_2(j))^{\mathrm{Outbds}(C_2(i))}$, where i, j are elements of $I : i \neq j\}$, and

(ii) for every elements i, j of I such that $i \neq j$ holds $it(\langle i, j\rangle)$ is a function from Outbds$(C_2(i))$ into the carrier of $C_2(j)$.

Now we state the proposition:

(5) Let us consider colored place/transition nets C_4, C_5, a function O_{12} from Outbds C_4 into the carrier of C_5, and a function q_{12}. Suppose

(i) dom q_{12} = Outbds C_4, and

(ii) for every transition t_{01} of C_4 such that t_{01} is outbound holds $q_{12}(t_{01})$ is a function from the thin cylinders of the colored set of C_4 and ${}^*\{t_{01}\}$ into the thin cylinders of the colored set of C_4 and $O_{12}{}^{\circ}t_{01}$.

Then $q_{12} \in (\bigcup\{$(the thin cylinders of the colored set of C_4 and $O_{12}{}^{\circ}t_{01})^{\alpha}$, where t_{01} is a transition of $C_4 : t_{01}$ is outbound$\})^{\mathrm{Outbds}\,C_4}$, where α is the thin cylinders of the colored set of C_4 and ${}^*\{t_{01}\}$.

Let I be a non trivial finite set, C_2 be a colored place/transition net family of I, and O be a connecting mapping of C_2. A connecting firing rule of O is a many sorted set indexed by XorDelta I and is defined by

(Def. 5) Let us consider elements i, j of I. Suppose $i \neq j$. Then there exists a function O_6 from Outbds$(C_2(i))$ into the carrier of $C_2(j)$ and there exists a function q_8 such that $q_8 = it(\langle i, j\rangle)$ and $O_6 = O(\langle i, j\rangle)$ and dom q_8 = Outbds$(C_2(i))$ and for every transition t_{01} of $C_2(i)$ such that t_{01} is outbound holds $q_8(t_{01})$ is a function from the thin cylinders of the colored set of $C_2(i)$ and ${}^*\{t_{01}\}$ into the thin cylinders of the colored set of $C_2(i)$ and $O_6{}^{\circ}t_{01}$.

3. Extension to a Family of Colored Petri Nets

Let I be a non trivial finite set, C_2 be a colored place/transition net family of I, O be a connecting mapping of C_2, and q be a connecting firing rule of O. Assume C_2 is disjoint valued and for every elements i, j_1, j_2 of I such that $i \neq j_1$ and $i \neq j_2$ and there exist elements x, y_1, y_2 such that $\langle x, y_1 \rangle \in q(\langle i, j_1 \rangle)$ and $\langle x, y_2 \rangle \in q(\langle i, j_2 \rangle)$ holds $j_1 = j_2$. The functor synthesis q yielding a strict colored place/transition net is defined by

(Def. 6) There exist many sorted sets P, T, S_9, T_8, C_3, F indexed by I and there exist functions U_9, U_8 such that for every element i of I, $P(i) =$ the carrier of $C_2(i)$ and $T(i) =$ the carrier' of $C_2(i)$ and $S_9(i) =$ the S-T arcs of $C_2(i)$ and $T_8(i) =$ the T-S arcs of $C_2(i)$ and $C_3(i) =$ the colored set of $C_2(i)$ and $F(i) =$ the firing-rule of $C_2(i)$ and $U_9 = \bigcup \operatorname{rng} F$ and $U_8 = \bigcup \operatorname{rng} q$ and the carrier of $it = \bigcup \operatorname{rng} P$ and the carrier' of $it = \bigcup \operatorname{rng} T$ and the S-T arcs of $it = \bigcup \operatorname{rng} S_9$ and the T-S arcs of $it = \bigcup \operatorname{rng} T_8 \cup \bigcup \operatorname{rng} O$ and the colored set of $it = \bigcup \operatorname{rng} C_3$ and the firing-rule of $it = U_9 {+\cdot} U_8$.

4. Definition of Cell Petri Nets

Let I be a non empty finite set and C_2 be a colored place/transition net family of I. We say that C_2 is cell Petri nets if and only if

(Def. 7) There exists a function N from I into $2^{\operatorname{rng} C_2}$ such that for every element i of I, $N(i) = \{C_2(j),$ where j is an element of $I : j \neq i\}$.

Let N be a function from I into $2^{\operatorname{rng} C_2}$ and O be a connecting mapping of C_2. We say that (N, O) is cell Petri nets if and only if

(Def. 8) Let us consider an element i of I. Then $N(i) = \{C_2(j),$ where j is an element of $I : j \neq i$ and there exists a transition t of $C_2(i)$ and there exists an element s such that $\langle t, s \rangle \in O(\langle i, j \rangle)\}$.

Now we state the proposition:

(6) Let us consider a non trivial finite set I, a colored place/transition net family C_2 of I, a function N from I into $2^{\operatorname{rng} C_2}$, and a connecting mapping O of C_2. Suppose

(i) C_2 is one-to-one, and

(ii) (N, O) is cell Petri nets.

Let us consider an element i of I. Then $C_2(i) \notin N(i)$.

5. ACTIVATION OF PETRI NETS

Let C_6 be a colored place/transition net structure. We say that C_6 has nontrivial colored set if and only if

(Def. 9) The colored set of C_6 is not trivial.

One can verify that there exists a strict colored-PT-net-like colored Petri net which has nontrivial colored set.

Let C_2 be a colored place/transition net with nontrivial colored set. One can verify that the colored set of C_2 is non trivial.

Let C_6 be a colored place/transition net with nontrivial colored set, S be a subset of the carrier of C_6, and D be a thin cylinder of the colored set of C_6 and S. A color threshold of D is a function from $\operatorname{loc} D$ into the colored set of C_6. Let C_6 be a colored place/transition net. A color count of C_6 is a function from the colored set of C_6 into $\mathbb{N}$. The colored states of C_6 yielding a non empty set is defined by the term

(Def. 10) the set of all e where e is a color count of C_6.

A colored state of C_6 is a function from C_6 into the colored states of C_6. From now on C_6 denotes a colored place/transition net with nontrivial colored set, m denotes a colored state of C_6, and t denotes an element of the carrier' of C_6.

Let C_6 be a colored place/transition net with nontrivial colored set, m be a colored state of C_6, and p be a place of C_6. Observe that the functor $m(p)$ yields a color count of C_6. Let m_1 be a color count of C_6 and x be an element. Let us observe that the functor $m_1(x)$ yields an element of $\mathbb{N}$. Let us consider C_6, m, and t. Let D be a thin cylinder of the colored set of C_6 and $^*\{t\}$ and C_a be a color threshold of D. We say that t is firable on m and C_a if and only if

(Def. 11) (i) (the firing-rule of C_6)($\langle t, D\rangle$) $\neq \emptyset$, and

(ii) for every place p of C_6 such that $p \in \operatorname{loc} D$ holds $1 \leqslant m(p)(C_a(p))$.

The firable set on m and t yielding a set is defined by the term

(Def. 12) $\{D$, where D is a thin cylinder of the colored set of C_6 and $^*\{t\}$: there exists a color threshold C_a of D such that t is firable on m and $C_a\}$.

Now we state the proposition:

(7) Let us consider a thin cylinder D of the colored set of C_6 and $^*\{t\}$. Then there exists a color threshold C_a of D such that t is firable on m and C_a if and only if $D \in$ the firable set on m and t.

Let us consider C_6, m, and t. Let D be a thin cylinder of the colored set of C_6 and $^*\{t\}$, C_a be a color threshold of D, and p be an element of C_6. Assume t is firable on m and C_a. The Petri subtraction(C_a,m,p) yielding a function from the colored set of C_6 into $\mathbb{N}$ is defined by

(Def. 13) Let us consider an element x of the colored set of C_6. Then

(i) if $p \in \operatorname{loc} D$ and $x = C_a(p)$, then $it(x) = m(p)(x) - 1$, and

(ii) if it is not true that $p \in \operatorname{loc} D$ and $x = C_a(p)$, then $it(x) = m(p)(x)$.

Let D be a thin cylinder of the colored set of C_6 and $\overline{\{t\}}$. The Petri addition(C_a,m,p) yielding a function from the colored set of C_6 into $\mathbb{N}$ is defined by

(Def. 14) Let us consider an element x of the colored set of C_6. Then

(i) if $p \in \operatorname{loc} D$ and $x = C_a(p)$, then $it(x) = m(p)(x) + 1$, and

(ii) if it is not true that $p \in \operatorname{loc} D$ and $x = C_a(p)$, then $it(x) = m(p)(x)$.

Let D be a thin cylinder of the colored set of C_6 and $^*\{t\}$ and E be a thin cylinder of the colored set of C_6 and $\overline{\{t\}}$. Let C_d be a color threshold of E. The firing result(C_a,C_d,m,p) yielding a function from the colored set of C_6 into $\mathbb{N}$ is defined by the term

(Def. 15) $\begin{cases} \text{the Petri subtraction}(C_a,m,p), & \textbf{if } t \text{ is firable on } m \text{ and } C_a \text{ and } p \in \operatorname{loc} D \setminus \operatorname{loc} E, \\ \text{the Petri addition}(C_d,m,p), & \textbf{if } t \text{ is firable on } m \text{ and } C_a \text{ and } p \in \operatorname{loc} E \setminus \operatorname{loc} D, \\ m(p), & \textbf{otherwise}. \end{cases}$

Let us consider a thin cylinder D_0 of the colored set of C_6 and $^*\{t\}$, a thin cylinder D_1 of the colored set of C_6 and $\overline{\{t\}}$, a color threshold C_b of D_0, a color threshold C_c of D_1, an element x of the colored set of C_6, and an element p of C_6. Now we state the propositions:

(8) $m(p)(x) - 1 \leqslant$ (the firing result(C_b,C_c,m,p))(x) $\leqslant m(p)(x) + 1$.

(9) If t is outbound, then $m(p)(x) - 1 \leqslant$ (the firing result(C_b,C_c,m,p))(x) $\leqslant$ $m(p)(x)$.

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Isometric Differentiable Functions on Real Normed Space[1]

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Summary. In this article, we formalize isometric differentiable functions on real normed space [**?**], and their properties.

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The notation and terminology used in this paper have been introduced in the following articles: [3], [2], [8], [4], [5], [17], [10], [11], [18], [14], [16], [1], [6], [9], [15], [22], [23], [20], [21], [13], [24], and [7].

From now on S, T, W denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, i, n denote natural numbers, and Y denotes a real normed space.

Let us consider a real norm space sequence G, a real normed space F, a set i, partial functions f, g from $\prod G$ to F, and a subset X of $\prod G$. Now we state the propositions:

(1) Suppose X is open and $i \in \operatorname{dom} G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i. Then

(i) $f + g$ is partially differentiable on X w.r.t. i, and

(ii) $(f + g){\restriction}^i X = (f{\restriction}^i X) + (g{\restriction}^i X)$.

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(2) Suppose X is open and $i \in \operatorname{dom} G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i. Then

(i) $f - g$ is partially differentiable on X w.r.t. i, and

(ii) $(f-g){\restriction^{i}}X = (f{\restriction^{i}}X) - (g{\restriction^{i}}X)$.

Now we state the propositions:

(3) Let us consider a real norm space sequence G, a real normed space F, a set i, a partial function f from $\prod G$ to F, a real number r, and a subset X of $\prod G$. Suppose

(i) X is open, and

(ii) $i \in \operatorname{dom} G$, and

(iii) f is partially differentiable on X w.r.t. i.

Then

(iv) $r \cdot f$ is partially differentiable on X w.r.t. i, and

(v) $r \cdot f{\restriction^{i}}X = r \cdot (f{\restriction^{i}}X)$.

Proof: Set $h = r \cdot f$. For every point x of $\prod G$ such that $x \in X$ holds h is partially differentiable in x w.r.t. i and $\operatorname{partdiff}(h, x, i) = r \cdot \operatorname{partdiff}(f, x, i)$ by [18, (24), (30)]. Set $f_3 = f{\restriction^{i}}X$. For every point x of $\prod G$ such that $x \in X$ holds $(r \cdot f_3)_x = \operatorname{partdiff}(h, x, i)$. □

(4) Let us consider sets X, Y, Z, functions I, f, and a set X. Then $(f{\restriction}X) \cdot I = (f \cdot I){\restriction}I^{-1}(X)$.

Let us consider S and T. Let f be a function from S into T. We say that f is isometric if and only if

(Def. 1) Let us consider an element x of S. Then $\|f(x)\| = \|x\|$.

Now we state the propositions:

(5) Let us consider a linear operator I from S into T. If I is isometric, then for every point x of S, I is continuous in x.

(6) Let us consider a linear operator I from S into T and a subset Z of S. If I is isometric, then I is continuous on Z. The theorem is a consequence of (5).

(7) Let us consider a linear operator I from S into T. Suppose I is one-to-one, onto, and isometric. Then there exists a linear operator J from T into S such that

(i) $J = I^{-1}$, and

(ii) J is one-to-one, onto, and isometric.

Proof: Reconsider $J = I^{-1}$ as a function from T into S. For every points v, w of T, $J(v + w) = J(v) + J(w)$ by [5, (113)], [4, (34)]. For every point v of T and for every real number r, $J(r \cdot v) = r \cdot J(v)$ by [5, (113)], [4, (34)]. For every point v of T, $\|J(v)\| = \|v\|$ by [5, (113)], [4, (34)]. □

Let us consider a linear operator I from S into T and a sequence s_1 of S. Now we state the propositions:

(8) If I is isometric and s_1 is convergent, then $I \cdot s_1$ is convergent and $\lim(I \cdot s_1) = I(\lim s_1)$.

(9) If I is one-to-one, onto, and isometric, then s_1 is convergent iff $I \cdot s_1$ is convergent.

Let us consider a linear operator I from S into T and a subset Z of S. Now we state the propositions:

(10) If I is one-to-one, onto, and isometric, then Z is closed iff $I^\circ Z$ is closed.

(11) If I is one-to-one, onto, and isometric, then Z is open iff $I^\circ Z$ is open.

(12) If I is one-to-one, onto, and isometric, then Z is compact iff $I^\circ Z$ is compact.

Now we state the propositions:

(13) Let us consider a partial function f from T to W, a function g from S into T, and a point x of S. Suppose

(i) $x \in \operatorname{dom} g$, and

(ii) $g_x \in \operatorname{dom} f$, and

(iii) g is continuous in x, and

(iv) f is continuous in g_x.

Then $f \cdot g$ is continuous in x. Proof: Set $h = f \cdot g$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in \operatorname{dom} h$ and $\|x_1 - x\| < s$ holds $\|h_{x_1} - h_x\| < r$ by [14, (7)], [12, (3), (4)]. □

(14) Let us consider a partial function f from T to W and a linear operator I from S into T. Suppose I is one-to-one, onto, and isometric. Let us consider a point x of S. Suppose $I(x) \in \operatorname{dom} f$. Then $f \cdot I$ is continuous in x if and only if f is continuous in $I(x)$. The theorem is a consequence of (7), (5), and (13).

(15) Let us consider a partial function f from T to W, a linear operator I from S into T, and a set X. Suppose

(i) $X \subseteq$ the carrier of T, and

(ii) I is one-to-one, onto, and isometric.

Then f is continuous on X if and only if $f \cdot I$ is continuous on $I^{-1}(X)$. The theorem is a consequence of (14) and (4). Proof: For every point y of T such that $y \in X$ holds $f{\restriction}X$ is continuous in y by [5, (113)], [22, (57)]. □

Let X, Y be real normed spaces. The functor IsoCPNrSP(X, Y) yielding a linear operator from $X \times Y$ into $\prod\langle X, Y\rangle$ is defined by

(Def. 2) (i) *it* is one-to-one and onto, and

(ii) for every point x of X and for every point y of Y, $it(x,y) = \langle x,y \rangle$, and

(iii) $0_{\prod\langle X,Y\rangle} = it(0_{X\times Y})$, and

(iv) *it* is isometric.

The functor IsoPCNrSP(X,Y) yielding a linear operator from $\prod\langle X,Y\rangle$ into $X \times Y$ is defined by

(Def. 3) (i) $it = (\text{IsoCPNrSP}(X,Y))^{-1}$, and

(ii) *it* is one-to-one and onto, and

(iii) for every point x of X and for every point y of Y, $it(\langle x,y\rangle) = \langle x, y \rangle$, and

(iv) $0_{X\times Y} = it(0_{\prod\langle X,Y\rangle})$, and

(v) *it* is isometric.

Now we state the propositions:

(16) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then IsoCPNrSP(X,Y) is continuous in z. The theorem is a consequence of (5).

(17) Let us consider real normed spaces X, Y and a point z of $\prod\langle X,Y\rangle$. Then IsoPCNrSP(X,Y) is continuous in z. The theorem is a consequence of (5).

(18) Let us consider real normed spaces X, Y and a subset Z of $X \times Y$. Then

(i) IsoCPNrSP(X,Y) is continuous on Z, and

(ii) Z is closed iff $(\text{IsoCPNrSP}(X,Y))^\circ Z$ is closed, and

(iii) Z is open iff $(\text{IsoCPNrSP}(X,Y))^\circ Z$ is open, and

(iv) Z is compact iff $(\text{IsoCPNrSP}(X,Y))^\circ Z$ is compact.

The theorem is a consequence of (6), (10), (11), and (12).

(19) Let us consider real normed spaces X, Y and a subset Z of $\prod\langle X,Y\rangle$. Then

(i) IsoPCNrSP(X,Y) is continuous on Z, and

(ii) Z is closed iff $(\text{IsoPCNrSP}(X,Y))^\circ Z$ is closed, and

(iii) Z is open iff $(\text{IsoPCNrSP}(X,Y))^\circ Z$ is open, and

(iv) Z is compact iff $(\text{IsoPCNrSP}(X,Y))^\circ Z$ is compact.

The theorem is a consequence of (6), (10), (11), and (12).

(20) Let us consider real normed spaces S, T, W, a point f of the real norm space of bounded linear operators from S into W, a point g of the real norm space of bounded linear operators from T into W, and a linear operator I from S into T. Suppose

(i) I is one-to-one, onto, and isometric, and

(ii) $f = g \cdot I$.

Then $\|f\| = \|g\|$. The theorem is a consequence of (7). Proof: Consider J being a linear operator from T into S such that $J = I^{-1}$ and J is one-to-one, onto, and isometric. Reconsider $g_0 = g$ as a Lipschitzian linear operator from T into W. Reconsider $g_4 = g \cdot I$ as a Lipschitzian linear operator from S into W. For every element x, $x \in \{\|g_0(t)\|$, where t is a vector of $T : \|t\| \leqslant 1\}$ iff $x \in \{\|g_4(w)\|$, where w is a vector of $S : \|w\| \leqslant 1\}$ by [4, (13), (35)]. □

(21) Let us consider real normed spaces X, Y, a partial function f from $\prod\langle X, Y\rangle$ to W, and a point z of $X \times Y$. Suppose $(\mathrm{IsoCPNrSP}(X, Y))(z) \in \mathrm{dom}\, f$. Then $f \cdot \mathrm{IsoCPNrSP}(X, Y)$ is continuous in z if and only if f is continuous in $(\mathrm{IsoCPNrSP}(X, Y))(z)$. The theorem is a consequence of (14).

(22) Let us consider real normed spaces X, Y, a partial function f from $X \times Y$ to W, and a point z of $\prod\langle X, Y\rangle$. Suppose $(\mathrm{IsoPCNrSP}(X, Y))(z) \in \mathrm{dom}\, f$. Then $f \cdot \mathrm{IsoPCNrSP}(X, Y)$ is continuous in z if and only if f is continuous in $(\mathrm{IsoPCNrSP}(X, Y))(z)$. The theorem is a consequence of (14).

(23) Let us consider real normed spaces X, Y, a partial function f from $\prod\langle X, Y\rangle$ to W, and a set D. Suppose $D \subseteq$ the carrier of $\prod\langle X, Y\rangle$. Then $f \cdot \mathrm{IsoCPNrSP}(X, Y)$ is continuous on $(\mathrm{IsoCPNrSP}(X, Y))^{-1}(D)$ if and only if f is continuous on D. The theorem is a consequence of (15).

(24) Let us consider real normed spaces X, Y, a partial function f from $X \times Y$ to W, and a set D. Suppose $D \subseteq$ the carrier of $X \times Y$. Then $f \cdot \mathrm{IsoPCNrSP}(X, Y)$ is continuous on $(\mathrm{IsoPCNrSP}(X, Y))^{-1}(D)$ if and only if f is continuous on D. The theorem is a consequence of (15).

(25) Let us consider a linear operator I from S into T. If I is isometric, then I is a Lipschitzian linear operator from S into T.

Let us consider real normed spaces X, Y. Now we state the propositions:

(26) $\mathrm{IsoCPNrSP}(X, Y)$ is a Lipschitzian linear operator from $X \times Y$ into $\prod\langle X, Y\rangle$.

(27) $\mathrm{IsoPCNrSP}(X, Y)$ is a Lipschitzian linear operator from $\prod\langle X, Y\rangle$ into $X \times Y$.

Let X, Y be real normed spaces. Note that the functor $\mathrm{IsoCPNrSP}(X, Y)$ yields a Lipschitzian linear operator from $X \times Y$ into $\prod\langle X, Y\rangle$. Let us observe that the functor $\mathrm{IsoPCNrSP}(X, Y)$ yields a Lipschitzian linear operator from $\prod\langle X, Y\rangle$ into $X \times Y$.

Let us consider real normed spaces X, Y, W, a point f of the real norm space of bounded linear operators from $X \times Y$ into W, and a point g of the real norm space of bounded linear operators from $\prod\langle X, Y\rangle$ into W. Now we state the propositions:

(28) If $f = g \cdot \mathrm{IsoCPNrSP}(X, Y)$, then $\|f\| = \|g\|$.

(29) If $g = f \cdot \text{IsoPCNrSP}(X, Y)$, then $\|f\| = \|g\|$.

Now we state the propositions:

(30) Let us consider real normed spaces S, T, a Lipschitzian linear operator L from S into T, and a point x_0 of S. Then

(i) L is differentiable in x_0, and

(ii) $L'(x_0) = L$.

Proof: Reconsider $L = L0$ as a point of the real norm space of bounded linear operators from S into T. Reconsider $R =$ (the carrier of S) $\longmapsto 0_T$ as a partial function from S to T. Set $N =$ the neighbourhood of x_0. For every point x of S such that $x \in N$ holds $L0_x - L0_{x_0} = L(x - x_0) + R_{x-x_0}$ by [19, (7)], [20, (4)]. □

(31) Let us consider real normed spaces X, Y and a point x_0 of $X \times Y$. Then

(i) $\text{IsoCPNrSP}(X, Y)$ is differentiable in x_0, and

(ii) $(\text{IsoCPNrSP}(X, Y))'(x_0) = \text{IsoCPNrSP}(X, Y)$.

(32) Let us consider real normed spaces X, Y and a point x_0 of $\prod\langle X, Y\rangle$. Then

(i) $\text{IsoPCNrSP}(X, Y)$ is differentiable in x_0, and

(ii) $(\text{IsoPCNrSP}(X, Y))'(x_0) = \text{IsoPCNrSP}(X, Y)$.

(33) Let us consider a partial function f from T to W, a Lipschitzian linear operator I from S into T, and a point I_0 of the real norm space of bounded linear operators from S into T. Suppose $I_0 = I$. Let us consider a point x of S. Suppose f is differentiable in $I(x)$. Then

(i) $f \cdot I$ is differentiable in x, and

(ii) $(f \cdot I)'(x) = f'(I(x)) \cdot I_0$.

The theorem is a consequence of (30).

(34) Let us consider real normed spaces X, Y, a partial function f from $\prod\langle X, Y\rangle$ to W, and a point I of the real norm space of bounded linear operators from $X \times Y$ into $\prod\langle X, Y\rangle$. Suppose $I = \text{IsoCPNrSP}(X, Y)$. Let us consider a point z of $X \times Y$. Suppose f is differentiable in $(\text{IsoCPNrSP}(X, Y))(z)$. Then

(i) $f \cdot \text{IsoCPNrSP}(X, Y)$ is differentiable in z, and

(ii) $(f \cdot \text{IsoCPNrSP}(X, Y))'(z) = f'((\text{IsoCPNrSP}(X, Y))(z)) \cdot I$.

(35) Let us consider real normed spaces X, Y, a partial function f from $X \times Y$ to W, and a point I of the real norm space of bounded linear operators from $\prod\langle X, Y\rangle$ into $X \times Y$. Suppose $I = \text{IsoPCNrSP}(X, Y)$. Let us consider a point z of $\prod\langle X, Y\rangle$. Suppose f is differentiable in $(\text{IsoPCNrSP}(X, Y))(z)$. Then

(i) $f \cdot \text{IsoPCNrSP}(X, Y)$ is differentiable in z, and

(ii) $(f \cdot \text{IsoPCNrSP}(X, Y))'(z) = f'((\text{IsoPCNrSP}(X, Y))(z)) \cdot I$.

(36) Let us consider a partial function f from T to W and a linear operator I from S into T. Suppose I is one-to-one, onto, and isometric. Let us consider a point x of S. Then $f \cdot I$ is differentiable in x if and only if f is differentiable in $I(x)$. The theorem is a consequence of (7), (25), (30), and (33).

(37) Let us consider real normed spaces X, Y, a partial function f from $\prod\langle X, Y\rangle$ to W, and a point z of $X \times Y$. Then $f \cdot \text{IsoCPNrSP}(X, Y)$ is differentiable in z if and only if f is differentiable in $(\text{IsoCPNrSP}(X, Y))(z)$. The theorem is a consequence of (36).

(38) Let us consider a partial function f from T to W, a linear operator I from S into T, and a set X. Suppose

(i) $X \subseteq$ the carrier of T, and

(ii) I is one-to-one, onto, and isometric.

Then f is differentiable on X if and only if $f \cdot I$ is differentiable on $I^{-1}(X)$. The theorem is a consequence of (36) and (4). PROOF: For every point y of T such that $y \in X$ holds $f{\restriction}X$ is differentiable in y by [5, (113)]. □

(39) Let us consider real normed spaces X, Y, a partial function f from $X \times Y$ to W, and a point z of $\prod\langle X, Y\rangle$. Then $f \cdot \text{IsoPCNrSP}(X, Y)$ is differentiable in z if and only if f is differentiable in $(\text{IsoPCNrSP}(X, Y))(z)$. The theorem is a consequence of (36).

(40) Let us consider real normed spaces X, Y, a partial function f from $\prod\langle X, Y\rangle$ to W, and a set D. Suppose $D \subseteq$ the carrier of $\prod\langle X, Y\rangle$. Then $f \cdot \text{IsoCPNrSP}(X, Y)$ is differentiable on $(\text{IsoCPNrSP}(X, Y))^{-1}(D)$ if and only if f is differentiable on D. The theorem is a consequence of (38).

(41) Let us consider real normed spaces X, Y, a partial function f from $X \times Y$ to W, and a set D. Suppose $D \subseteq$ the carrier of $X \times Y$. Then $f \cdot \text{IsoPCNrSP}(X, Y)$ is differentiable on $(\text{IsoPCNrSP}(X, Y))^{-1}(D)$ if and only if f is differentiable on D. The theorem is a consequence of (38).

(42) Let us consider real normed spaces X, Y, a partial function f from $\prod\langle X, Y\rangle$ to W, and a subset D of $\prod\langle X, Y\rangle$. Suppose f is differentiable on D. Let us consider a point z of $\prod\langle X, Y\rangle$. Suppose $z \in \operatorname{dom} f'_{\restriction D}$. Then $f'_{\restriction D}(z) = ((f \cdot \text{IsoCPNrSP}(X, Y))'_{\restriction(\text{IsoCPNrSP}(X,Y))^{-1}(D)})_{(\text{IsoPCNrSP}(X,Y))(z)} \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$. The theorem is a consequence of (40) and (33). PROOF: Set $I = \text{IsoCPNrSP}(X, Y)$. Set $J = \text{IsoPCNrSP}(X, Y)$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $\prod\langle X, Y\rangle$ such that $z \in \operatorname{dom} f'_{\restriction D}$ holds $f'_{\restriction D}(z) = (g'_{\restriction E})_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [22, (36)]. □

(43) Let us consider real normed spaces X, Y, a partial function f from $X \times Y$ to W, and a subset D of $X \times Y$. Suppose f is differentiable on

D. Let us consider a point z of $X \times Y$. Suppose $z \in \operatorname{dom} f'_{\restriction D}$. Then $f'_{\restriction D}(z) = ((f \cdot \mathrm{IsoPCNrSP}(X, Y))'_{\restriction (\mathrm{IsoPCNrSP}(X,Y))^{-1}(D)})_{(\mathrm{IsoCPNrSP}(X,Y))(z)} \cdot (\mathrm{IsoPCNrSP}(X, Y))^{-1}$. The theorem is a consequence of (41) and (33).
PROOF: Set $I = \mathrm{IsoPCNrSP}(X, Y)$. Set $J = \mathrm{IsoCPNrSP}(X, Y)$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $X \times Y$ such that $z \in \operatorname{dom} f'_{\restriction D}$ holds $f'_{\restriction D}(z) = (g'_{\restriction E})_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [22, (36)]. □

Let X, Y be real normed spaces and x be an element of $X \times Y$. The functor reproj1 x yielding a function from X into $X \times Y$ is defined by

(Def. 4) Let us consider an element r of X. Then $it(r) = \langle r, x_{\mathbf{2}} \rangle$.

The functor reproj2 x yielding a function from Y into $X \times Y$ is defined by

(Def. 5) Let us consider an element r of Y. Then $it(r) = \langle x_{\mathbf{1}}, r \rangle$.

Now we state the proposition:

(44) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then

(i) reproj1 $z = \mathrm{IsoPCNrSP}(X, Y) \cdot \mathrm{reproj}(1(\in \operatorname{dom}\langle X, Y \rangle), (\mathrm{IsoCPNrSP}(X, Y))(z))$, and

(ii) reproj2 $z = \mathrm{IsoPCNrSP}(X, Y) \cdot \mathrm{reproj}(2(\in \operatorname{dom}\langle X, Y \rangle), (\mathrm{IsoCPNrSP}(X, Y))(z))$.

Let X, Y be real normed spaces and z be a point of $X \times Y$. Observe that the functor $z_{\mathbf{1}}$ yields a point of X. Let us note that the functor $z_{\mathbf{2}}$ yields a point of Y. Let X, Y, W be real normed spaces. Let f be a partial function from $X \times Y$ to W. We say that f is partial differentiable in'1 z if and only if

(Def. 6) $f \cdot$ reproj1 z is differentiable in $z_{\mathbf{1}}$.

We say that f is partial differentiable in'2 z if and only if

(Def. 7) $f \cdot$ reproj2 z is differentiable in $z_{\mathbf{2}}$.

Now we state the propositions:

(45) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then

(i) $z_{\mathbf{1}}$ = the projection onto $1(\in \operatorname{dom}\langle X, Y \rangle)((\mathrm{IsoCPNrSP}(X, Y))(z))$, and

(ii) $z_{\mathbf{2}}$ = the projection onto $2(\in \operatorname{dom}\langle X, Y \rangle)((\mathrm{IsoCPNrSP}(X, Y))(z))$.

(46) Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, and a partial function f from $X \times Y$ to W. Then

(i) f is partial differentiable in'1 z iff $f \cdot \mathrm{IsoPCNrSP}(X, Y)$ is partially differentiable in $(\mathrm{IsoCPNrSP}(X, Y))(z)$ w.r.t. 1, and

(ii) f is partial differentiable in'2 z iff $f \cdot \mathrm{IsoPCNrSP}(X, Y)$ is partially differentiable in $(\mathrm{IsoCPNrSP}(X, Y))(z)$ w.r.t. 2.

The theorem is a consequence of (44) and (45).

Let X, Y, W be real normed spaces, z be a point of $X \times Y$, and f be a partial function from $X \times Y$ to W. The functor partdiff'1(f, z) yielding a point of the real norm space of bounded linear operators from X into W is defined by the term

(Def. 8) $(f \cdot \text{reproj1}\, z)'(z_{\mathbf{1}})$.

The functor partdiff‘2(f, z) yielding a point of the real norm space of bounded linear operators from Y into W is defined by the term

(Def. 9) $(f \cdot \text{reproj2}\, z)'(z_{\mathbf{2}})$.

Now we state the propositions:

(47) Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, and a partial function f from $X \times Y$ to W. Then

(i) $\text{partdiff‘1}(f, z) = \text{partdiff}(f \cdot \text{IsoPCNrSP}(X, Y), (\text{IsoCPNrSP}(X, Y))(z), 1)$, and

(ii) $\text{partdiff‘2}(f, z) = \text{partdiff}(f \cdot \text{IsoPCNrSP}(X, Y), (\text{IsoCPNrSP}(X, Y))(z), 2)$.

The theorem is a consequence of (44) and (45).

(48) Let us consider real normed spaces X, Y, W, a function I from X into Y, and partial functions f_1, f_2 from Y to W. Then

(i) $(f_1 + f_2) \cdot I = f_1 \cdot I + f_2 \cdot I$, and

(ii) $(f_1 - f_2) \cdot I = f_1 \cdot I - f_2 \cdot I$.

PROOF: Set D_1 = the carrier of X. For every element s of D_1, $s \in \text{dom}((f_1+f_2) \cdot I)$ iff $s \in \text{dom}(f_1 \cdot I + f_2 \cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \text{dom}((f_1+f_2) \cdot I)$ holds $((f_1+f_2) \cdot I)(z) = (f_1 \cdot I + f_2 \cdot I)(z)$ by [4, (11), (12)]. For every element s of D_1, $s \in \text{dom}((f_1 - f_2) \cdot I)$ iff $s \in \text{dom}(f_1 \cdot I - f_2 \cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \text{dom}((f_1 - f_2) \cdot I)$ holds $((f_1 - f_2) \cdot I)(z) = (f_1 \cdot I - f_2 \cdot I)(z)$ by [4, (11), (12)]. $\square$

(49) Let us consider real normed spaces X, Y, W, a function I from X into Y, a partial function f from Y to W, and a real number r. Then $r \cdot (f \cdot I) = (r \cdot f) \cdot I$. PROOF: Set D_1 = the carrier of X. For every element s of D_1, $s \in \text{dom}((r \cdot f) \cdot I)$ iff $s \in \text{dom}(f \cdot I)$ by [4, (11)]. For every element s of D_1, $s \in \text{dom}((r \cdot f) \cdot I)$ iff $I(s) \in \text{dom}(r \cdot f)$ by [4, (11)]. For every element z of D_1 such that $z \in \text{dom}(r \cdot (f \cdot I))$ holds $(r \cdot (f \cdot I))(z) = ((r \cdot f) \cdot I)(z)$ by [4, (12)]. $\square$

Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, and partial functions f_1, f_2 from $X \times Y$ to W. Now we state the propositions:

(50) Suppose f_1 is partial differentiable in‘1 z and f_2 is partial differentiable in‘1 z. Then

(i) $f_1 + f_2$ is partial differentiable in‘1 z, and

(ii) $\text{partdiff‘1}((f_1 + f_2), z) = \text{partdiff‘1}(f_1, z) + \text{partdiff‘1}(f_2, z)$, and

(iii) $f_1 - f_2$ is partial differentiable in‘1 z, and

(iv) partdiff‘1$((f_1 - f_2), z)$ = partdiff‘1(f_1, z) − partdiff‘1(f_2, z).

(51) Suppose f_1 is partial differentiable in‘2 z and f_2 is partial differentiable in‘2 z. Then

(i) $f_1 + f_2$ is partial differentiable in‘2 z, and

(ii) partdiff‘2$((f_1 + f_2), z)$ = partdiff‘2(f_1, z) + partdiff‘2(f_2, z), and

(iii) $f_1 - f_2$ is partial differentiable in‘2 z, and

(iv) partdiff‘2$((f_1 - f_2), z)$ = partdiff‘2(f_1, z) − partdiff‘2(f_2, z).

Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, a real number r, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(52) If f is partial differentiable in‘1 z, then $r \cdot f$ is partial differentiable in‘1 z and partdiff‘1$((r \cdot f), z) = r \cdot$ partdiff‘1(f, z).

(53) If f is partial differentiable in‘2 z, then $r \cdot f$ is partial differentiable in‘2 z and partdiff‘2$((r \cdot f), z) = r \cdot$ partdiff‘2(f, z).

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W. We say that f is partial differentiable on‘1 Z if and only if

(Def. 10) (i) $Z \subseteq \operatorname{dom} f$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f{\restriction}Z$ is partial differentiable in‘1 z.

We say that f is partial differentiable on‘2 Z if and only if

(Def. 11) (i) $Z \subseteq \operatorname{dom} f$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f{\restriction}Z$ is partial differentiable in‘2 z.

Now we state the proposition:

(54) Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W. Then

(i) f is partial differentiable on‘1 Z iff $f \cdot \mathrm{IsoPCNrSP}(X, Y)$ is partially differentiable on $(\mathrm{IsoPCNrSP}(X, Y))^{-1}(Z)$ w.r.t. 1, and

(ii) f is partial differentiable on‘2 Z iff $f \cdot \mathrm{IsoPCNrSP}(X, Y)$ is partially differentiable on $(\mathrm{IsoPCNrSP}(X, Y))^{-1}(Z)$ w.r.t. 2.

The theorem is a consequence of (46) and (4). PROOF: Set $I = \mathrm{IsoPCNrSP}(X, Y)$. Set $g = f \cdot I$. Set $E = I^{-1}(Z)$. f is partial differentiable on‘1 Z iff g is partially differentiable on E w.r.t. 1 by [5, (113)], [4, (34)], [5, (38)]. f is partial differentiable on‘2 Z iff g is partially differentiable on E w.r.t. 2 by [5, (113)], [4, (34)], [5, (38)]. □

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W. Assume f is partial differentiable on‘1 Z. The functor f ‘partial‘1$|$ Z yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from X into W is defined by

(Def. 12) (i) dom $it = Z$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \text{partdiff`1}(f, z)$.

Assume f is partial differentiable on`2 $Z$. The functor $f$ `partial`2| Z yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from Y into W is defined by

(Def. 13) (i) dom $it = Z$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \text{partdiff`2}(f, z)$.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(55) Suppose f is partial differentiable on`1 $Z$. Then $f$ `partial`1| $Z = (f \cdot \text{IsoPCNrSP}(X, Y){\restriction}^1(\text{IsoPCNrSP}(X, Y))^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y)$.

(56) Suppose f is partial differentiable on`2 $Z$. Then $f$ `partial`2| $Z = (f \cdot \text{IsoPCNrSP}(X, Y){\restriction}^2(\text{IsoPCNrSP}(X, Y))^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y)$.

(57) Suppose Z is open. Then f is partial differentiable on`1 $Z$ if and only if $Z \subseteq \text{dom} f$ and for every point $x$ of $X \times Y$ such that $x \in Z$ holds $f$ is partial differentiable in`1 x.

(58) Suppose Z is open. Then f is partial differentiable on`2 $Z$ if and only if $Z \subseteq \text{dom} f$ and for every point $x$ of $X \times Y$ such that $x \in Z$ holds $f$ is partial differentiable in`2 x.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and partial functions f, g from $X \times Y$ to W. Now we state the propositions:

(59) Suppose Z is open and f is partial differentiable on`1 $Z$ and $g$ is partial differentiable on`1 Z. Then

(i) $f + g$ is partial differentiable on`1 Z, and

(ii) $(f + g)$ `partial`1| $Z = (f$ `partial`1| $Z) + (g$ `partial`1| $Z)$.

(60) Suppose Z is open and f is partial differentiable on`1 $Z$ and $g$ is partial differentiable on`1 Z. Then

(i) $f - g$ is partial differentiable on`1 Z, and

(ii) $(f - g)$ `partial`1| $Z = (f$ `partial`1| $Z) - (g$ `partial`1| $Z)$.

(61) Suppose Z is open and f is partial differentiable on`2 $Z$ and $g$ is partial differentiable on`2 Z. Then

(i) $f + g$ is partial differentiable on`2 Z, and

(ii) $(f + g)$ `partial`2| $Z = (f$ `partial`2| $Z) + (g$ `partial`2| $Z)$.

(62) Suppose Z is open and f is partial differentiable on`2 $Z$ and $g$ is partial differentiable on`2 Z. Then

(i) $f - g$ is partial differentiable on`2 Z, and

(ii) $(f - g)$ `partial`2| $Z = (f$ `partial`2| $Z) - (g$ `partial`2| $Z)$.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, a real number r, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(63) Suppose Z is open and f is partial differentiable on'1 Z. Then

(i) $r \cdot f$ is partial differentiable on'1 Z, and

(ii) $r \cdot f$ 'partial'1$|$ $Z = r \cdot (f$ 'partial'1$|$ $Z)$.

(64) Suppose Z is open and f is partial differentiable on'2 Z. Then

(i) $r \cdot f$ is partial differentiable on'2 Z, and

(ii) $r \cdot f$ 'partial'2$|$ $Z = r \cdot (f$ 'partial'2$|$ $Z)$.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(65) Suppose f is differentiable on Z. Then $f'_{\restriction Z}$ is continuous on Z if and only if $(f \cdot \mathrm{IsoPCNrSP}(X, Y))'_{\restriction (\mathrm{IsoPCNrSP}(X,Y))^{-1}(Z)}$ is continuous on $(\mathrm{IsoPCNrSP}(X, Y))^{-1}(Z)$.

(66) Suppose Z is open. Then f is partial differentiable on'1 Z and f is partial differentiable on'2 Z and f 'partial'1$|$ Z is continuous on Z and f 'partial'2$|$ Z is continuous on Z if and only if f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z.

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Differential Equations on Functions from ℝ into Real Banach Space[1]

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Summary. In this article, we described the differential equations on functions from ℝ into real Banach space. The descriptions were based on the article [20]. As preliminary to prove these theorems, we proved some properties of differentiable functions on real normed space. For the proof we referred to descriptions and theorems in the article [21] and the article [31]. And applying the theorems of Riemann integral introduced in the article [22], we proved the ordinary differential equations on real Banach space. We referred to the methods of proof in [?].

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The notation and terminology used in this paper have been introduced in the following articles: [29], [5], [11], [3], [6], [7], [19], [13], [33], [30], [32], [1], [15], [25], [31], [18], [24], [23], [26], [27], [20], [2], [8], [14], [16], [28], [12], [36], [37], [9], [34], [35], [17], and [10].

1. Some Properties of Differentiable Functions on Real Normed Space

From now on Y denotes a real normed space.

Now we state the propositions:

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(1) Let us consider a real normed space Y, a function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y. Suppose

(i) $J = \text{proj}(1, 1)$, and

(ii) $x_0 \in \text{dom}\, f$, and

(iii) $y_0 \in \text{dom}\, g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f = g \cdot J$.

Then f is continuous in x_0 if and only if g is continuous in y_0. PROOF: If f is continuous in x_0, then g is continuous in y_0 by [14, (2)], [6, (39)], [36, (36)]. □

(2) Let us consider a real normed space Y, a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y. Suppose

(i) $I = (\text{proj}(1, 1)$ **qua** function$)^{-1}$, and

(ii) $x_0 \in \text{dom}\, f$, and

(iii) $y_0 \in \text{dom}\, g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$.

Then f is continuous in x_0 if and only if g is continuous in y_0. PROOF: If f is continuous in x_0, then g is continuous in y_0 by [14, (1)], [21, (33)], [26, (15)]. □

(3) Let us consider a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Suppose $I = (\text{proj}(1, 1)$ **qua** function$)^{-1}$. Then

(i) for every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y, $R \cdot I$ is a rest of Y, and

(ii) for every linear operator L from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y, $L \cdot I$ is a linear of Y.

PROOF: For every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y, $R \cdot I$ is a rest of Y by [15, (23)], [5, (47)], [14, (3)]. Reconsider $L_0 = L$ as a function from $\mathcal{R}^1$ into Y. Reconsider $L_1 = L_0 \cdot I$ as a partial function from $\mathbb{R}$ to Y. Reconsider $j_0 = 1$ as an element of $\mathbb{R}$. Reconsider $r = L_1(j_0)$ as a point of Y. For every real number p, $L_{1p} = p \cdot r$ by [6, (13)], [14, (3)], [6, (12)]. □

(4) Let us consider a function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$. Suppose $J = \text{proj}(1, 1)$. Then

(i) for every rest R of Y, $R \cdot J$ is a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y, and

(ii) for every linear L of Y, $L \cdot J$ is a Lipschitzian linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y.

PROOF: For every rest R of Y, $R \cdot J$ is a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, Y by [14, (4)], [15, (6)], [5, (47)]. Consider r being a point of Y such that for every real number p, $L_p = p \cdot r$. □

(5) Let us consider a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to Y. Suppose

(i) $I = (\text{proj}(1,1)$ **qua** function$)^{-1}$, and

(ii) $x_0 \in \text{dom} f$, and

(iii) $y_0 \in \text{dom} g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$, and

(vi) f is differentiable in x_0.

Then

(vii) g is differentiable in y_0, and

(viii) $g'(y_0) = f'(x_0)(\langle 1 \rangle)$, and

(ix) for every element r of $\mathbb{R}$, $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$.

The theorem is a consequence of (3). PROOF: Consider N_1 being a neighbourhood of x_0 such that $N_1 \subseteq \text{dom} f$ and there exists a point L of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into Y and there exists a rest R of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, Y such that for every point x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x \in N_1$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x - x_0}$. Consider e being a real number such that $0 < e$ and $\{z,$ where z is a point of $\langle \mathcal{E}^1, \| \cdot \| \rangle : \|z - x_0\| < e\} \subseteq N_1$. Consider L being a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into Y, R being a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, Y such that for every point x_3 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x_3 \in N_1$ holds $f_{x_3} - f_{x_0} = L(x_3 - x_0) + R_{x_3 - x_0}$. Reconsider $R_0 = R \cdot I$ as a rest of Y. Reconsider $L_0 = L \cdot I$ as a linear of Y. Set $N = \{z,$ where z is a point of $\langle \mathcal{E}^1, \| \cdot \| \rangle : \|z - x_0\| < e\}$. $N \subseteq$ the carrier of $\langle \mathcal{E}^1, \| \cdot \| \rangle$. Set $N_0 = \{z,$ where z is an element of $\mathbb{R} : |z - y_0| < e\}$. $]y_0 - e, y_0 + e[\subseteq N_0$ by [28, (1)]. $N_0 \subseteq]y_0 - e, y_0 + e[$ by [28, (1)]. For every real number y_1 such that $y_1 \in N_0$ holds $(f \cdot I)_{y_1} - (f \cdot I)_{y_0} = L_{0 y_1 - y_0} + R_{0 y_1 - y_0}$ by [6, (12)], [7, (35)], [14, (3)]. □

(6) Let us consider a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a real number y_0, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to Y. Suppose

(i) $I = (\text{proj}(1,1)$ **qua** function$)^{-1}$, and

(ii) $x_0 \in \text{dom} f$, and

(iii) $y_0 \in \text{dom} g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$.

Then f is differentiable in x_0 if and only if g is differentiable in y_0. The theorem is a consequence of (5) and (4). PROOF: Reconsider $J = \mathrm{proj}(1,1)$ as a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$. Consider N_0 being a neighbourhood of y_0 such that $N_0 \subseteq \mathrm{dom}(f \cdot I)$ and there exists a linear L of Y and there exists a rest R of Y such that for every real number y such that $y \in N_0$ holds $(f \cdot I)_y - (f \cdot I)_{y_0} = L_{y-y_0} + R_{y-y_0}$. Consider e_0 being a real number such that $0 < e_0$ and $N_0 =]y_0 - e_0, y_0 + e_0[$. Reconsider $e = e_0$ as an element of $\mathbb{R}$. Set $N = \{z,$ where z is a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle : \|z - x_0\| < e\}$. Consider L being a linear of Y, R being a rest of Y such that for every real number y_1 such that $y_1 \in N_0$ holds $(f \cdot I)_{y_1} - (f \cdot I)_{y_0} = L_{y_1 - y_0} + R_{y_1 - y_0}$. Reconsider $R_0 = R \cdot J$ as a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y. Reconsider $L_0 = L \cdot J$ as a Lipschitzian linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y. $N \subseteq$ the carrier of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. For every point y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $y \in N$ holds $f_y - f_{x_0} = L_0(y - x_0) + R_{0y-x_0}$ by [6, (13)], [7, (35)], [14, (4)]. □

(7) Let us consider a function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y. Suppose

(i) $J = \mathrm{proj}(1,1)$, and

(ii) $x_0 \in \mathrm{dom}\, f$, and

(iii) $y_0 \in \mathrm{dom}\, g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f = g \cdot J$.

Then f is differentiable in x_0 if and only if g is differentiable in y_0. The theorem is a consequence of (6).

(8) Let us consider a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y. Suppose

(i) $I = (\mathrm{proj}(1,1)$ **qua** function$)^{-1}$, and

(ii) $x_0 \in \mathrm{dom}\, f$, and

(iii) $y_0 \in \mathrm{dom}\, g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$, and

(vi) f is differentiable in x_0.

Then $\|g'(y_0)\| = \|f'(x_0)\|$. The theorem is a consequence of (5). PROOF: Reconsider $d_1 = f'(x_0)$ as a Lipschitzian linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y. Set $A = \mathrm{PreNorms}(d_1)$. For every real number r such that $r \in A$ holds $r \leqslant \|g'(y_0)\|$ by [14, (1), (4)]. □

Let us consider real numbers a, b, z and points p, q, x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$. Now we state the propositions:

(9) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then

(i) if $z \in]a, b[$, then $x \in]p, q[$, and

(ii) if $x \in]p, q[$, then $a \neq b$ and if $a < b$, then $z \in]a, b[$ and if $a > b$, then $z \in]b, a[$.

(10) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then

(i) if $z \in [a, b]$, then $x \in [p, q]$, and

(ii) if $x \in [p, q]$, then if $a \leqslant b$, then $z \in [a, b]$ and if $a \geqslant b$, then $z \in [b, a]$.

Now we state the propositions:

(11) Let us consider real numbers a, b, points p, q of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, and a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$. Suppose

(i) $p = \langle a \rangle$, and

(ii) $q = \langle b \rangle$, and

(iii) $I = (\text{proj}(1, 1)$ **qua** function$)^{-1}$.

Then

(iv) if $a \leqslant b$, then $I^{\circ}[a, b] = [p, q]$, and

(v) if $a < b$, then $I^{\circ}]a, b[=]p, q[$.

The theorem is a consequence of (10) and (9).

(12) Let us consider a real normed space Y, a partial function g from $\mathbb{R}$ to the carrier of Y, and real numbers a, b, M. Suppose

(i) $a \leqslant b$, and

(ii) $[a, b] \subseteq \text{dom}\, g$, and

(iii) for every real number x such that $x \in [a, b]$ holds g is continuous in x, and

(iv) for every real number x such that $x \in]a, b[$ holds g is differentiable in x, and

(v) for every real number x such that $x \in]a, b[$ holds $\|g'(x)\| \leqslant M$.

Then $\|g_b - g_a\| \leqslant M \cdot |b - a|$. The theorem is a consequence of (11), (10), (1), (9), (7), and (8).

2. Differential Equations

In the sequel X, Y denote real Banach spaces, Z denotes an open subset of $\mathbb{R}$, a, b, c, d, e, r, x_0 denote real numbers, y_0 denotes a vector of X, and G denotes a function from X into X.

Now we state the propositions:

(13) Let us consider a real Banach space X, a partial function F from $\mathbb{R}$ to the carrier of X, and a continuous partial function f from $\mathbb{R}$ to the carrier of X. Suppose

(i) $[a, b] \subseteq \operatorname{dom} f$, and

(ii) $]a, b[\subseteq \operatorname{dom} F$, and

(iii) for every real number x such that $x \in]a, b[$ holds $F_x = \int\limits_a^x f(x)dx$, and

(iv) $x_0 \in]a, b[$, and

(v) f is continuous in x_0.

Then

(vi) F is differentiable in x_0, and

(vii) $F'(x_0) = f_{x_0}$.

(14) Let us consider a partial function F from $\mathbb{R}$ to the carrier of X and a continuous partial function f from $\mathbb{R}$ to the carrier of X. Suppose

(i) $\operatorname{dom} f = [a, b]$, and

(ii) $\operatorname{dom} F = [a, b]$, and

(iii) for every real number t such that $t \in [a, b]$ holds $F_t = \int\limits_a^t f(x)dx$.

Let us consider a real number x. If $x \in [a, b]$, then F is continuous in x.

(15) Let us consider a continuous partial function f from $\mathbb{R}$ to the carrier of X. If $a \in \operatorname{dom} f$, then $\int\limits_a^a f(x)dx = 0_X$.

Let us consider a continuous partial function f from $\mathbb{R}$ to the carrier of X and a partial function g from $\mathbb{R}$ to the carrier of X. Now we state the propositions:

(16) Suppose $a \leqslant b$ and $\operatorname{dom} f = [a, b]$ and for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int\limits_a^t f(x)dx$. Then $g_a = y_0$.

(17) Suppose $\operatorname{dom} f = [a, b]$ and $\operatorname{dom} g = [a, b]$ and $Z =]a, b[$ and for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int\limits_a^t f(x)dx$. Then

(i) g is continuous and differentiable on Z, and

(ii) for every real number t such that $t \in Z$ holds $g'(t) = f_t$.

Let us consider a partial function f from $\mathbb{R}$ to the carrier of X. Now we state the propositions:

(18) Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and for every real number x such that $x \in [a, b]$ holds f is continuous in x and f is differentiable on $]a, b[$ and for every real number x such that $x \in]a, b[$ holds $f'(x) = 0_X$. Then $f_b = f_a$.

(19) Suppose $[a, b] \subseteq \operatorname{dom} f$ and for every real number x such that $x \in [a, b]$ holds f is continuous in x and f is differentiable on $]a, b[$ and for every real number x such that $x \in]a, b[$ holds $f'(x) = 0_X$. Then $f{\restriction}]a, b[$ is constant.

Now we state the propositions:

(20) Let us consider a continuous partial function f from $\mathbb{R}$ to the carrier of X. Suppose

(i) $[a, b] = \operatorname{dom} f$, and

(ii) $f{\restriction}]a, b[$ is constant.

Let us consider a real number x. If $x \in [a, b]$, then $f_x = f_a$.

(21) Let us consider continuous partial functions y, G_1 from $\mathbb{R}$ to the carrier of X and a partial function g from $\mathbb{R}$ to the carrier of X. Suppose

(i) $a \leqslant b$, and

(ii) $Z =]a, b[$, and

(iii) $\operatorname{dom} y = [a, b]$, and

(iv) $\operatorname{dom} g = [a, b]$, and

(v) $\operatorname{dom} G_1 = [a, b]$, and

(vi) y is differentiable on Z, and

(vii) $y_a = y_0$, and

(viii) for every real number t such that $t \in Z$ holds $y'(t) = {G_1}_t$, and

(ix) for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int\limits_a^t G_1(x)dx$.

Then $y = g$. The theorem is a consequence of (17), (16), (19), and (20).
PROOF: Reconsider $h = y - g$ as a continuous partial function from $\mathbb{R}$ to the carrier of X. For every real number x such that $x \in \operatorname{dom} h$ holds $h_x = 0_X$ by [34, (15)]. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom} y$ holds $y(x) = g(x)$ by [34, (21)]. □

Let X be a real Banach space, y_0 be a vector of X, G be a function from X into X, and a, b be real numbers. Assume $a \leqslant b$ and G is continuous on $\operatorname{dom} G$. The functor Fredholm(G, a, b, y_0) yielding a function from the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X into the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X is defined by

(Def. 1) Let us consider a vector x of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X. Then there exist continuous partial functions f, g, G_1 from $\mathbb{R}$ to the carrier of X such that

(i) $x = f$, and

(ii) $it(x) = g$, and

(iii) $\operatorname{dom} f = [a, b]$, and

(iv) $\operatorname{dom} g = [a, b]$, and

(v) $G_1 = G \cdot f$, and

(vi) for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int\limits_a^t G_1(x)dx$.

Now we state the propositions:

(22) Suppose $a \leqslant b$ and $0 < r$ and for every vectors y_1, y_2 of X, $\|G_{y_1} - G_{y_2}\| \leqslant r \cdot \|y_1 - y_2\|$. Let us consider vectors u, v of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X and continuous partial functions g, h from $\mathbb{R}$ to the carrier of X. Suppose

(i) $g = (\text{Fredholm}(G, a, b, y_0))(u)$, and

(ii) $h = (\text{Fredholm}(G, a, b, y_0))(v)$.

Let us consider a real number t. Suppose $t \in [a, b]$. Then $\|g_t - h_t\| \leqslant (r \cdot (t - a)) \cdot \|u - v\|$. Proof: Set $F = \text{Fredholm}(G, a, b, y_0)$. Consider f_1, g_1, G_3 being continuous partial functions from $\mathbb{R}$ to the carrier of X such that $u = f_1$ and $F(u) = g_1$ and $\operatorname{dom} f_1 = [a, b]$ and $\operatorname{dom} g_1 = [a, b]$ and $G_3 = G \cdot f_1$ and for every real number t such that $t \in [a, b]$ holds ${g_1}_t = y_0 + \int\limits_a^t G_3(x)dx$. Consider f_2, g_2, G_5 being continuous partial functions from $\mathbb{R}$ to the carrier of X such that $v = f_2$ and $F(v) = g_2$ and $\operatorname{dom} f_2 = [a, b]$ and $\operatorname{dom} g_2 = [a, b]$ and $G_5 = G \cdot f_2$ and for every real number t such that $t \in [a, b]$ holds ${g_2}_t = y_0 + \int\limits_a^t G_5(x)dx$. Set $G_4 = G_3 - G_5$. For every real number x such that $x \in [a, t]$ holds $\|{G_4}_x\| \leqslant r \cdot \|u - v\|$ by [20, (26)], [6, (12)]. □

(23) Suppose $a \leqslant b$ and $0 < r$ and for every vectors y_1, y_2 of X, $\|G_{y_1} - G_{y_2}\| \leqslant r \cdot \|y_1 - y_2\|$. Let us consider vectors u, v of the $\mathbb{R}$-norm space of

continuous functions of $[a, b]$ and X, an element m of $\mathbb{N}$, and continuous partial functions g, h from $\mathbb{R}$ to the carrier of X. Suppose

(i) $g = (\text{Fredholm}(G, a, b, y_0))^{m+1}(u)$, and

(ii) $h = (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)$.

Let us consider a real number t. Suppose $t \in [a, b]$. Then $\|g_t - h_t\| \leqslant \frac{(r\cdot(t-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$. The theorem is a consequence of (22). PROOF: Set $F = \text{Fredholm}(G, a, b, y_0)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every continuous partial functions g, h from $\mathbb{R}$ to the carrier of X such that $g = F^{\$_1+1}(u_1)$ and $h = F^{\$_1+1}(v_1)$ for every real number t such that $t \in [a, b]$ holds $\|g_t - h_t\| \leqslant \frac{(r\cdot(t-a))^{\$_1+1}}{(\$_1+1)!} \cdot \|u_1 - v_1\|$. $\mathcal{P}[0]$ by [4, (70)], [18, (5), (13)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (71)], [6, (13)], [36, (27)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(24) Let us consider a natural number m. Suppose

(i) $a \leqslant b$, and

(ii) $0 < r$, and

(iii) for every vectors y_1, y_2 of X, $\|G_{y_1} - G_{y_2}\| \leqslant r \cdot \|y_1 - y_2\|$.

Let us consider vectors u, v of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X. Then $\|(\text{Fredholm}(G, a, b, y_0))^{m+1}(u) - (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)\| \leqslant \frac{(r\cdot(b-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$. The theorem is a consequence of (23).

(25) If $a < b$ and G is Lipschitzian on the carrier of X, then there exists a natural number m such that $(\text{Fredholm}(G, a, b, y_0))^{m+1}$ is contraction. The theorem is a consequence of (24).

(26) If $a < b$ and G is Lipschitzian on the carrier of X, then $\text{Fredholm}(G, a, b, y_0)$ has unique fixpoint. The theorem is a consequence of (25).

(27) Let us consider continuous partial functions f, g from $\mathbb{R}$ to the carrier of X. Suppose

(i) $\text{dom}\, f = [a, b]$, and

(ii) $\text{dom}\, g = [a, b]$, and

(iii) $Z =]a, b[$, and

(iv) $a < b$, and

(v) G is Lipschitzian on the carrier of X, and

(vi) $g = (\text{Fredholm}(G, a, b, y_0))(f)$.

Then

(vii) $g_a = y_0$, and

(viii) g is differentiable on Z, and

(ix) for every real number t such that $t \in Z$ holds $g'(t) = (G \cdot f)_t$.

The theorem is a consequence of (17) and (16).

(28) Let us consider a continuous partial function y from $\mathbb{R}$ to the carrier of X. Suppose

(i) $a < b$, and

(ii) $Z =]a, b[$, and

(iii) G is Lipschitzian on the carrier of X, and

(iv) $\operatorname{dom} y = [a, b]$, and

(v) y is differentiable on Z, and

(vi) $y_a = y_0$, and

(vii) for every real number t such that $t \in Z$ holds $y'(t) = G(y_t)$.

Then y is a fixpoint of Fredholm(G, a, b, y_0). The theorem is a consequence of (21). Proof: Consider f, g, G_1 being continuous partial functions from $\mathbb{R}$ to the carrier of X such that $y = f$ and $(\text{Fredholm}(G, a, b, y_0))(y) = g$ and $\operatorname{dom} f = [a, b]$ and $\operatorname{dom} g = [a, b]$ and $G_1 = G \cdot f$ and for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int\limits_a^t G_1(x)dx$. For every real number t such that $t \in Z$ holds $y'(t) = G_{1t}$ by [6, (13)]. □

(29) Let us consider continuous partial functions y_1, y_2 from $\mathbb{R}$ to the carrier of X. Suppose

(i) $a < b$, and

(ii) $Z =]a, b[$, and

(iii) G is Lipschitzian on the carrier of X, and

(iv) $\operatorname{dom} y_1 = [a, b]$, and

(v) y_1 is differentiable on Z, and

(vi) $y_{1a} = y_0$, and

(vii) for every real number t such that $t \in Z$ holds $y_1{}'(t) = G(y_{1t})$, and

(viii) $\operatorname{dom} y_2 = [a, b]$, and

(ix) y_2 is differentiable on Z, and

(x) $y_{2a} = y_0$, and

(xi) for every real number t such that $t \in Z$ holds $y_2{}'(t) = G(y_{2t})$.

Then $y_1 = y_2$. The theorem is a consequence of (26) and (28).

(30) Suppose $a < b$ and $Z =]a, b[$ and G is Lipschitzian on the carrier of X. Then there exists a continuous partial function y from $\mathbb{R}$ to the carrier of X such that

(i) $\operatorname{dom} y = [a, b]$, and

(ii) y is differentiable on Z, and

(iii) $y_a = y_0$, and

(iv) for every real number t such that $t \in Z$ holds $y'(t) = G(y_t)$.

The theorem is a consequence of (26) and (27).

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Submodule of free $\mathbb{Z}$-module[1]

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Summary. In this article, we formalize a free $\mathbb{Z}$-module and its property. Specially, we formalize a vector space of rational field corresponding to a free $\mathbb{Z}$-module and prove formally that submodules of a free $\mathbb{Z}$-module are free. $\mathbb{Z}$-module is necassary for lattice problems, LLL (Lenstra, Lenstra and Lovász) base reduction algorithm and cryptographic systems with lattice [20]. Some theorems in this article are described by translating theorems in [11] into theorems of Z-module, however their proofs are different.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [1], [24], [22], [5], [12], [7], [8], [16], [26], [19], [23], [21], [3], [4], [9], [17], [31], [33], [32], [27], [30], [18], [28], [29], [34], [10], [13], [14], and [15].

1. Vector Space of Rational Field Generated by a Free $\mathbb{Z}$-module

From now on V denotes a $\mathbb{Z}$-module and W, W_1, W_2 denote submodules of V.

Let us consider a $\mathbb{Z}$-module V, submodules W_1, W_2 of V, and submodules W_5, W_6 of $W_1 + W_2$. Now we state the propositions:

(1) If $W_5 = W_1$ and $W_6 = W_2$, then $W_1 + W_2 = W_5 + W_6$.

(2) If $W_5 = W_1$ and $W_6 = W_2$, then $W_1 \cap W_2 = W_5 \cap W_6$.

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Let V be a $\mathbb{Z}$-module. Note that (the carrier of V) $\times$ ($\mathbb{Z} \setminus \{0\}$) is non empty.

Assume V is cancelable on multiplication. The functor EQRZM V yielding an equivalence relation of (the carrier of V) $\times$ ($\mathbb{Z} \setminus \{0\}$) is defined by

(Def. 1) Let us consider elements S, T. Then $\langle S, T\rangle \in it$ if and only if $S, T \in$ (the carrier of V) $\times$ ($\mathbb{Z} \setminus \{0\}$) and there exist elements z_1, z_2 of V and there exist integers i_1, i_2 such that $S = \langle z_1, i_1\rangle$ and $T = \langle z_2, i_2\rangle$ and $i_1 \neq 0$ and $i_2 \neq 0$ and $i_2 \cdot z_1 = i_1 \cdot z_2$.

Now we state the proposition:

(3) Let us consider a $\mathbb{Z}$-module V, elements z_1, z_2 of V, and integers i_1, i_2. Suppose V is cancelable on multiplication. Then $\langle\langle z_1, i_1\rangle, \langle z_2, i_2\rangle\rangle \in$ EQRZM V if and only if $i_1 \neq 0$ and $i_2 \neq 0$ and $i_2 \cdot z_1 = i_1 \cdot z_2$.

Let V be a $\mathbb{Z}$-module. Assume V is cancelable on multiplication. The functor addCoset V yielding a binary operation on Classes EQRZM V is defined by

(Def. 2) Let us consider elements A, B. Suppose $A, B \in$ Classes EQRZM V. Let us consider elements z_1, z_2 of V and integers i_1, i_2. Suppose

(i) $i_1 \neq 0$, and

(ii) $i_2 \neq 0$, and

(iii) $A = [\langle z_1, i_1\rangle]_{\mathrm{EQRZM}\,V}$, and

(iv) $B = [\langle z_2, i_2\rangle]_{\mathrm{EQRZM}\,V}$.

Then $it(A, B) = [\langle i_2 \cdot z_1 + i_1 \cdot z_2, i_1 \cdot i_2\rangle]_{\mathrm{EQRZM}\,V}$.

Assume V is cancelable on multiplication. The functor zeroCoset V yielding an element of Classes EQRZM V is defined by

(Def. 3) Let us consider an integer i. Suppose $i \neq 0$. Then $it = [\langle 0_V, i\rangle]_{\mathrm{EQRZM}\,V}$.

Assume V is cancelable on multiplication. The functor lmultCoset V yielding a function from (the carrier of FRat) $\times$ Classes EQRZM V into Classes EQRZM V is defined by

(Def. 4) Let us consider an element q and an element A. Suppose

(i) $q \in \mathbb{Q}$, and

(ii) $A \in$ Classes EQRZM V.

Let us consider integers m, n, i and an element z of V. Suppose

(iii) $n \neq 0$, and

(iv) $q = \frac{m}{n}$, and

(v) $i \neq 0$, and

(vi) $A = [\langle z, i\rangle]_{\mathrm{EQRZM}\,V}$.

Then $it(q, A) = [\langle m \cdot z, n \cdot i\rangle]_{\mathrm{EQRZM}\,V}$.

Now we state the propositions:

(4) Let us consider a $\mathbb{Z}$-module V, an element z of V, and integers i, n. Suppose

(i) $i \neq 0$, and

(ii) $n \neq 0$, and

(iii) V is cancelable on multiplication.

Then $[\langle z, i\rangle]_{\mathrm{EQRZM}\,V} = [\langle n \cdot z, n \cdot i\rangle]_{\mathrm{EQRZM}\,V}$. The theorem is a consequence of (3).

(5) Let us consider a $\mathbb{Z}$-module V and an element v of $\langle$Classes EQRZM V, addCoset V, zeroCoset V, lmultCo Suppose V is cancelable on multiplication. Then there exists an integer i and there exists an element z of V such that $i \neq 0$ and $v = [\langle z, i\rangle]_{\mathrm{EQRZM}\,V}$.

Let V be a $\mathbb{Z}$-module. Assume V is cancelable on multiplication. The functor ZMQVectSp V yielding a vector space over FRat is defined by the term

(Def. 5) $\langle$Classes EQRZM V, addCoset V, zeroCoset V, lmultCoset $V\rangle$.

Assume V is cancelable on multiplication. The functor MorphsZQ V yielding a function from V into ZMQVectSp V is defined by

(Def. 6) (i) *it* is one-to-one, and

(ii) for every element v of V, $it(v) = [\langle v, 1\rangle]_{\mathrm{EQRZM}\,V}$, and

(iii) for every elements v, w of V, $it(v + w) = it(v) + it(w)$, and

(iv) for every element v of V and for every integer i and for every element q of FRat such that $i = q$ holds $it(i \cdot v) = q \cdot it(v)$, and

(v) $it(0_V) = 0_{\mathrm{ZMQVectSp}\,V}$.

Now we state the propositions:

(6) Let us consider a $\mathbb{Z}$-module V. Suppose V is cancelable on multiplication. Let us consider a finite sequence s of elements of V and a finite sequence t of elements of ZMQVectSp V. Suppose

(i) $\mathrm{len}\, s = \mathrm{len}\, t$, and

(ii) for every element i of $\mathbb{N}$ such that $i \in \mathrm{dom}\, s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $t(i) = (\mathrm{MorphsZQ}\,V)(s_1)$.

Then $\sum t = (\mathrm{MorphsZQ}\,V)(\sum s)$. Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence s of elements of V for every finite sequence t of elements of ZMQVectSp V such that $\mathrm{len}\, s = \$_1$ and $\mathrm{len}\, s = \mathrm{len}\, t$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{dom}\, s$ there exists a vector s_1 of V such that $s_1 = s(i)$ and $t(i) = (\mathrm{MorphsZQ}\,V)(s_1)$ holds $\sum t = (\mathrm{MorphsZQ}\,V)(\sum s)$. $\mathcal{P}[0]$ by [27, (43)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [5, (59)], [3, (11)], [5, (4)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. $\square$

(7) Let us consider a $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of ZMQVectSp V, a z linear combination l of I, and a linear combination l_5 of I_6. Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\mathrm{MorphsZQ}\,V)^\circ I$, and

(iii) $l = l_5 \cdot \mathrm{MorphsZQ}\,V$.

Then $\sum l_5 = (\mathrm{MorphsZQ}\,V)(\sum l)$. The theorem is a consequence of (6).

(8) Let us consider a $\mathbb{Z}$-module V, a subset I_6 of ZMQVectSp V, and a linear combination l_5 of I_6. Then there exists an integer m and there exists an element a of FRat such that $m \neq 0$ and $m = a$ and $\mathrm{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every linear combination l_5 of I_6 such that $\overline{\overline{\text{the support of } l_5}} = \$_1$ there exists an integer m and there exists an element a of FRat such that $m \neq 0$ and $m = a$ and $\mathrm{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. $\mathcal{P}[0]$ by [28, (28)], [8, (113)], [28, (3)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [2, (44)], [10, (31)], [2, (42)]. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. □

(9) Let us consider a $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of ZMQVectSp V, and a linear combination l_5 of I_6. Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\mathrm{MorphsZQ}\,V)^\circ I$.

Then there exists an integer m and there exists an element a of FRat and there exists a z linear combination l of I such that $m \neq 0$ and $m = a$ and $l = (a \cdot l_5) \cdot \mathrm{MorphsZQ}\,V$ and $(\mathrm{MorphsZQ}\,V)^{-1}(\text{the support of } l_5) =$ the support of l. The theorem is a consequence of (8). Proof: Consider m being an integer, a being an element of FRat such that $m \neq 0$ and $m = a$ and $\mathrm{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. Reconsider $l = (a \cdot l_5) \cdot \mathrm{MorphsZQ}\,V$ as an element of $\mathbb{Z}^{\text{the carrier of } V}$. Set $T = \{v, \text{ where } v \text{ is an element of } V : l(v) \neq 0\}$. Set $F = \mathrm{MorphsZQ}\,V$. $T \subseteq F^{-1}(\text{the support of } l_5)$ by [7, (13)], [8, (38)]. $F^{-1}(\text{the support of } l_5) \subseteq T$ by [8, (38)], [7, (13)]. □

(10) Let us consider a $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of ZMQVectSp V, a linear combination l_5 of I_6, an integer m, an element a of FRat, and a z linear combination l of I. Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\mathrm{MorphsZQ}\,V)^\circ I$, and

(iii) $m \neq 0$, and

(iv) $m = a$, and

(v) $l = (a \cdot l_5) \cdot \mathrm{MorphsZQ}\,V$.

Then $a \cdot \sum l_5 = (\mathrm{MorphsZQ}\,V)(\sum l)$. The theorem is a consequence of (7).

(11) Let us consider a $\mathbb{Z}$-module V, a subset I of V, and a subset I_6 of ZMQVectSp V. Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\text{MorphsZQ}\, V)^\circ I$, and

(iii) I is linearly independent.

Then I_6 is linearly independent. The theorem is a consequence of (9) and (10).

(12) Let us consider a $\mathbb{Z}$-module V, a subset I of V, a z linear combination l of I, and a subset I_6 of ZMQVectSp V. Suppose

(i) V is cancelable on multiplication, and

(ii) $I_6 = (\text{MorphsZQ}\, V)^\circ I$.

Then there exists a linear combination l_5 of I_6 such that

(iii) $l = l_5 \cdot \text{MorphsZQ}\, V$, and

(iv) the support of $l_5 = (\text{MorphsZQ}\, V)^\circ$the support of l.

PROOF: Reconsider $I_0 =$ the support of l as a finite subset of V. Reconsider $I_7 = (\text{MorphsZQ}\, V)^\circ I_0$ as a finite subset of ZMQVectSp V. Define $\mathcal{P}[\text{element}, \text{element}] \equiv \$_1 \in I_7$ and there exists an element v of V such that $v \in I_0$ and $\$_1 = (\text{MorphsZQ}\, V)(v)$ and $\$_2 = l(v)$ or $\$_1 \notin I_7$ and $\$_2 = 0_{\text{FRat}}$. For every element x such that $x \in$ the carrier of ZMQVectSp V there exists an element y such that $y \in \mathbb{Q}$ and $\mathcal{P}[x, y]$ by [8, (64)], [25, (14)]. Consider l_5 being a function from the carrier of ZMQVectSp V into $\mathbb{Q}$ such that for every element x such that $x \in$ the carrier of ZMQVectSp V holds $\mathcal{P}[x, l_5(x)]$ from [8, Sch. 1]. The support of $l_5 \subseteq I_7$. For every element x such that $x \in \text{dom}\, l$ holds $l(x) = (l_5 \cdot \text{MorphsZQ}\, V)(x)$ by [8, (35), (19)], [7, (12)]. $I_7 \subseteq$ the support of l_5 by [8, (64)], [7, (12)], [14, (8)]. □

(13) Let us consider a free $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of ZMQVectSp V, a z linear combination l of I, and an integer i. Suppose

(i) $i \neq 0$, and

(ii) $I_6 = (\text{MorphsZQ}\, V)^\circ I$.

Then $[\langle \sum l, i \rangle]_{\text{EQRZM}\, V} \in \text{Lin}(I_6)$. The theorem is a consequence of (12) and (7).

Let us consider a free $\mathbb{Z}$-module V, a subset I of V, and a subset I_6 of ZMQVectSp V. Now we state the propositions:

(14) If $I_6 = (\text{MorphsZQ}\, V)^\circ I$, then $\overline{\overline{I}} = \overline{\overline{I_6}}$.

(15) If $I_6 = (\text{MorphsZQ}\, V)^\circ I$ and I is a basis of V, then I_6 is a basis of ZMQVectSp V.

Let V be a finite-rank free $\mathbb{Z}$-module. Note that ZMQVectSp V is finite dimensional.

Now we state the propositions:

(16) Let us consider a finite-rank free $\mathbb{Z}$-module V. Then $\text{rank}\, V = \dim(\text{ZMQVectSp}\, V)$. The theorem is a consequence of (15) and (14).

(17) Let us consider a free $\mathbb{Z}$-module V and finite subsets I, A of V. Suppose

(i) I is a basis of V, and

(ii) $\overline{\overline{I}} + 1 = \overline{\overline{A}}$.

Then A is linearly dependent. The theorem is a consequence of (15), (11), and (14).

(18) Let us consider a free $\mathbb{Z}$-module V and subsets A, B of V. If A is linearly dependent and $A \subseteq B$, then B is linearly dependent.

(19) Let us consider a free $\mathbb{Z}$-module V and subsets D, A of V. Suppose

(i) D is basis of V and finite, and

(ii) $\overline{\overline{D}} \subset \overline{\overline{A}}$.

Then A is linearly dependent. The theorem is a consequence of (17) and (18).

(20) Let us consider a free $\mathbb{Z}$-module V and subsets I, A of V. Suppose

(i) I is basis of V and finite, and

(ii) A is linearly independent.

Then $\overline{\overline{A}} \subseteq \overline{\overline{I}}$.

2. Submodule of Free $\mathbb{Z}$-module

Now we state the proposition:

(21) Let us consider a $\mathbb{Z}$-module V. If Ω_V is free, then V is free.

Let us consider a $\mathbb{Z}$-module V, submodules W_1, W_2 of V, and strict submodules W_3, W_4 of V. Now we state the propositions:

(22) If $W_3 = \Omega_{W_1}$ and $W_4 = \Omega_{W_2}$, then $W_3 + W_4 = W_1 + W_2$.

(23) If $W_3 = \Omega_{W_1}$ and $W_4 = \Omega_{W_2}$, then $W_3 \cap W_4 = W_1 \cap W_2$.

Now we state the propositions:

(24) Let us consider a $\mathbb{Z}$-module V and a strict submodule W of V. Suppose $W \neq \mathbf{0}_V$. Then there exists a vector v of V such that

(i) $v \in W$, and

(ii) $v \neq 0_V$.

(25) Let us consider a subset A of V and z linear combinations l_1, l_2 of A. Suppose (the support of l_1) $\cap$ (the support of l_2) $= \emptyset$. Then the support of $l_1 + l_2 =$ (the support of l_1) $\cup$ (the support of l_2). Proof: (The support of l_1) $\cup$ (the support of l_2) $\subseteq$ the support of $l_1 + l_2$ by [14, (8)]. $\square$

(26) Let us consider subsets A_1, A_2 of V and a z linear combination l of $A_1 \cup A_2$. Suppose $A_1 \cap A_2 = \emptyset$. Then there exists a z linear combination l_1 of A_1 and there exists a z linear combination l_2 of A_2 such that $l = l_1 + l_2$.
PROOF: Define $\mathcal{P}[\text{element}, \text{element}] \equiv$ if $\$_1$ is a vector of V, then $\$_1 \in A_1$ and $\$_2 = l(\$_1)$ or $\$_1 \notin A_1$ and $\$_2 = 0$. For every element x such that $x \in$ the carrier of V there exists an element y such that $y \in \mathbb{Z}$ and $\mathcal{P}[x, y]$. There exists a function l_1 from the carrier of V into $\mathbb{Z}$ such that for every element x such that $x \in$ the carrier of V holds $\mathcal{P}[x, l_1(x)]$ from [8, Sch. 1]. Consider l_1 being a function from the carrier of V into $\mathbb{Z}$ such that for every element x such that $x \in$ the carrier of V holds $\mathcal{P}[x, l_1(x)]$. For every element x such that $x \in$ the support of l_1 holds $x \in A_1$ by [14, (8)]. Define $\mathcal{Q}[\text{element}, \text{element}] \equiv$ if $\$_1$ is a vector of V, then $\$_1 \in A_2$ and $\$_2 = l(\$_1)$ or $\$_1 \notin A_2$ and $\$_2 = 0$. For every element x such that $x \in$ the carrier of V there exists an element y such that $y \in \mathbb{Z}$ and $\mathcal{Q}[x, y]$. There exists a function l_2 from the carrier of V into $\mathbb{Z}$ such that for every element x such that $x \in$ the carrier of V holds $\mathcal{Q}[x, l_2(x)]$ from [8, Sch. 1]. Consider l_2 being a function from the carrier of V into $\mathbb{Z}$ such that for every element x such that $x \in$ the carrier of V holds $\mathcal{Q}[x, l_2(x)]$. For every element x such that $x \in$ the support of l_2 holds $x \in A_2$ by [14, (8)]. For every vector v of V, $l(v) = (l_1 + l_2)(v)$. $\square$

(27) Let us consider a $\mathbb{Z}$-module V, free submodules W_1, W_2 of V, a basis I_1 of W_1, and a basis I_2 of W_2. If V is the direct sum of W_1 and W_2, then $I_1 \cap I_2 = \emptyset$.

Let us consider a $\mathbb{Z}$-module V, free submodules W_1, W_2 of V, a basis I_1 of W_1, a basis I_2 of W_2, and a subset I of V. Now we state the propositions:

(28) If V is the direct sum of W_1 and W_2 and $I = I_1 \cup I_2$, then $\text{Lin}(I) =$ the $\mathbb{Z}$-module structure of V.

(29) If V is the direct sum of W_1 and W_2 and $I = I_1 \cup I_2$, then I is linearly independent.

Let us consider a $\mathbb{Z}$-module V and free submodules W_1, W_2 of V. Now we state the propositions:

(30) If V is the direct sum of W_1 and W_2, then V is free.

(31) If $W_1 \cap W_2 = \mathbf{0}_V$, then $W_1 + W_2$ is free.

Let us consider a free $\mathbb{Z}$-module V, a basis I of V, and a vector v of V. Now we state the propositions:

(32) If $v \in I$, then $\text{Lin}(I \setminus \{v\})$ is free and $\text{Lin}(\{v\})$ is free.

(33) If $v \in I$, then V is the direct sum of $\text{Lin}(I \setminus \{v\})$ and $\text{Lin}(\{v\})$.

Let V be a finite-rank free $\mathbb{Z}$-module. One can verify that every submodule of V is free.

Now we state the propositions:

(34) Let us consider a $\mathbb{Z}$-module V, a submodule W of V, and free submodules W_1, W_2 of V. Suppose

(i) $W_1 \cap W_2 = \mathbf{0}_V$, and

(ii) the $\mathbb{Z}$-module structure of $W = W_1 + W_2$.

Then W is free. The theorem is a consequence of (31).

(35) Let us consider a prime number p and a free $\mathbb{Z}$-module V. If $\mathrm{Z_M Q_V ectSp}(V, p)$ is finite dimensional, then V is finite-rank.

(36) Let us consider a prime number p, a $\mathbb{Z}$-module V, an element s of V, an integer a, and an element b of $\mathrm{GF}(p)$. Suppose $b = a \bmod p$. Then $b \cdot \mathrm{ZMtoMQV}(V, p, s) = \mathrm{ZMtoMQV}(V, p, a \cdot s)$.

(37) Let us consider a prime number p, a free $\mathbb{Z}$-module V, a subset I of V, a subset I_6 of $\mathrm{Z_M Q_V ectSp}(V, p)$, and a z linear combination l of I. Suppose $I_6 = \{\mathrm{ZMtoMQV}(V, p, u)$, where u is a vector of $V : u \in I\}$. Then $\mathrm{ZMtoMQV}(V, p, \sum l) \in \mathrm{Lin}(I_6)$.

(38) Let us consider a prime number p, a free $\mathbb{Z}$-module V, a subset I of V, and a subset I_6 of $\mathrm{Z_M Q_V ectSp}(V, p)$. Suppose

(i) $\mathrm{Lin}(I) =$ the $\mathbb{Z}$-module structure of V, and

(ii) $I_6 = \{\mathrm{ZMtoMQV}(V, p, u)$, where u is a vector of $V : u \in I\}$.

Then $\mathrm{Lin}(I_6) =$ the vector space structure of $\mathrm{Z_M Q_V ectSp}(V, p)$. The theorem is a consequence of (37). PROOF: For every element v_3 of $\mathrm{Z_M Q_V ectSp}(V, p)$, $v_3 \in \mathrm{Lin}(I_6)$ by [15, (22)], [14, (64)]. □

(39) Let us consider a finitely-generated free $\mathbb{Z}$-module V. Then there exists a finite subset A of V such that A is a basis of V. The theorem is a consequence of (38). PROOF: Set $p =$ the prime number. Consider B being a finite subset of V such that $\mathrm{Lin}(B) =$ the $\mathbb{Z}$-module structure of V. Set $B_1 = \{\mathrm{ZMtoMQV}(V, p, u)$, where u is a vector of $V : u \in B\}$. Define $\mathcal{F}(\text{element of } V) = \mathrm{ZMtoMQV}(V, p, \$_1)$. Consider f being a function from the carrier of V into $\mathrm{Z_M Q_V ectSp}(V, p)$ such that for every element x of V, $f(x) = \mathcal{F}(x)$ from [8, Sch. 4]. For every element y such that $y \in B_1$ there exists an element x such that $x \in \mathrm{dom}(f{\restriction}B)$ and $y = (f{\restriction}B)(x)$ by [31, (62)], [7, (47)]. Consider I_6 being a basis of $\mathrm{Z_M Q_V ectSp}(V, p)$ such that $I_6 \subseteq B_1$. □

One can verify that every finitely-generated free $\mathbb{Z}$-module is finite-rank and every finite-rank free $\mathbb{Z}$-module is finitely-generated.

Now we state the proposition:

(40) Let us consider a finite-rank free $\mathbb{Z}$-module V and a subset A of V. If A is linearly independent, then A is finite. The theorem is a consequence of (19).

Let V be a $\mathbb{Z}$-module and W_1, W_2 be finite-rank free submodules of V. One can check that $W_1 \cap W_2$ is free.

Note that $W_1 \cap W_2$ is finite-rank.

Let V be a finite-rank free $\mathbb{Z}$-module. Note that every submodule of V is finite-rank.

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