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The Jordan's Property for Certain Subsets of the Plane

Yatsuka Nakamura
Shinshu University
Nagano

Jarosław Kotowicz[1]
Warsaw University
Białystok

Summary. Let S be a subset of the topological Euclidean plane $\mathcal{E}^2_{\rm T}$. We say that S has Jordan's property if there exist two non-empty, disjoint and connected subsets G_1 and G_2 of $\mathcal{E}^2_{\rm T}$ such that $S^{\rm c} = G_1 \cup G_2$ and $\overline{G_1} \setminus G_1 = \overline{G_2} \setminus G_2$ (see [19], [10]). The aim is to prove that the boundaries of some special polygons in $\mathcal{E}^2_{\rm T}$ have this property (see Section 3). Moreover, it is proved that both the interior and the exterior of the boundary of any rectangle in $\mathcal{E}^2_{\rm T}$ is open and connected.

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The articles [22], [24], [11], [17], [1], [4], [5], [20], [3], [16], [7], [15], [23], [18], [12], [2], [21], [14], [13], [8], [6], and [9] provide the notation and terminology for this paper.

1. Selected theorems on connected spaces

In the sequel G_1, G_2 are topological spaces and A is a subset of G_1. The following propositions are true:

(1) If $A \neq \emptyset$, then the carrier of $G_1 \restriction A = A$.

(2) For every topological space G_1 if for every points x, y of G_1 there exists G_2 such that G_2 is connected and there exists a map f from G_2 into G_1 such that f is continuous and $x \in \operatorname{rng} f$ and $y \in \operatorname{rng} f$, then G_1 is connected.

The following propositions are true:

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(3) For every topological space G_1 if for all points x, y of G_1 such that $x \neq y$ there exists a map h from $\mathbb{I}$ into G_1 such that h is continuous and $x = h(0)$ and $y = h(1)$, then G_1 is connected.

(4) Let A be a subset of G_1. Then if $A \neq \emptyset_{G_1}$ and for all points x_1, y_1 of G_1 such that $x_1 \in A$ and $y_1 \in A$ and $x_1 \neq y_1$ there exists a map h from $\mathbb{I}$ into $G_1 \restriction A$ such that h is continuous and $x_1 = h(0)$ and $y_1 = h(1)$, then A is connected.

(5) For every G_1 and for every subset A_0 of G_1 and for every subset A_1 of G_1 such that A_0 is connected and A_1 is connected and $A_0 \cap A_1 \neq \emptyset$ holds $A_0 \cup A_1$ is connected.

(6) For every G_1 and for all subsets A_0, A_1, A_2 of G_1 such that A_0 is connected and A_1 is connected and A_2 is connected and $A_0 \cap A_1 \neq \emptyset$ and $A_1 \cap A_2 \neq \emptyset$ holds $A_0 \cup A_1 \cup A_2$ is connected.

(7) For every G_1 and for all subsets A_0, A_1, A_2, A_3 of G_1 such that A_0 is connected and A_1 is connected and A_2 is connected and A_3 is connected and $A_0 \cap A_1 \neq \emptyset$ and $A_1 \cap A_2 \neq \emptyset$ and $A_2 \cap A_3 \neq \emptyset$ holds $A_0 \cup A_1 \cup A_2 \cup A_3$ is connected.

2. Certain connected and open subsets in the Euclidean plane

We follow a convention: P, Q, P_1, P_2 denote subsets of $\mathcal{E}^2_{\mathrm{T}}$ and w_1, w_2 denote points of $\mathcal{E}^2_{\mathrm{T}}$. One can prove the following proposition

(8) For every P such that $P \neq \emptyset_{\mathcal{E}^2_{\mathrm{T}}}$ and for all w_1, w_2 such that $w_1 \in P$ and $w_2 \in P$ and $w_1 \neq w_2$ holds $\mathcal{L}(w_1, w_2) \subseteq P$ holds P is connected.

We adopt the following rules: p_1, p_2 will be points of $\mathcal{E}^2_{\mathrm{T}}$ and s_1, t_1, s_2, t_2, s, t, s_3, t_3, s_4, t_4, s_5, t_5, s_6, t_6, l, s_7, t_7 will be real numbers. Next we state two propositions:

(9) If $s_1 < s_3$ and $s_1 < s_4$ and $0 \leq l$ and $l \leq 1$, then $s_1 < (1-l) \cdot s_3 + l \cdot s_4$.

(10) If $s_3 < s_1$ and $s_4 < s_1$ and $0 \leq l$ and $l \leq 1$, then $(1-l) \cdot s_3 + l \cdot s_4 < s_1$.

In the sequel s_8, t_8 denote real numbers. The following propositions are true:

(11) $\{[s,t] : s_1 < s \wedge s < s_2 \wedge t_1 < t \wedge t < t_2\} = \{[s_3,t_3] : s_1 < s_3\} \cap \{[s_4, t_4] : s_4 < s_2\} \cap \{[s_5,t_5] : t_1 < t_5\} \cap \{[s_6,t_6] : t_6 < t_2\}$.

(12) $\{[s,t] : \neg(s_1 \leq s \wedge s \leq s_2 \wedge t_1 \leq t \wedge t \leq t_2)\} = \{[s_3,t_3] : s_3 < s_1\} \cup \{[s_4, t_4] : t_4 < t_1\} \cup \{[s_5,t_5] : s_2 < s_5\} \cup \{[s_6,t_6] : t_2 < t_6\}$.

(13) For all s_1, t_1, s_2, t_2, P such that $s_1 < s_2$ and $t_1 < t_2$ and $P = \{[s, t] : s_1 < s \wedge s < s_2 \wedge t_1 < t \wedge t < t_2\}$ holds P is connected.

(14) For all s_1, P such that $P = \{[s,t] : s_1 < s\}$ holds P is connected.

(15) For all s_2, P such that $P = \{[s,t] : s < s_2\}$ holds P is connected.

(16) For all t_1, P such that $P = \{[s,t] : t_1 < t\}$ holds P is connected.

(17) For all t_2, P such that $P = \{[s,t] : t < t_2\}$ holds P is connected.

(18) For all s_1, t_1, s_2, t_2, P such that $P = \{[s,t] : \neg(s_1 \leq s \wedge s \leq s_2 \wedge t_1 \leq t \wedge t \leq t_2)\}$ holds P is connected.

(19) For all s_1, P such that $P = \{[s,t] : s_1 < s\}$ holds P is open.

(20) For all s_1, P such that $P = \{[s,t] : s_1 > s\}$ holds P is open.

(21) For all s_1, P such that $P = \{[s,t] : s_1 < t\}$ holds P is open.

(22) For all s_1, P such that $P = \{[s,t] : s_1 > t\}$ holds P is open.

(23) For all s_1, t_1, s_2, t_2, P such that $P = \{[s,t] : s_1 < s \wedge s < s_2 \wedge t_1 < t \wedge t < t_2\}$ holds P is open.

(24) For all s_1, t_1, s_2, t_2, P such that $P = \{[s,t] : \neg(s_1 \leq s \wedge s \leq s_2 \wedge t_1 \leq t \wedge t \leq t_2)\}$ holds P is open.

(25) Given s_1, t_1, s_2, t_2, P, Q. Suppose $P = \{[s_7,t_7] : s_1 < s_7 \wedge s_7 < s_2 \wedge t_1 < t_7 \wedge t_7 < t_2\}$ and $Q = \{[s_8,t_8] : \neg(s_1 \leq s_8 \wedge s_8 \leq s_2 \wedge t_1 \leq t_8 \wedge t_8 \leq t_2)\}$. Then $P \cap Q = \emptyset_{\mathcal{E}^2_{\mathrm{T}}}$.

(26) For all real numbers s_1, s_2, t_1, t_2 holds $\{p : s_1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < s_2 \wedge t_1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < t_2\} = \{[s_7,t_7] : s_1 < s_7 \wedge s_7 < s_2 \wedge t_1 < t_7 \wedge t_7 < t_2\}$, where p ranges over points of $\mathcal{E}^2_{\mathrm{T}}$.

(27) For all s_1, s_2, t_1, t_2 holds $\{q_1 : \neg(s_1 \leq q_{1\mathbf{1}} \wedge q_{1\mathbf{1}} \leq s_2 \wedge t_1 \leq q_{1\mathbf{2}} \wedge q_{1\mathbf{2}} \leq t_2)\} = \{[s_8,t_8] : \neg(s_1 \leq s_8 \wedge s_8 \leq s_2 \wedge t_1 \leq t_8 \wedge t_8 \leq t_2)\}$, where q_1 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$.

(28) For all s_1, s_2, t_1, t_2 holds $\{p_0 : s_1 < p_{0\mathbf{1}} \wedge p_{0\mathbf{1}} < s_2 \wedge t_1 < p_{0\mathbf{2}} \wedge p_{0\mathbf{2}} < t_2\}$, where p_0 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$, is a subset of $\mathcal{E}^2_{\mathrm{T}}$.

(29) For all s_1, s_2, t_1, t_2 holds $\{p_3 : \neg(s_1 \leq p_{3\mathbf{1}} \wedge p_{3\mathbf{1}} \leq s_2 \wedge t_1 \leq p_{3\mathbf{2}} \wedge p_{3\mathbf{2}} \leq t_2)\}$, where p_3 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$, is a subset of $\mathcal{E}^2_{\mathrm{T}}$.

(30) For all s_1, t_1, s_2, t_2, P such that $s_1 < s_2$ and $t_1 < t_2$ and $P = \{p_0 : s_1 < p_{0\mathbf{1}} \wedge p_{0\mathbf{1}} < s_2 \wedge t_1 < p_{0\mathbf{2}} \wedge p_{0\mathbf{2}} < t_2\}$, where p_0 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$ holds P is connected.

(31) For all s_1, t_1, s_2, t_2, P such that $P = \{p_3 : \neg(s_1 \leq p_{3\mathbf{1}} \wedge p_{3\mathbf{1}} \leq s_2 \wedge t_1 \leq p_{3\mathbf{2}} \wedge p_{3\mathbf{2}} \leq t_2)\}$, where p_3 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$ holds P is connected.

(32) For all s_1, t_1, s_2, t_2, P such that $P = \{p_0 : s_1 < p_{0\mathbf{1}} \wedge p_{0\mathbf{1}} < s_2 \wedge t_1 < p_{0\mathbf{2}} \wedge p_{0\mathbf{2}} < t_2\}$, where p_0 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$ holds P is open.

(33) For all s_1, t_1, s_2, t_2, P such that $P = \{p_3 : \neg(s_1 \leq p_{3\mathbf{1}} \wedge p_{3\mathbf{1}} \leq s_2 \wedge t_1 \leq p_{3\mathbf{2}} \wedge p_{3\mathbf{2}} \leq t_2)\}$, where p_3 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$ holds P is open.

(34) Given s_1, t_1, s_2, t_2, P, Q. Suppose $P = \{p : s_1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < s_2 \wedge t_1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < t_2\}$, where p ranges over points of $\mathcal{E}^2_{\mathrm{T}}$ and $Q = \{q_1 : \neg(s_1 \leq q_{1\mathbf{1}} \wedge q_{1\mathbf{1}} \leq s_2 \wedge t_1 \leq q_{1\mathbf{2}} \wedge q_{1\mathbf{2}} \leq t_2)\}$, where q_1 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$. Then $P \cap Q = \emptyset_{\mathcal{E}^2_{\mathrm{T}}}$.

(35) Given s_1, t_1, s_2, t_2, P, P_1, P_2. Suppose that

(i) $s_1 < s_2$,

(ii) $t_1 < t_2$,

(iii) $P = \{p : p_{\mathbf{1}} = s_1 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1 \ \vee\ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_2 \ \vee\ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_1 \ \vee\ p_{\mathbf{1}} = s_2 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1\}$, where p ranges over points of $\mathcal{E}^2_{\mathrm{T}}$,

(iv) $P_1 = \{p_1 : s_1 < p_{1\mathbf{1}} \wedge p_{1\mathbf{1}} < s_2 \wedge t_1 < p_{1\mathbf{2}} \wedge p_{1\mathbf{2}} < t_2\}$, where p_1 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$,

(v) $P_2 = \{p_2 : \neg(s_1 \leq p_{2\mathbf{1}} \wedge p_{2\mathbf{1}} \leq s_2 \wedge t_1 \leq p_{2\mathbf{2}} \wedge p_{2\mathbf{2}} \leq t_2)\}$, where p_2 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$.

Then

(vi) $P^{\mathrm{c}} = P_1 \cup P_2$,

(vii) $P^{\mathrm{c}} \neq \emptyset$,

(viii) $P_1 \cap P_2 = \emptyset$,

(ix) for all subsets P_3, P_4 of $(\mathcal{E}^2_{\mathrm{T}}) \restriction P^{\mathrm{c}}$ such that $P_3 = P_1$ and $P_4 = P_2$ holds P_3 is a component of $(\mathcal{E}^2_{\mathrm{T}}) \restriction P^{\mathrm{c}}$ and P_4 is a component of $(\mathcal{E}^2_{\mathrm{T}}) \restriction P^{\mathrm{c}}$.

(36) Given s_1, t_1, s_2, t_2, P, P_1, P_2. Suppose that

(i) $s_1 < s_2$,

(ii) $t_1 < t_2$,

(iii) $P = \{p : p_{\mathbf{1}} = s_1 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_2 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_1 \ \vee \ p_{\mathbf{1}} = s_2 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1\}$, where p ranges over points of $\mathcal{E}^2_{\mathrm{T}}$,

(iv) $P_1 = \{p_1 : s_1 < p_{1\mathbf{1}} \wedge p_{1\mathbf{1}} < s_2 \wedge t_1 < p_{1\mathbf{2}} \wedge p_{1\mathbf{2}} < t_2\}$, where p_1 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$,

(v) $P_2 = \{p_2 : \neg(s_1 \leq p_{2\mathbf{1}} \wedge p_{2\mathbf{1}} \leq s_2 \wedge t_1 \leq p_{2\mathbf{2}} \wedge p_{2\mathbf{2}} \leq t_2)\}$, where p_2 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$.

Then $P = \overline{P_1} \setminus P_1$ and $P = \overline{P_2} \setminus P_2$.

(37) Given s_1, s_2, t_1, t_2, P, P_1. Suppose that

(i) $s_1 < s_2$,

(ii) $t_1 < t_2$,

(iii) $P = \{p : p_{\mathbf{1}} = s_1 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_2 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_1 \ \vee \ p_{\mathbf{1}} = s_2 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1\}$, where p ranges over points of $\mathcal{E}^2_{\mathrm{T}}$,

(iv) $P_1 = \{p_1 : s_1 < p_{1\mathbf{1}} \wedge p_{1\mathbf{1}} < s_2 \wedge t_1 < p_{1\mathbf{2}} \wedge p_{1\mathbf{2}} < t_2\}$, where p_1 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$.

Then $P_1 \subseteq \Omega_{(\mathcal{E}^2_{\mathrm{T}}) \restriction P^{\mathrm{c}}}$.

(38) Given s_1, s_2, t_1, t_2, P, P_1. Suppose that

(i) $s_1 < s_2$,

(ii) $t_1 < t_2$,

(iii) $P = \{p : p_{\mathbf{1}} = s_1 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_2 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_1 \ \vee \ p_{\mathbf{1}} = s_2 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1\}$, where p ranges over points of $\mathcal{E}^2_{\mathrm{T}}$,

(iv) $P_1 = \{p_1 : s_1 < p_{1\mathbf{1}} \wedge p_{1\mathbf{1}} < s_2 \wedge t_1 < p_{1\mathbf{2}} \wedge p_{1\mathbf{2}} < t_2\}$, where p_1 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$.

Then P_1 is a subset of $(\mathcal{E}^2_{\mathrm{T}}) \restriction P^{\mathrm{c}}$.

(39) Given s_1, s_2, t_1, t_2, P, P_2. Suppose that

(i) $s_1 < s_2$,

(ii) $t_1 < t_2$,

(iii) $P = \{p : p_{\mathbf{1}} = s_1 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_2 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_1 \ \vee \ p_{\mathbf{1}} = s_2 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1\}$, where p ranges over points of $\mathcal{E}^2_{\mathrm{T}}$,

(iv) $P_2 = \{p_2 : \neg(s_1 \leq p_{2\mathbf{1}} \wedge p_{2\mathbf{1}} \leq s_2 \wedge t_1 \leq p_{2\mathbf{2}} \wedge p_{2\mathbf{2}} \leq t_2)\}$, where p_2 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$.

Then $P_2 \subseteq \Omega_{(\mathcal{E}^2_{\mathrm{T}}) \restriction P^{\mathrm{c}}}$.

(40) Given s_1, s_2, t_1, t_2, P, P_2. Suppose that

(i) $s_1 < s_2$,

(ii) $t_1 < t_2$,

(iii) $P = \{p : p_{\mathbf{1}} = s_1 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_2 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_1 \ \vee \ p_{\mathbf{1}} = s_2 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1\}$, where p ranges over points of $\mathcal{E}^2_{\mathrm{T}}$,

(iv) $P_2 = \{p_2 : \neg(s_1 \leq p_{2\mathbf{1}} \wedge p_{2\mathbf{1}} \leq s_2 \wedge t_1 \leq p_{2\mathbf{2}} \wedge p_{2\mathbf{2}} \leq t_2)\}$, where p_2 ranges over points of $\mathcal{E}^2_{\mathrm{T}}$.

Then P_2 is a subset of $(\mathcal{E}^2_{\mathrm{T}}) \restriction P^{\mathrm{c}}$.

3. Jordan's property

In the sequel S, A_1, A_2 will be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Let us consider S. We say that S has Jordan's property if and only if the conditions (Def.1) is satisfied.

(Def.1) (i) $S^{\mathrm{c}} \neq \emptyset$,

(ii) there exist A_1, A_2 such that $S^{\mathrm{c}} = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$ and $\overline{A_1} \setminus A_1 = \overline{A_2} \setminus A_2$ and for all subsets C_1, C_2 of $(\mathcal{E}^2_{\mathrm{T}}) \restriction S^{\mathrm{c}}$ such that $C_1 = A_1$ and $C_2 = A_2$ holds C_1 is a component of $(\mathcal{E}^2_{\mathrm{T}}) \restriction S^{\mathrm{c}}$ and C_2 is a component of $(\mathcal{E}^2_{\mathrm{T}}) \restriction S^{\mathrm{c}}$.

The following propositions are true:

(41) Suppose S has Jordan's property. Then

(i) $S^{\mathrm{c}} \neq \emptyset$,

(ii) there exist subsets A_1, A_2 of $\mathcal{E}^2_{\mathrm{T}}$ and there exist subsets C_1, C_2 of $(\mathcal{E}^2_{\mathrm{T}}) \restriction S^{\mathrm{c}}$ such that $S^{\mathrm{c}} = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$ and $\overline{A_1} \setminus A_1 = \overline{A_2} \setminus A_2$ and $C_1 = A_1$ and $C_2 = A_2$ and C_1 is a component of $(\mathcal{E}^2_{\mathrm{T}}) \restriction S^{\mathrm{c}}$ and C_2 is a component of $(\mathcal{E}^2_{\mathrm{T}}) \restriction S^{\mathrm{c}}$ and for every subset C_3 of $(\mathcal{E}^2_{\mathrm{T}}) \restriction S^{\mathrm{c}}$ such that C_3 is a component of $(\mathcal{E}^2_{\mathrm{T}}) \restriction S^{\mathrm{c}}$ holds $C_3 = C_1$ or $C_3 = C_2$.

(42) Given s_1, s_2, t_1, t_2, P. Suppose that

(i) $s_1 < s_2$,

(ii) $t_1 < t_2$,

(iii) $P = \{p : p_{\mathbf{1}} = s_1 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_2 \ \vee \ p_{\mathbf{1}} \leq s_2 \wedge p_{\mathbf{1}} \geq s_1 \wedge p_{\mathbf{2}} = t_1 \ \vee \ p_{\mathbf{1}} = s_2 \wedge p_{\mathbf{2}} \leq t_2 \wedge p_{\mathbf{2}} \geq t_1\}$, where p ranges over points of $\mathcal{E}^2_{\mathrm{T}}$.

Then P has Jordan's property.

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The Lattice of Domains of an Extremally Disconnected Space [1]

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X be a topological space and let A be a subset of X. Recall that A is said to be a *domain* in X provided $\operatorname{Int}\overline{A} \subseteq A \subseteq \overline{\operatorname{Int} A}$ (see [24], [11]). Recall also that A is said to be a(n) *closed* (*open*) *domain* in X if $A = \overline{\operatorname{Int} A}$ ($A = \operatorname{Int}\overline{A}$, resp.) (see e.g. [14], [24]). It is well-known that for a given topological space all its closed domains form a Boolean lattice, and similarly all its open domains form a Boolean lattice, too (see e.g., [15], [3]). In [23] it is proved that all domains of a given topological space form a complemented lattice. One may ask whether the lattice of all domains is Boolean. The aim is to give an answer to this question.

To present the main results we first recall the definition of a class of topological spaces which is important here. X is called *extremally disconnected* if for every open subset A of X the closure $\overline{A}$ is open in X [18] (comp. [10]). It is shown here, using Mizar System, that *the lattice of all domains of a topological space X is modular iff X is extremally disconnected.* Moreover, for every extremally disconnected space the lattice of all its domains coincides with both the lattice of all its closed domains and the lattice of all its open domains. From these facts it follows that *the lattice of all domains of a topological space X is Boolean iff X is extremally disconnected.*

Note that we also review some of the standard facts on discrete, anti-discrete, almost discrete, extremally disconnected and hereditarily extremally disconnected topological spaces (comp. [14], [10]).

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The notation and terminology used in this paper are introduced in the following articles: [20], [22], [21], [16], [6], [7], [17], [24], [9], [4], [19], [12], [5], [25], [8], [2], [1], [23], and [13].

[1] *Editor's Note:* This work has won the 1992 Śleszyński's Award of the Mizar Society.

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1. Selected Properties of Subsets of a Topological Space

In the sequel X will be a topological space. We now state the proposition

(1) For every set B and for every subset A of X such that $B \subseteq A$ holds B is a subset of X.

In the sequel C denotes a subset of X. We now state three propositions:

(2) $\overline{C} = (\operatorname{Int}(C^{\mathrm{c}}))^{\mathrm{c}}$.

(3) $\overline{C^{\mathrm{c}}} = (\operatorname{Int} C)^{\mathrm{c}}$.

(4) $\operatorname{Int}(C^{\mathrm{c}}) = \overline{C}^{\mathrm{c}}$.

In the sequel A, B denote subsets of X. Next we state several propositions:

(5) If $A \cap B = \emptyset$, then if A is open, then $A \cap \overline{B} = \emptyset$ and also if B is open, then $\overline{A} \cap B = \emptyset$.

(6) If $A \cup B =$ the carrier of X, then if A is closed, then $A \cup \operatorname{Int} B =$ the carrier of X and also if B is closed, then $\operatorname{Int} A \cup B =$ the carrier of X.

(7) A is open and A is closed if and only if $\overline{A} = \operatorname{Int} A$.

(8) If A is open and A is closed, then $\operatorname{Int} \overline{A} = \overline{\operatorname{Int} A}$.

(9) If A is a domain and $\overline{\operatorname{Int} A} \subseteq \operatorname{Int} \overline{A}$, then A is an open domain and A is a closed domain.

(10) If A is a domain and $\overline{\operatorname{Int} A} \subseteq \operatorname{Int} \overline{A}$, then A is open and A is closed.

(11) If A is a domain, then $\operatorname{Int} \overline{A} = \operatorname{Int} A$ and $\overline{A} = \overline{\operatorname{Int} A}$.

2. Discrete Topological Structures

We now define two new attributes. A topological structure is discrete if:

(Def.1) the topology of it $= 2^{\text{the carrier of it}}$.

A topological structure is anti-discrete if:

(Def.2) the topology of it $= \{\emptyset, \text{the carrier of it}\}$.

Next we state two propositions:

(12) For every Y being a topological structure such that Y is discrete and Y is anti-discrete holds $2^{\text{the carrier of } Y} = \{\emptyset, \text{the carrier of } Y\}$.

(13) For every Y being a topological structure such that $\emptyset \in$ the topology of Y and the carrier of $Y \in$ the topology of Y holds if $2^{\text{the carrier of } Y} = \{\emptyset, \text{the carrier of } Y\}$, then Y is discrete and Y is anti-discrete.

Let us mention that there exists a topological structure which is discrete anti-discrete and strict.

Next we state two propositions:

(14) For every Y being a discrete topological structure and for every subset A of the carrier of Y holds (the carrier of Y) $\setminus A \in$ the topology of Y.

(15) For every Y being an anti-discrete topological structure and for every subset A of the carrier of Y such that $A \in$ the topology of Y holds (the carrier of Y) $\setminus A \in$ the topology of Y.

Let us observe that every topological structure which is discrete is also topological space-like and every anti-discrete topological structure is topological space-like.

One can prove the following proposition

(16) For every Y being a topological space-like topological structure such that $2^{\text{the carrier of } Y} = \{\emptyset, \text{the carrier of } Y\}$ holds Y is discrete and Y is anti-discrete.

A topological structure is almost discrete if:

(Def.3) for every subset A of the carrier of it such that $A \in$ the topology of it holds (the carrier of it) $\setminus A \in$ the topology of it.

One can verify the following observations:

* every topological structure which is discrete is also almost discrete,
* every topological structure which is anti-discrete is also almost discrete, and
* there exists an almost discrete strict topological structure.

3. Discrete Topological Spaces

Let us mention that there exists a discrete anti-discrete strict topological space.

In the sequel X denotes a topological space. Next we state three propositions:

(17) X is discrete if and only if every subset of X is open.

(18) X is discrete if and only if every subset of X is closed.

(19) If for every subset A of X and for every point x of X such that $A = \{x\}$ holds A is open, then X is discrete.

Let X be a discrete topological space. Note that every subspace of X is open closed and discrete.

Let X be a discrete topological space. Observe that there exists a discrete strict subspace of X.

Next we state three propositions:

(20) X is anti-discrete if and only if for every subset A of X such that A is open holds $A = \emptyset$ or $A =$ the carrier of X.

(21) X is anti-discrete if and only if for every subset A of X such that A is closed holds $A = \emptyset$ or $A =$ the carrier of X.

(22) If for every subset A of X and for every point x of X such that $A = \{x\}$ holds $\overline{A} =$ the carrier of X, then X is anti-discrete.

Let X be an anti-discrete topological space. Observe that every subspace of X is anti-discrete.

Let X be an anti-discrete topological space. Note that there exists an anti-discrete subspace of X.

One can prove the following propositions:

(23) X is almost discrete if and only if for every subset A of X such that A is open holds A is closed.

(24) X is almost discrete if and only if for every subset A of X such that A is closed holds A is open.

(25) X is almost discrete if and only if for every subset A of X such that A is open holds $\overline{A} = A$.

(26) X is almost discrete if and only if for every subset A of X such that A is closed holds $\operatorname{Int} A = A$.

Let us observe that there exists an almost discrete strict topological space.

One can prove the following two propositions:

(27) If for every subset A of X and for every point x of X such that $A = \{x\}$ holds $\overline{A}$ is open, then X is almost discrete.

(28) X is discrete if and only if X is almost discrete and for every subset A of X and for every point x of X such that $A = \{x\}$ holds A is closed.

Let us observe that every discrete topological space is almost discrete and every anti-discrete topological space is almost discrete.

Let X be an almost discrete topological space. Observe that every subspace of X is almost discrete.

Let X be an almost discrete topological space. One can verify that every open subspace of X is closed and every closed subspace of X is open.

Let X be an almost discrete topological space. Note that there exists a subspace of X which is almost discrete and strict.

4. Extremally Disconnected Topological Spaces

A topological space is extremally disconnected if:

(Def.4) for every subset A of it such that A is open holds $\overline{A}$ is open.

Let us note that there exists a topological space which is extremally disconnected and strict.

In the sequel X denotes a topological space. The following propositions are true:

(29) X is extremally disconnected if and only if for every subset A of X such that A is closed holds $\operatorname{Int} A$ is closed.

(30) X is extremally disconnected if and only if for all subsets A, B of X such that A is open and B is open holds if $A \cap B = \emptyset$, then $\overline{A} \cap \overline{B} = \emptyset$.

(31) X is extremally disconnected if and only if for all subsets A, B of X such that A is closed and B is closed holds if $A \cup B =$ the carrier of X, then $\operatorname{Int} A \cup \operatorname{Int} B =$ the carrier of X.

(32) X is extremally disconnected if and only if for every subset A of X such that A is open holds $\overline{A} = \operatorname{Int} \overline{A}$.

(33) X is extremally disconnected if and only if for every subset A of X such that A is closed holds $\operatorname{Int} A = \overline{\operatorname{Int} A}$.

(34) X is extremally disconnected if and only if for every subset A of X such that A is a domain holds A is closed and A is open.

(35) X is extremally disconnected if and only if for every subset A of X such that A is a domain holds A is a closed domain and A is an open domain.

(36) X is extremally disconnected if and only if for every subset A of X such that A is a domain holds $\operatorname{Int} \overline{A} = \overline{\operatorname{Int} A}$.

(37) X is extremally disconnected if and only if for every subset A of X such that A is a domain holds $\operatorname{Int} A = \overline{A}$.

(38) X is extremally disconnected if and only if for every subset A of X holds if A is an open domain, then A is a closed domain and also if A is a closed domain, then A is an open domain.

A topological space is hereditarily extremally disconnected if:

(Def.5) every subspace of it is extremally disconnected.

One can check the following observations:

* there exists a hereditarily extremally disconnected strict topological space,
* every hereditarily extremally disconnected topological space is extremally disconnected, and
* every topological space which is almost discrete is also hereditarily extremally disconnected.

One can prove the following proposition

(39) For every extremally disconnected topological space X and for every subspace X_0 of X and for every subset A of X such that $A =$ the carrier of X_0 and A is dense holds X_0 is extremally disconnected.

Let X be an extremally disconnected topological space. One can check that every open subspace of X is extremally disconnected.

Let X be an extremally disconnected topological space. Note that there exists an extremally disconnected strict subspace of X.

Let X be a hereditarily extremally disconnected topological space. Note that every subspace of X is hereditarily extremally disconnected.

Let X be a hereditarily extremally disconnected topological space. Note that there exists a hereditarily extremally disconnected strict subspace of X.

One can prove the following proposition

(40) If every closed subspace of X is extremally disconnected, then X is hereditarily extremally disconnected.

5. The Lattice of Domains of Extremally Disconnected Spaces

In the sequel Y is an extremally disconnected topological space. The following propositions are true:

(41) The domains of Y = the closed domains of Y.

(42) D-Union(Y) = CLD-Union(Y) and D-Meet(Y) = CLD-Meet(Y).

(43) The lattice of domains of Y = the lattice of closed domains of Y.

(44) The domains of Y = the open domains of Y.

(45) D-Union(Y) = OPD-Union(Y) and D-Meet(Y) = OPD-Meet(Y).

(46) The lattice of domains of Y = the lattice of open domains of Y.

(47) For all elements A, B of the domains of Y holds (D-Union(Y))(A, B) = $A \cup B$ and (D-Meet(Y))(A, B) = $A \cap B$.

(48) For all elements a, b of the lattice of domains of Y and for all elements A, B of the domains of Y such that $a = A$ and $b = B$ holds $a \sqcup b = A \cup B$ and $a \sqcap b = A \cap B$.

(49) For every family F of subsets of Y such that F is domains-family and for every subset S of the lattice of domains of Y such that $S = F$ holds $\bigsqcup_{(\text{the lattice of domains of } Y)} S = \overline{\bigcup F}$.

(50) For every family F of subsets of Y such that F is domains-family and for every subset S of the lattice of domains of Y such that $S = F$ holds if $S \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } Y)} S = \operatorname{Int} \bigcap F$ and also if $S = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } Y)} S = \Omega_Y$.

In the sequel X will denote a topological space. One can prove the following propositions:

(51) X is extremally disconnected if and only if the lattice of domains of X is a modular lattice.

(52) If the lattice of domains of X = the lattice of closed domains of X, then X is extremally disconnected.

(53) If the lattice of domains of X = the lattice of open domains of X, then X is extremally disconnected.

(54) If the lattice of closed domains of X = the lattice of open domains of X, then X is extremally disconnected.

(55) X is extremally disconnected if and only if the lattice of domains of X is a Boolean lattice.

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A Mathematical Model of CPU

Yatsuka Nakamura
Shinshu University
Nagano

Andrzej Trybulec[1]
Warsaw University
Białystok

Summary. This paper is based on a previous work of the first author [12] in which a mathematical model of the computer has been presented. The model deals with random access memory, such as RASP of C. C. Elgot and A. Robinson [11], however, it allows for a more realistic modeling of real computers. This new model of computers has been named by the author (Y. Nakamura, [12]) Architecture Model for Instructions (AMI). It is more developed than previous models, both in the description of hardware (e.g., the concept of the program counter, the structure of memory) as well as in the description of instructions (instruction codes, addresses). The structure of AMI over an arbitrary collection of mathematical domains N consists of:

- a non-empty set of objects,
- the instruction counter,
- a non-empty set of objects called instruction locations,
- a non-empty set of instruction codes,
- an instruction code for halting,
- a set of instructions that are ordered pairs with the first element being an instruction code and the second a finite sequence in which members are either objects of the AMI or elements of one of the domains included in N,
- a function that assigns to every object of AMI its kind that is either *an instruction* or *an instruction location* or an element of N,
- a function that assigns to every instruction its execution that is again a function mapping states of AMI into the set of states.

By a state of AMI we mean a function that assigns to every object of AMI an element of the same kind. In this paper we develop the theory of AMI. Some properties of AMI are introduced ensuring it to have some properties of real computers:

- a von Neumann AMI, in which only addresses to instruction locations are stored in the program counter,
- data oriented, those in which instructions cannot be stored in data locations,
- halting, in which the execution of the halt instruction is the identity mapping of the states of an AMI,
- steady programmed, the condition in which the contents of the instruction locations do not change during execution,

[1]The work has been done while the second author was visiting Nagano in autumn 1992.

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- definite, a property in which only instructions may be stored in instruction locations.

We present an example of AMI called a Small Concrete Model which has been constructed in [12]. The Small Concrete Model has only one kind of data: integers and a set of instructions, small but sufficient to cope with integers.

MML Identifier: `AMI_1`.

The terminology and notation used here have been introduced in the following articles: [19], [5], [6], [15], [2], [20], [14], [3], [17], [16], [10], [1], [4], [18], [13], [7], [9], [21], and [8].

1. Preliminaries

In the sequel x is arbitrary. Next we state several propositions:

(1) $\mathbb{N} \neq \mathbb{Z}$.

(2) For arbitrary a, b holds $1 \neq \langle a, b \rangle$.

(3) For arbitrary a, b holds $2 \neq \langle a, b \rangle$.

(4) For arbitrary a, b, c, d and for every function g such that $\operatorname{dom} g = \{a, b\}$ and $g(a) = c$ and $g(b) = d$ holds $g = [a \longmapsto c, b \longmapsto d]$.

(5) For arbitrary a, b, c, d such that $a \neq b$ holds $\prod[a \longmapsto \{c\}, b \longmapsto \{d\}] = \{[a \longmapsto c, b \longmapsto d]\}$.

Let A be a set, and let B be a non-empty set. Then $A \cup B$ is a non-empty set. Let A be a non-empty set, and let B be a set. Then $A \cup B$ is a non-empty set. A set has non-empty elements if:

(Def.1) $\emptyset \notin$ it.

One can verify that there exists a set which is non-empty with and non-empty elements.

Let A be a non-empty set. Then $\{A\}$ is a non-empty set with non-empty elements. Let B be a non-empty set. Then $\{A, B\}$ is a non-empty set with non-empty elements. Let A, B be non-empty sets with non-empty elements. Then $A \cup B$ is a non-empty set with non-empty elements.

2. General concepts

In the sequel N will be a non-empty set with non-empty elements.

We now define several new constructions. Let us consider N. We consider AMI's over N which are systems

⟨objects, a instruction counter, instruction locations, instruction codes, a halt instruction, instructions, a object kind, a execution⟩,
where the objects constitute a non-empty set, the instruction counter is an element of the objects, the instruction locations constitute a non-empty subset of the objects, the instruction codes constitute a non-empty set, the halt instruction is an element of the instruction codes, the instructions constitute a non-empty subset of $[\!:$ the instruction codes, $(\bigcup N \cup \text{the objects})^{*}\,:\!]$, the object kind is a function from the objects into $N \cup \{$the instructions,the instruction locations$\}$, and the execution is a function from the instructions into $(\prod(\text{the object kind}))^{\prod(\text{the object kind})}$. Let us consider N, and let S be an AMI over N. An object of S is an element of the objects of S.

An instruction of S is an element of the instructions of S.

An instruction-location of S is an element of the instruction locations of S.

Let us consider N, and let S be an AMI over N. The functor $\mathbf{IC}_S$ yields an object of S and is defined by:

(Def.2) $\mathbf{IC}_S =$ the instruction counter of S.

Let us consider N, and let S be an AMI over N, and let o be an object of S. The functor ObjectKind(o) yielding an element of $N \cup \{$the instructions of S, the instruction locations of $S\}$ is defined by:

(Def.3) ObjectKind(o) = (the object kind of S)(o).

Let A be a set, and let B be a non-empty set with non-empty elements, and let f be a function from A into B. Then $\prod f$ is a non-empty set of functions. Let P be a non-empty set of functions. We see that the element of P is a function. Let us consider N, and let S be an AMI over N. A state of S is an element of $\prod$(the object kind of S).

Let us consider N, and let S be an AMI over N, and let I be an instruction of S, and let s be a state of S. The functor Exec(I, s) yielding a state of S is defined by:

(Def.4) Exec(I, s) = (the execution of S **qua** a function from the instructions of S into $(\prod(\text{the object kind of } S))^{\prod(\text{the object kind of } S)})(I)(s)$.

Let us consider N, and let S be an AMI over N satisfying the condition: ⟨the halt instruction of S, ε⟩ $\in$ the instructions of S. The functor $\mathbf{halt}_S$ yields an instruction of S and is defined as follows:

(Def.5) $\mathbf{halt}_S =$ ⟨the halt instruction of S, ε⟩.

Let us consider N. An AMI over N is von Neumann if:

(Def.6) ObjectKind($\mathbf{IC}_{\mathrm{it}}$) = the instruction locations of it.

An AMI over N is data-oriented if:

(Def.7) (the object kind of it) $^{-1}$ {the instructions of it} $\subseteq$ the instruction locations of it.

An AMI over N is halting if:

(Def.8) for every state s of it holds Exec($\mathbf{halt}_{\mathrm{it}}, s$) $= s$.

An AMI over N is steady-programmed if:

(Def.9) for every state s of it and for every instruction i of it and for every instruction-location l of it holds $(\mathrm{Exec}(i,s))(l) = s(l)$.

An AMI over N is definite if:

(Def.10) for every instruction-location l of it holds $\mathrm{ObjectKind}(l) =$ the instructions of it.

Let us consider N. Note that there exists a von Neumann data-oriented halting steady-programmed definite strict AMI over N.

Let us consider N, and let S be a von Neumann AMI over N, and let s be a state of S. The functor $\mathbf{IC}_s$ yields an instruction-location of S and is defined as follows:

(Def.11) $\mathbf{IC}_s = s(\mathbf{IC}_S)$.

3. A small concrete model

In the sequel i, k will be natural numbers. We now define four new functors. The non-empty subset $\mathrm{Loc}_{\mathrm{SCM}}$ of $\mathbb{N}$ is defined by:

(Def.12) $\mathrm{Loc}_{\mathrm{SCM}} = \mathbb{N} \setminus \{0\}$.

The element $\mathrm{Halt}_{\mathrm{SCM}}$ of $\mathbb{Z}_9$ is defined as follows:

(Def.13) $\mathrm{Halt}_{\mathrm{SCM}} = 0$.

The non-empty subset $\text{Data-Loc}_{\mathrm{SCM}}$ of $\mathrm{Loc}_{\mathrm{SCM}}$ is defined as follows:

(Def.14) $\text{Data-Loc}_{\mathrm{SCM}} = \{2 \cdot k + 1\}$.

The non-empty subset $\text{Instr-Loc}_{\mathrm{SCM}}$ of $\mathbb{N}$ is defined by:

(Def.15) $\text{Instr-Loc}_{\mathrm{SCM}} = \{2 \cdot k : k > 0\}$.

We adopt the following convention: I, J, K are elements of $\mathbb{Z}_9$, a, a_1, a_2 are elements of $\text{Instr-Loc}_{\mathrm{SCM}}$, and b, b_1, b_2, c, c_1 are elements of $\text{Data-Loc}_{\mathrm{SCM}}$. The non-empty subset $\mathrm{Instr}_{\mathrm{SCM}}$ of $[\!:\mathbb{Z}_9, \bigcup\{\mathbb{Z}\} \cup \mathbb{N}^*:\!]$ is defined as follows:

(Def.16) $\mathrm{Instr}_{\mathrm{SCM}} = \{\langle\mathrm{Halt}_{\mathrm{SCM}}, \varepsilon\rangle\} \cup \{\langle J, \langle a\rangle\rangle : J = 6\} \cup \{\langle K, \langle a_1, b_1\rangle\rangle : K \in \{7,8\}\} \cup \{\langle I, \langle b, c\rangle\rangle : I \in \{1,2,3,4,5\}\}$.

The following propositions are true:

(6) $\mathrm{Instr}_{\mathrm{SCM}} = \{\langle\mathrm{Halt}_{\mathrm{SCM}}, \varepsilon\rangle\} \cup \{\langle J, \langle a\rangle\rangle : J = 6\} \cup \{\langle K, \langle a_1, b_1\rangle\rangle : K \in \{7,8\}\} \cup \{\langle I, \langle b, c\rangle\rangle : I \in \{1,2,3,4,5\}\}$.

(7) $\langle 0, \varepsilon\rangle \in \mathrm{Instr}_{\mathrm{SCM}}$.

(8) $\langle 6, \langle a_2\rangle\rangle \in \mathrm{Instr}_{\mathrm{SCM}}$.

(9) If $x \in \{7,8\}$, then $\langle x, \langle a_2, b_2\rangle\rangle \in \mathrm{Instr}_{\mathrm{SCM}}$.

(10) If $x \in \{1,2,3,4,5\}$, then $\langle x, \langle b_1, c_1\rangle\rangle \in \mathrm{Instr}_{\mathrm{SCM}}$.

The function $\mathrm{OK}_{\mathrm{SCM}}$ from $\mathbb{N}$ into $\{\mathbb{Z}\} \cup \{\mathrm{Instr}_{\mathrm{SCM}}, \text{Instr-Loc}_{\mathrm{SCM}}\}$ is defined by:

(Def.17) $\mathrm{OK}_{\mathrm{SCM}}(0) = \text{Instr-Loc}_{\mathrm{SCM}}$ and for every natural number k holds $\mathrm{OK}_{\mathrm{SCM}}(2 \cdot k + 1) = \mathbb{Z}$ and $\mathrm{OK}_{\mathrm{SCM}}(2 \cdot k + 2) = \mathrm{Instr}_{\mathrm{SCM}}$.

The following four propositions are true:

(11) Instr-Loc$_{\mathrm{SCM}} \neq \mathbb{Z}$ and Instr$_{\mathrm{SCM}} \neq \mathbb{Z}$ and Instr-Loc$_{\mathrm{SCM}} \neq$ Instr$_{\mathrm{SCM}}$.

(12) For every i holds $\mathrm{OK}_{\mathrm{SCM}}(i) =$ Instr-Loc$_{\mathrm{SCM}}$ if and only if $i = 0$.

(13) For every i holds $\mathrm{OK}_{\mathrm{SCM}}(i) = \mathbb{Z}$ if and only if there exists k such that $i = 2 \cdot k + 1$.

(14) For every i holds $\mathrm{OK}_{\mathrm{SCM}}(i) =$ Instr$_{\mathrm{SCM}}$ if and only if there exists k such that $i = 2 \cdot k + 2$.

A state$_{\mathrm{SCM}}$ is an element of $\prod(\mathrm{OK}_{\mathrm{SCM}})$.

In the sequel s is a state$_{\mathrm{SCM}}$. We now state several propositions:

(15) For every element a of Data-Loc$_{\mathrm{SCM}}$ holds $\mathrm{OK}_{\mathrm{SCM}}(a) = \mathbb{Z}$.

(16) For every element a of Instr-Loc$_{\mathrm{SCM}}$ holds $\mathrm{OK}_{\mathrm{SCM}}(a) =$ Instr$_{\mathrm{SCM}}$.

(17) For every element a of Instr-Loc$_{\mathrm{SCM}}$ and for every element t of Data-Loc$_{\mathrm{SCM}}$ holds $a \neq t$.

(18) $\pi_0 \prod(\mathrm{OK}_{\mathrm{SCM}}) =$ Instr-Loc$_{\mathrm{SCM}}$.

(19) For every element a of Data-Loc$_{\mathrm{SCM}}$ holds $\pi_a \prod(\mathrm{OK}_{\mathrm{SCM}}) = \mathbb{Z}$.

(20) For every element a of Instr-Loc$_{\mathrm{SCM}}$ holds $\pi_a \prod(\mathrm{OK}_{\mathrm{SCM}}) =$ Instr$_{\mathrm{SCM}}$.

We now define two new functors. Let s be a state$_{\mathrm{SCM}}$. The functor $\mathbf{IC}_s$ yielding an element of Instr-Loc$_{\mathrm{SCM}}$ is defined by:

(Def.18) $\mathbf{IC}_s = s(0)$.

Let s be a state$_{\mathrm{SCM}}$, and let u be an element of Instr-Loc$_{\mathrm{SCM}}$. The functor $\mathrm{Chg}_{\mathrm{SCM}}(s, u)$ yields a state$_{\mathrm{SCM}}$ and is defined as follows:

(Def.19) $\mathrm{Chg}_{\mathrm{SCM}}(s, u) = s +\cdot (0 \dot{\longmapsto} u)$.

The following three propositions are true:

(21) For every state$_{\mathrm{SCM}}$ s and for every element u of Instr-Loc$_{\mathrm{SCM}}$ holds $(\mathrm{Chg}_{\mathrm{SCM}}(s, u))(0) = u$.

(22) For every state$_{\mathrm{SCM}}$ s and for every element u of Instr-Loc$_{\mathrm{SCM}}$ and for every element m_1 of Data-Loc$_{\mathrm{SCM}}$ holds $(\mathrm{Chg}_{\mathrm{SCM}}(s, u))(m_1) = s(m_1)$.

(23) For every state$_{\mathrm{SCM}}$ s and for all elements u, v of Instr-Loc$_{\mathrm{SCM}}$ holds $(\mathrm{Chg}_{\mathrm{SCM}}(s, u))(v) = s(v)$.

Let s be a state$_{\mathrm{SCM}}$, and let t be an element of Data-Loc$_{\mathrm{SCM}}$, and let u be an integer. The functor $\mathrm{Chg}_{\mathrm{SCM}}(s, t, u)$ yielding a state$_{\mathrm{SCM}}$ is defined by:

(Def.20) $\mathrm{Chg}_{\mathrm{SCM}}(s, t, u) = s +\cdot (t \dot{\longmapsto} u)$.

The following four propositions are true:

(24) For every state$_{\mathrm{SCM}}$ s and for every element t of Data-Loc$_{\mathrm{SCM}}$ and for every integer u holds $(\mathrm{Chg}_{\mathrm{SCM}}(s, t, u))(0) = s(0)$.

(25) For every state$_{\mathrm{SCM}}$ s and for every element t of Data-Loc$_{\mathrm{SCM}}$ and for every integer u holds $(\mathrm{Chg}_{\mathrm{SCM}}(s, t, u))(t) = u$.

(26) For every state$_{\mathrm{SCM}}$ s and for every element t of Data-Loc$_{\mathrm{SCM}}$ and for every integer u and for every element m_1 of Data-Loc$_{\mathrm{SCM}}$ such that $m_1 \neq t$ holds $(\mathrm{Chg}_{\mathrm{SCM}}(s, t, u))(m_1) = s(m_1)$.

(27) For every state$_{\rm SCM}$ s and for every element t of Data-Loc$_{\rm SCM}$ and for every integer u and for every element v of Instr-Loc$_{\rm SCM}$ holds $(\mathrm{Chg}_{\rm SCM}(s,t,u))(v) = s(v)$.

We now define two new functors. Let x be an element of Instr$_{\rm SCM}$. Let us assume that there exist m_1, m_2 of the type elements of Data-Loc$_{\rm SCM}$; I such that $x = \langle I, \langle m_1, m_2\rangle\rangle$. The functor $x$address$_1$ yields an element of Data-Loc$_{\rm SCM}$ and is defined by:

(Def.21) there exists a finite sequence f of elements of Data-Loc$_{\rm SCM}$ such that $f = x_{\mathbf{2}}$ and $x$address$_1 = \pi_1 f$.

The functor $x$address$_2$ yields an element of Data-Loc$_{\rm SCM}$ and is defined by:

(Def.22) there exists a finite sequence f of elements of Data-Loc$_{\rm SCM}$ such that $f = x_{\mathbf{2}}$ and $x$address$_2 = \pi_2 f$.

One can prove the following proposition

(28) For every element x of Instr$_{\rm SCM}$ and for all elements m_1, m_2 of Data-Loc$_{\rm SCM}$ and for every I such that $x = \langle I, \langle m_1, m_2\rangle\rangle$ holds $x$address$_1 = m_1$ and $x$address$_2 = m_2$.

Let x be an element of Instr$_{\rm SCM}$. Let us assume that there exist m_1 of the type an element of Instr-Loc$_{\rm SCM}$; I such that $x = \langle I, \langle m_1\rangle\rangle$. The functor $x$address$_{\rm j}$ yielding an element of Instr-Loc$_{\rm SCM}$ is defined as follows:

(Def.23) there exists a finite sequence f of elements of Instr-Loc$_{\rm SCM}$ such that $f = x_{\mathbf{2}}$ and $x$address$_{\rm j} = \pi_1 f$.

We now state the proposition

(29) For every element x of Instr$_{\rm SCM}$ and for every element m_1 of Instr-Loc$_{\rm SCM}$ and for every I such that $x = \langle I, \langle m_1\rangle\rangle$ holds $x$address$_{\rm j} = m_1$.

We now define two new functors. Let x be an element of Instr$_{\rm SCM}$. Let us assume that there exist m_1 of the type an element of Instr-Loc$_{\rm SCM}$; m_2 of the type an element of Data-Loc$_{\rm SCM}$; I such that $x = \langle I, \langle m_1, m_2\rangle\rangle$. The functor $x$address$_{\rm j}$ yields an element of Instr-Loc$_{\rm SCM}$ and is defined as follows:

(Def.24) there exists an element m_1 of Instr-Loc$_{\rm SCM}$ and there exists an element m_2 of Data-Loc$_{\rm SCM}$ such that $\langle m_1, m_2\rangle = x_{\mathbf{2}}$ and $x$address$_{\rm j} = \pi_1\langle m_1, m_2\rangle$.

The functor $x$address$_{\rm c}$ yielding an element of Data-Loc$_{\rm SCM}$ is defined by:

(Def.25) there exists an element m_1 of Instr-Loc$_{\rm SCM}$ and there exists an element m_2 of Data-Loc$_{\rm SCM}$ such that $\langle m_1, m_2\rangle = x_{\mathbf{2}}$ and $x$address$_{\rm c} = \pi_2\langle m_1, m_2\rangle$.

The following proposition is true

(30) For every element x of Instr$_{\rm SCM}$ and for every element m_1 of Instr-Loc$_{\rm SCM}$ and for every element m_2 of Data-Loc$_{\rm SCM}$ and for every I such that $x = \langle I, \langle m_1, m_2\rangle\rangle$ holds $x$address$_{\rm j} = m_1$ and $x$address$_{\rm c} = m_2$.

We now define five new functors. Let s be a state$_{\rm SCM}$, and let a be an element of Data-Loc$_{\rm SCM}$. Then $s(a)$ is an integer. Let D be a non-empty set, and let x, y be arbitrary, and let a, b be elements of D. Then $(x = y \rightarrow a, b)$ is an element of D. Let D be a non-empty set, and let x, y be real numbers, and let a, b be elements of D. The functor $(x > y \to a, b)$ yields an element of D and is defined as follows:

(Def.26)

$$(x > y \to a, b) = \begin{cases} a, & \text{if } x > y, \\ b, & \text{otherwise.} \end{cases}$$

Let d be an element of $\text{Instr-Loc}_{\text{SCM}}$. The functor $\text{Next}(d)$ yields an element of $\text{Instr-Loc}_{\text{SCM}}$ and is defined as follows:

(Def.27) $\text{Next}(d) = d + 2$.

Let x be an element of $\text{Instr}_{\text{SCM}}$, and let s be a $\text{state}_{\text{SCM}}$. The functor $\text{Exec-Res}_{\text{SCM}}(x, s)$ yielding a $\text{state}_{\text{SCM}}$ is defined as follows:

(Def.28) (i) $\text{Exec-Res}_{\text{SCM}}(x, s) = \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\text{address}_1, s(x\text{address}_2)), \text{Next}(\mathbf{IC}_s))$ if there exist elements m_1, m_2 of $\text{Data-Loc}_{\text{SCM}}$ such that $x = \langle\!\langle 1, \langle m_1, m_2\rangle\rangle\!\rangle$,

(ii) $\text{Exec-Res}_{\text{SCM}}(x, s) = \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\text{address}_1, s(x\text{address}_1) + s(x\text{address}_2)), \text{Next}(\mathbf{IC}_s))$ if there exist elements m_1, m_2 of $\text{Data-Loc}_{\text{SCM}}$ such that $x = \langle\!\langle 2, \langle m_1, m_2\rangle\rangle\!\rangle$,

(iii) $\text{Exec-Res}_{\text{SCM}}(x, s) = \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\text{address}_1, s(x\text{address}_1) - s(x\text{address}_2)), \text{Next}(\mathbf{IC}_s))$ if there exist elements m_1, m_2 of $\text{Data-Loc}_{\text{SCM}}$ such that $x = \langle\!\langle 3, \langle m_1, m_2\rangle\rangle\!\rangle$,

(iv) $\text{Exec-Res}_{\text{SCM}}(x, s) = \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\text{address}_1, s(x\text{address}_1) \cdot s(x\text{address}_2)), \text{Next}(\mathbf{IC}_s))$ if there exist elements m_1, m_2 of $\text{Data-Loc}_{\text{SCM}}$ such that $x = \langle\!\langle 4, \langle m_1, m_2\rangle\rangle\!\rangle$,

(v) $\text{Exec-Res}_{\text{SCM}}(x, s) = \text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(\text{Chg}_{\text{SCM}}(s, x\text{address}_1, s(x\text{address}_1) \div s(x\text{address}_2)), x\text{address}_2, s(x\text{address}_1) \bmod s(x\text{address}_2)), \text{Next}(\mathbf{IC}_s))$ if there exist elements m_1, m_2 of $\text{Data-Loc}_{\text{SCM}}$ such that $x = \langle\!\langle 5, \langle m_1, m_2\rangle\rangle\!\rangle$,

(vi) $\text{Exec-Res}_{\text{SCM}}(x, s) = \text{Chg}_{\text{SCM}}(s, x\text{address}_{\text{j}})$ if there exists an element m_1 of $\text{Instr-Loc}_{\text{SCM}}$ such that $x = \langle\!\langle 6, \langle m_1\rangle\rangle\!\rangle$,

(vii) $\text{Exec-Res}_{\text{SCM}}(x, s) = \text{Chg}_{\text{SCM}}(s, (s(x\text{address}_{\text{c}}) = 0 \to x\text{address}_{\text{j}}, \text{Next}(\mathbf{IC}_s)))$ if there exists an element m_1 of $\text{Instr-Loc}_{\text{SCM}}$ and there exists an element m_2 of $\text{Data-Loc}_{\text{SCM}}$ such that $x = \langle\!\langle 7, \langle m_1, m_2\rangle\rangle\!\rangle$,

(viii) $\text{Exec-Res}_{\text{SCM}}(x, s) = \text{Chg}_{\text{SCM}}(s, (s(x\text{address}_{\text{c}}) > 0 \to x\text{address}_{\text{j}}, \text{Next}(\mathbf{IC}_s)))$ if there exists an element m_1 of $\text{Instr-Loc}_{\text{SCM}}$ and there exists an element m_2 of $\text{Data-Loc}_{\text{SCM}}$ such that $x = \langle\!\langle 8, \langle m_1, m_2\rangle\rangle\!\rangle$,

(ix) $\text{Exec-Res}_{\text{SCM}}(x, s) = s$, otherwise.

The function Exec_{SCM} from $\text{Instr}_{\text{SCM}}$ into $\prod \text{OK}_{\text{SCM}}^{\prod \text{OK}_{\text{SCM}}}$ is defined by:

(Def.29) for every element x of $\text{Instr}_{\text{SCM}}$ and for every $\text{state}_{\text{SCM}}$ y holds $(\text{Exec}_{\text{SCM}}(x)\ \mathbf{qua}$ an element of $(\prod(\text{OK}_{\text{SCM}}))^{\prod(\text{OK}_{\text{SCM}})})(y) = \text{Exec-Res}_{\text{SCM}}(x, y)$.

The von Neumann strict AMI **SCM** is defined by:

(Def.30) $\mathbf{SCM} = \langle \mathbb{N}, 0, \text{Instr-Loc}_{\text{SCM}}, \mathbb{Z}_9, \text{Halt}_{\text{SCM}}, \text{Instr}_{\text{SCM}}, \text{OK}_{\text{SCM}}, \text{Exec}_{\text{SCM}} \rangle$.

Next we state three propositions:

(31) **SCM** is data-oriented.

(32) **SCM** is definite.

(33) The objects of $\mathbf{SCM} = \mathbb{N}$ and the instruction counter of $\mathbf{SCM} = 0$ and the instruction locations of $\mathbf{SCM} = \text{Instr-Loc}_{\text{SCM}}$ and the instruction codes of $\mathbf{SCM} = \mathbb{Z}_9$ and the halt instruction of $\mathbf{SCM} = \text{Halt}_{\text{SCM}}$ and the instructions of $\mathbf{SCM} = \text{Instr}_{\text{SCM}}$ and the object kind of $\mathbf{SCM} = \text{OK}_{\text{SCM}}$ and the execution of $\mathbf{SCM} = \text{Exec}_{\text{SCM}}$.

An object of **SCM** is said to be a data-location if:

(Def.31) it $\in \text{Data-Loc}_{\text{SCM}}$.

Let s be a state of **SCM**, and let d be a data-location. Then $s(d)$ is an integer.

We adopt the following convention: a, b, c denote data-locations, l_1 denotes an instruction-location of **SCM**, and I denotes an instruction of **SCM**. We now define several new functors. Let us consider a, b. The functor $a{:=}b$ yielding an instruction of **SCM** is defined by:

(Def.32) $a{:=}b = \langle 1, \langle a, b \rangle \rangle$.

The functor $\text{AddTo}(a, b)$ yielding an instruction of **SCM** is defined by:

(Def.33) $\text{AddTo}(a, b) = \langle 2, \langle a, b \rangle \rangle$.

The functor $\text{SubFrom}(a, b)$ yielding an instruction of **SCM** is defined by:

(Def.34) $\text{SubFrom}(a, b) = \langle 3, \langle a, b \rangle \rangle$.

The functor $\text{MultBy}(a, b)$ yields an instruction of **SCM** and is defined by:

(Def.35) $\text{MultBy}(a, b) = \langle 4, \langle a, b \rangle \rangle$.

The functor $\text{Divide}(a, b)$ yields an instruction of **SCM** and is defined as follows:

(Def.36) $\text{Divide}(a, b) = \langle 5, \langle a, b \rangle \rangle$.

Let us consider l_1. The functor goto l_1 yields an instruction of **SCM** and is defined by:

(Def.37) $\text{goto } l_1 = \langle 6, \langle l_1 \rangle \rangle$.

Let us consider a. The functor **if** $a = 0$ **goto** l_1 yielding an instruction of **SCM** is defined as follows:

(Def.38) **if** $a = 0$ **goto** $l_1 = \langle 7, \langle l_1, a \rangle \rangle$.

The functor **if** $a > 0$ **goto** l_1 yields an instruction of **SCM** and is defined as follows:

(Def.39) **if** $a > 0$ **goto** $l_1 = \langle 8, \langle l_1, a \rangle \rangle$.

In the sequel s will denote a state of **SCM**. Next we state two propositions:

(34) $\mathbf{IC}_{\mathbf{SCM}} = 0$.

(35) For every $\text{state}_{\text{SCM}}$ S such that $S = s$ holds $\mathbf{IC}_s = \mathbf{IC}_S$.

Let l_1 be an instruction-location of **SCM**. The functor $\text{Next}(l_1)$ yielding an instruction-location of **SCM** is defined by:

(Def.40) there exists an element m_3 of Instr-Loc$_{\mathrm{SCM}}$ such that $m_3 = l_1$ and $\mathrm{Next}(l_1) = \mathrm{Next}(m_3)$.

Next we state two propositions:

(36) For every instruction-location l_1 of **SCM** and for every element m_3 of Instr-Loc$_{\mathrm{SCM}}$ such that $m_3 = l_1$ holds $\mathrm{Next}(m_3) = \mathrm{Next}(l_1)$.

(37) For every element i of Instr$_{\mathrm{SCM}}$ such that $i = I$ and for every state$_{\mathrm{SCM}}$ S such that $S = s$ holds $\mathrm{Exec}(I, s) = \text{Exec-Res}_{\mathrm{SCM}}(i, S)$.

4. Users guide

One can prove the following propositions:

(38) $(\mathrm{Exec}(a{:=}b, s))(\mathbf{IC}_{\mathbf{SCM}}) = \mathrm{Next}(\mathbf{IC}_s)$ and $(\mathrm{Exec}(a{:=}b, s))(a) = s(b)$ and for every c such that $c \neq a$ holds $(\mathrm{Exec}(a{:=}b, s))(c) = s(c)$.

(39) $(\mathrm{Exec}(\mathrm{AddTo}(a, b), s))(\mathbf{IC}_{\mathbf{SCM}}) = \mathrm{Next}(\mathbf{IC}_s)$ and $(\mathrm{Exec}(\mathrm{AddTo}(a, b), s))(a) = s(a) + s(b)$ and for every c such that $c \neq a$ holds $(\mathrm{Exec}(\mathrm{AddTo}(a, b), s))(c) = s(c)$.

(40) $(\mathrm{Exec}(\mathrm{SubFrom}(a, b), s))(\mathbf{IC}_{\mathbf{SCM}}) = \mathrm{Next}(\mathbf{IC}_s)$ and $(\mathrm{Exec}(\mathrm{SubFrom}(a, b), s))(a) = s(a) - s(b)$ and for every c such that $c \neq a$ holds $(\mathrm{Exec}(\mathrm{SubFrom}(a, b), s))(c) = s(c)$.

(41) $(\mathrm{Exec}(\mathrm{MultBy}(a, b), s))(\mathbf{IC}_{\mathbf{SCM}}) = \mathrm{Next}(\mathbf{IC}_s)$ and $(\mathrm{Exec}(\mathrm{MultBy}(a, b), s))(a) = s(a) \cdot s(b)$ and for every c such that $c \neq a$ holds $(\mathrm{Exec}(\mathrm{MultBy}(a, b), s))(c) = s(c)$.

(42) Suppose $a \neq b$. Then

(i) $(\mathrm{Exec}(\mathrm{Divide}(a, b), s))(\mathbf{IC}_{\mathbf{SCM}}) = \mathrm{Next}(\mathbf{IC}_s)$,

(ii) $(\mathrm{Exec}(\mathrm{Divide}(a, b), s))(a) = s(a) \div s(b)$,

(iii) $(\mathrm{Exec}(\mathrm{Divide}(a, b), s))(b) = s(a) \bmod s(b)$,

(iv) for every c such that $c \neq a$ and $c \neq b$ holds $(\mathrm{Exec}(\mathrm{Divide}(a, b), s))(c) = s(c)$.

(43) $(\mathrm{Exec}(\mathrm{goto}\ l_1, s))(\mathbf{IC}_{\mathbf{SCM}}) = l_1$ and $(\mathrm{Exec}(\mathrm{goto}\ l_1, s))(c) = s(c)$.

(44) If $s(a) = 0$, then $(\mathrm{Exec}(\mathbf{if}\ a = 0\ \mathbf{goto}\ l_1, s))(\mathbf{IC}_{\mathbf{SCM}}) = l_1$ and also if $s(a) \neq 0$, then $(\mathrm{Exec}(\mathbf{if}\ a = 0\ \mathbf{goto}\ l_1, s))(\mathbf{IC}_{\mathbf{SCM}}) = \mathrm{Next}(\mathbf{IC}_s)$ and $(\mathrm{Exec}(\mathbf{if}\ a = 0\ \mathbf{goto}\ l_1, s))(c) = s(c)$.

(45) If $s(a) > 0$, then $(\mathrm{Exec}(\mathbf{if}\ a > 0\ \mathbf{goto}\ l_1, s))(\mathbf{IC}_{\mathbf{SCM}}) = l_1$ and also if $s(a) \leq 0$, then $(\mathrm{Exec}(\mathbf{if}\ a > 0\ \mathbf{goto}\ l_1, s))(\mathbf{IC}_{\mathbf{SCM}}) = \mathrm{Next}(\mathbf{IC}_s)$ and $(\mathrm{Exec}(\mathbf{if}\ a > 0\ \mathbf{goto}\ l_1, s))(c) = s(c)$.

(46) $\mathrm{Exec}(\mathbf{halt}_{\mathbf{SCM}}, s) = s$.

(47) For every state s of **SCM** and for every instruction-location i of **SCM** holds $s(i)$ is an instruction of **SCM**.

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Cartesian Categories

Czesław Byliński
Warsaw University
Białystok

Summary. We define and prove some simple facts on Cartesian categories and its duals co-Cartesian categories. The Cartesian category is defined as a category with the fixed terminal object, the fixed projections, and the binary products. Category C has finite products if and only if C has a terminal object and for every pair a, b of objects of C the product $a \times b$ exists. We say that a category C has a finite product if every finite family of objects of C has a product. Our work is based on ideas of [13], where the algebraic properties of the proof theory are investigated. The terminal object of a Cartesian category C is denoted by $\mathbf{1}_C$. The binary product of a and b is written as $a \times b$. The projections of the product are written as $pr_1(a, b)$ and as $pr_2(a, b)$. We define the products $f \times g$ of arrows $f : a \to a'$ and $g : b \to b'$ as $< f \cdot pr_1, g \cdot pr_2 >: a \times b \to a' \times b'$

Co-Cartesian category is defined dually to the Cartesian category. Dual to a terminal object is an initial object, and to products are co-products. The initial object of a Cartesian category C is written as $\mathbf{0}_C$. Binary coproduct of a and b is written as $a+b$. Injections of the coproduct are written as $in_1(a, b)$ and as $in_2(a, b)$.

MML Identifier: `CAT_4`.

The terminology and notation used in this paper are introduced in the following papers: [16], [15], [11], [4], [5], [14], [9], [12], [2], [1], [3], [7], [6], [8], and [10].

1. Preliminaries

In the sequel o, m, r will be arbitrary. We now define two new constructions. Let us consider o, m, r. $[\langle o, m\rangle \mapsto r]$ is a function from $[\!:\{o\}, \{m\}:\!]$ into $\{r\}$.

Let C be a category, and let a, b be objects of C. Let us observe that a and b are isomorphic if:

(Def.1) $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$ and there exists a morphism f from a to b and there exists a morphism f' from b to a such that $f \cdot f' = \mathrm{id}_b$ and $f' \cdot f = \mathrm{id}_a$.

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2. Cartesian Categories

Let C be a category. We say that C has finite product if and only if:

(Def.2) for every set I and for every function F from I into the objects of C such that I is finite there exists an object a of C and there exists a projections family F' from a onto I such that $\mathrm{cod}_\kappa F'(\kappa) = F$ and a is a product w.r.t. F'.

We now state the proposition

(1) Let C be a category. Then C has finite product if and only if there exists an object of C which is a terminal object and for every objects a, b of C there exists an object c of C and there exist morphisms p_1, p_2 of C such that $\mathrm{dom}\, p_1 = c$ and $\mathrm{dom}\, p_2 = c$ and $\mathrm{cod}\, p_1 = a$ and $\mathrm{cod}\, p_2 = b$ and c is a product w.r.t. p_1 and p_2.

We now define several new constructions. We consider Cartesian category structures which are extension of category structures and are systems

⟨objects, morphisms, a dom-map, a cod-map, a composition, an id-map, a terminal, a product, a 1st-projection, a 2nd-projection⟩,

where the objects, the morphisms constitute non-empty sets, the dom-map, the cod-map are functions from the morphisms into the objects, the composition is a partial function from $[\!:$ the morphisms, the morphisms $:\!]$ to the morphisms, the id-map is a function from the objects into the morphisms, the terminal is an element of the objects, the product is a function from $[\!:$ the objects, the objects $:\!]$ into the objects, and the 1st-projection, the 2nd-projection are functions from $[\!:$ the objects, the objects $:\!]$ into the morphisms. Let C be a Cartesian category structure. The functor $\mathbf{1}_C$ yielding an object of C is defined by:

(Def.3) $\mathbf{1}_C =$ the terminal of C.

Let a, b be objects of C. The functor $a \times b$ yielding an object of C is defined as follows:

(Def.4) $a \times b = (\text{the product of } C)(\langle a, b \rangle)$.

The functor $\pi_1(a \times b)$ yielding a morphism of C is defined as follows:

(Def.5) $\pi_1(a \times b) = (\text{the 1st-projection of } C)(\langle a, b \rangle)$.

The functor $\pi_2(a \times b)$ yields a morphism of C and is defined as follows:

(Def.6) $\pi_2(a \times b) = (\text{the 2nd-projection of } C)(\langle a, b \rangle)$.

Let us consider o, m. The functor $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ yielding a strict Cartesian category structure is defined by:

(Def.7) $\dot{\circlearrowleft}_{\mathrm{c}}(o,m) = \langle \{o\}, \{m\}, \{m\} \longmapsto o, \{m\} \longmapsto o, \langle m, m\rangle \longmapsto m, \{o\} \longmapsto m, \mathrm{Extract}(o), [\langle o, o\rangle \mapsto o], [\langle o, o\rangle \mapsto m], [\langle o, o\rangle \mapsto m]\rangle$.

We now state the proposition

(2) The category structure of $\dot{\circlearrowleft}_{\mathrm{c}}(o,m) = \dot{\circlearrowleft}(o,m)$.

Let us note that there exists a Cartesian category structure which is strict and category-like.

Let o, m be arbitrary. Then $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ is a strict category-like Cartesian category structure.

The following propositions are true:

(3) For every object a of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $a = o$.

(4) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $a = b$.

(5) For every morphism f of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $f = m$.

(6) For all morphisms f, g of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $f = g$.

(7) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ and for every morphism f of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $f \in \hom(a, b)$.

(8) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ every morphism of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ is a morphism from a to b.

(9) For all objects a, b of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds $\hom(a, b) \neq \emptyset$.

(10) Every object of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ is a terminal object.

(11) For every object c of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ and for all morphisms p_1, p_2 of $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ holds c is a product w.r.t. p_1 and p_2.

A category-like Cartesian category structure is Cartesian if:

(Def.8) the terminal of it is a terminal object and for all objects a, b of it holds cod (the 1st-projection of it)($\langle a, b\rangle$) $= a$ and cod (the 2nd-projection of it)($\langle a, b\rangle$) $= b$ and (the product of it)($\langle a, b\rangle$) is a product w.r.t. (the 1st-projection of it)($\langle a, b\rangle$) and (the 2nd-projection of it)($\langle a, b\rangle$).

We now state the proposition

(12) For arbitrary o, m holds $\dot{\circlearrowleft}_{\mathrm{c}}(o, m)$ is Cartesian.

One can verify that there exists a strict Cartesian category-like Cartesian category structure.

A Cartesian category is a category-like Cartesian category structure.

We adopt the following convention: C denotes a Cartesian category and a, b, c, d, e, s denote objects of C. We now state three propositions:

(13) $\mathbf{1}_C$ is a terminal object.

(14) For all morphisms f_1, f_2 from a to $\mathbf{1}_C$ holds $f_1 = f_2$.

(15) $\hom(a, \mathbf{1}_C) \neq \emptyset$.

Let us consider C, a. $!_a$ is a morphism from a to $\mathbf{1}_C$.

Next we state several propositions:

(16) $!_a = |_{\mathbf{1}_C} a$.

(17) $\mathrm{dom}(!_a) = a$ and $\mathrm{cod}(!_a) = \mathbf{1}_C$.

(18) $\hom(a, \mathbf{1}_C) = \{!_a\}$.

(19) $\mathrm{dom}\, \pi_1(a \times b) = a \times b$ and $\mathrm{cod}\, \pi_1(a \times b) = a$.

(20) $\mathrm{dom}\, \pi_2(a \times b) = a \times b$ and $\mathrm{cod}\, \pi_2(a \times b) = b$.

Let us consider C, a, b. Then $\pi_1(a \times b)$ is a morphism from $a \times b$ to a. Then $\pi_2(a \times b)$ is a morphism from $a \times b$ to b.

The following four propositions are true:

(21) $\hom(a \times b, a) \neq \emptyset$ and $\hom(a \times b, b) \neq \emptyset$.

(22) $a \times b$ is a product w.r.t. $\pi_1(a \times b)$ and $\pi_2(a \times b)$.

(23) C has finite product.

(24) If $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$, then $\pi_1(a \times b)$ is retraction and $\pi_2(a \times b)$ is retraction.

Let us consider C, a, b, c, and let f be a morphism from c to a, and let g be a morphism from c to b. Let us assume that $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$. The functor $\langle f, g \rangle$ yields a morphism from c to $a \times b$ and is defined by:

(Def.9) $\pi_1(a \times b) \cdot \langle f, g \rangle = f$ and $\pi_2(a \times b) \cdot \langle f, g \rangle = g$.

The following propositions are true:

(25) If $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$, then $\hom(c, a \times b) \neq \emptyset$.

(26) $\langle \pi_1(a \times b), \pi_2(a \times b) \rangle = \mathrm{id}_{(a \times b)}$.

(27) For every morphism f from c to a and for every morphism g from c to b and for every morphism h from d to c such that $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$ and $\hom(d, c) \neq \emptyset$ holds $\langle f \cdot h, g \cdot h \rangle = \langle f, g \rangle \cdot h$.

(28) For all morphisms f, k from c to a and for all morphisms g, h from c to b such that $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$ and $\langle f, g \rangle = \langle k, h \rangle$ holds $f = k$ and $g = h$.

(29) For every morphism f from c to a and for every morphism g from c to b such that $\hom(c, a) \neq \emptyset$ and $\hom(c, b) \neq \emptyset$ and also f is monic or g is monic holds $\langle f, g \rangle$ is monic.

(30) $\hom(a, a \times \mathbf{1}_C) \neq \emptyset$ and $\hom(a, \mathbf{1}_C \times a) \neq \emptyset$.

We now define four new functors. Let us consider C, a. The functor $\lambda(a)$ yielding a morphism from $\mathbf{1}_C \times a$ to a is defined by:

(Def.10) $\lambda(a) = \pi_2(\mathbf{1}_C \times a)$.

The functor $\lambda^{-1}(a)$ yielding a morphism from a to $\mathbf{1}_C \times a$ is defined as follows:

(Def.11) $\lambda^{-1}(a) = \langle !_a, \mathrm{id}_a \rangle$.

The functor $\rho(a)$ yields a morphism from $a \times \mathbf{1}_C$ to a and is defined as follows:

(Def.12) $\rho(a) = \pi_1(a \times \mathbf{1}_C)$.

The functor $\rho^{-1}(a)$ yielding a morphism from a to $a \times \mathbf{1}_C$ is defined as follows:

(Def.13) $\rho^{-1}(a) = \langle \mathrm{id}_a, !_a \rangle$.

The following propositions are true:

(31) $\lambda(a) \cdot \lambda^{-1}(a) = \mathrm{id}_a$ and $\lambda^{-1}(a) \cdot \lambda(a) = \mathrm{id}_{(\mathbf{1}_C \times a)}$ and $\rho(a) \cdot \rho^{-1}(a) = \mathrm{id}_a$ and $\rho^{-1}(a) \cdot \rho(a) = \mathrm{id}_{(a \times \mathbf{1}_C)}$.

(32) a and $a \times \mathbf{1}_C$ are isomorphic and a and $\mathbf{1}_C \times a$ are isomorphic.

Let us consider C, a, b. The functor $\mathrm{Switch}(a)$ yielding a morphism from $a \times b$ to $b \times a$ is defined as follows:

(Def.14) $\mathrm{Switch}(a) = \langle \pi_2(a \times b), \pi_1(a \times b) \rangle$.

One can prove the following three propositions:

(33) $\hom(a \times b, b \times a) \neq \emptyset$.

(34) $\mathrm{Switch}(a) \cdot \mathrm{Switch}(b) = \mathrm{id}_{(b \times a)}$.

(35) $a \times b$ and $b \times a$ are isomorphic.

Let us consider C, a. The functor $\Delta(a)$ yielding a morphism from a to $a \times a$ is defined by:

(Def.15) $\Delta(a) = \langle \mathrm{id}_a, \mathrm{id}_a \rangle$.

We now state two propositions:

(36) $\hom(a, a \times a) \neq \emptyset$.

(37) For every morphism f from a to b such that $\hom(a,b) \neq \emptyset$ holds $\langle f, f \rangle = \Delta(b) \cdot f$.

We now define two new functors. Let us consider C, a, b, c. The functor $\alpha((a,b),c)$ yielding a morphism from $a \times b \times c$ to $a \times (b \times c)$ is defined by:

(Def.16) $\alpha((a,b),c) = \langle \pi_1(a \times b) \cdot \pi_1((a \times b) \times c), \langle \pi_2(a \times b) \cdot \pi_1((a \times b) \times c), \pi_2((a \times b) \times c) \rangle \rangle$.

The functor $\alpha(a,(b,c))$ yields a morphism from $a \times (b \times c)$ to $a \times b \times c$ and is defined as follows:

(Def.17) $\alpha(a,(b,c)) = \langle \langle \pi_1(a \times (b \times c)), \pi_1(b \times c) \cdot \pi_2(a \times (b \times c)) \rangle, \pi_2(b \times c) \cdot \pi_2(a \times (b \times c)) \rangle$.

The following three propositions are true:

(38) $\hom(a \times b \times c, a \times (b \times c)) \neq \emptyset$ and $\hom(a \times (b \times c), a \times b \times c) \neq \emptyset$.

(39) $\alpha((a,b),c) \cdot \alpha(a,(b,c)) = \mathrm{id}_{(a \times (b \times c))}$ and $\alpha(a,(b,c)) \cdot \alpha((a,b),c) = \mathrm{id}_{(a \times b \times c)}$.

(40) $(a \times b) \times c$ and $a \times (b \times c)$ are isomorphic.

Let us consider C, a, b, c, d, and let f be a morphism from a to b, and let g be a morphism from c to d. The functor $f \times g$ yields a morphism from $a \times c$ to $b \times d$ and is defined by:

(Def.18) $f \times g = \langle f \cdot \pi_1(a \times c), g \cdot \pi_2(a \times c) \rangle$.

One can prove the following propositions:

(41) If $\hom(a,c) \neq \emptyset$ and $\hom(b,d) \neq \emptyset$, then $\hom(a \times b, c \times d) \neq \emptyset$.

(42) $\mathrm{id}_a \times \mathrm{id}_b = \mathrm{id}_{(a \times b)}$.

(43) Let f be a morphism from a to b. Let h be a morphism from c to d. Then for every morphism g from e to a and for every morphism k from e to c such that $\hom(a,b) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$ and $\hom(e,a) \neq \emptyset$ and $\hom(e,c) \neq \emptyset$ holds $(f \times h) \cdot \langle g, k \rangle = \langle f \cdot g, h \cdot k \rangle$.

(44) For every morphism f from c to a and for every morphism g from c to b such that $\hom(c,a) \neq \emptyset$ and $\hom(c,b) \neq \emptyset$ holds $\langle f, g \rangle = (f \times g) \cdot \Delta(c)$.

(45) Let f be a morphism from a to b. Let h be a morphism from c to d. Then for every morphism g from e to a and for every morphism k from s to c such that $\hom(a,b) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$ and $\hom(e,a) \neq \emptyset$ and $\hom(s,c) \neq \emptyset$ holds $(f \times h) \cdot (g \times k) = (f \cdot g) \times (h \cdot k)$.

3. Co-Cartesian Categories

Let C be a category. We say that C has finite coproduct if and only if:

(Def.19) for every set I and for every function F from I into the objects of C such that I is finite there exists an object a of C and there exists a injections family F' into a on I such that $\mathrm{dom}_\kappa F'(\kappa) = F$ and a is a coproduct w.r.t. F'.

Next we state the proposition

(46) Let C be a category. Then C has finite coproduct if and only if there exists an object of C which is an initial object and for every objects a, b of C there exists an object c of C and there exist morphisms i_1, i_2 of C such that $\mathrm{dom}\, i_1 = a$ and $\mathrm{dom}\, i_2 = b$ and $\mathrm{cod}\, i_1 = c$ and $\mathrm{cod}\, i_2 = c$ and c is a coproduct w.r.t. i_1 and i_2.

We now define several new constructions. We consider cocartesian category structures which are extension of category structures and are systems

⟨objects, morphisms, a dom-map, a cod-map, a composition, an id-map, a initial, a coproduct, a 1st-coprojection, a 2nd-coprojection⟩,

where the objects, the morphisms constitute non-empty sets, the dom-map, the cod-map are functions from the morphisms into the objects, the composition is a partial function from $[\!:\text{the morphisms}, \text{the morphisms}:\!]$ to the morphisms, the id-map is a function from the objects into the morphisms, the initial is an element of the objects, the coproduct is a function from $[\!:\text{the objects}, \text{the objects}:\!]$ into the objects, and the 1st-coprojection, the 2nd-coprojection are functions from $[\!:\text{the objects}, \text{the objects}:\!]$ into the morphisms. Let C be a cocartesian category structure. The functor $\mathbf{0}_C$ yields an object of C and is defined as follows:

(Def.20) $\mathbf{0}_C =$ the initial of C.

Let a, b be objects of C. The functor $a + b$ yields an object of C and is defined as follows:

(Def.21) $a + b = (\text{the coproduct of } C)(\langle\!\langle a,\, b\rangle\!\rangle)$.

The functor $\mathrm{in}_1(a + b)$ yields a morphism of C and is defined as follows:

(Def.22) $\mathrm{in}_1(a + b) = (\text{the 1st-coprojection of } C)(\langle\!\langle a,\, b\rangle\!\rangle)$.

The functor $\mathrm{in}_2(a + b)$ yields a morphism of C and is defined by:

(Def.23) $\mathrm{in}_2(a + b) = (\text{the 2nd-coprojection of } C)(\langle\!\langle a,\, b\rangle\!\rangle)$.

Let us consider o, m. The functor $\dot{\circlearrowleft}_{\mathrm{c}}^{\mathrm{op}}(o, m)$ yielding a strict cocartesian category structure is defined by:

(Def.24) $\dot{\circlearrowleft}_{\mathrm{c}}^{\mathrm{op}}(o, m) = \langle \{o\}, \{m\}, \{m\} \longmapsto o, \{m\} \longmapsto o, \langle m, m\rangle \longmapsto m, \{o\} \longmapsto m, \mathrm{Extract}(o), [\langle o, o\rangle \mapsto o], [\langle o, o\rangle \mapsto m], [\langle o, o\rangle \mapsto m]\rangle$.

One can prove the following proposition

(47) The category structure of $\dot{\circlearrowleft}_{\mathrm{c}}^{\mathrm{op}}(o, m) = \dot{\circlearrowleft}(o, m)$.

Let us note that there exists a strict category-like cocartesian category structure.

Let o, m be arbitrary. Then $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ is a strict category-like cocartesian category structure.

One can prove the following propositions:

(48) For every object a of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ holds $a = o$.

(49) For all objects a, b of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ holds $a = b$.

(50) For every morphism f of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ holds $f = m$.

(51) For all morphisms f, g of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ holds $f = g$.

(52) For all objects a, b of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ and for every morphism f of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ holds $f \in \hom(a,b)$.

(53) For all objects a, b of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ every morphism of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ is a morphism from a to b.

(54) For all objects a, b of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ holds $\hom(a,b) \neq \emptyset$.

(55) Every object of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ is an initial object.

(56) For every object c of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ and for all morphisms i_1, i_2 of $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ holds c is a coproduct w.r.t. i_1 and i_2.

A category-like cocartesian category structure is cocartesian if:

(Def.25) the initial of it is an initial object and for all objects a, b of it holds dom (the 1st-coprojection of it)($\langle a,\, b\rangle$) $= a$ and dom (the 2nd-coprojection of it)($\langle a,\, b\rangle$) $= b$ and (the coproduct of it)($\langle a,\, b\rangle$) is a coproduct w.r.t. (the 1st-coprojection of it)($\langle a,\, b\rangle$) and (the 2nd-coprojection of it)($\langle a,\, b\rangle$).

One can prove the following proposition

(57) For arbitrary o, m holds $\dot{\circlearrowleft}^{\rm op}_{\rm c}(o,m)$ is cocartesian.

One can check that there exists a category-like cocartesian category structure which is strict and cocartesian.

A cocartesian category is a category-like cocartesian category structure.

We adopt the following rules: C is a cocartesian category and a, b, c, d, e, s are objects of C. Next we state two propositions:

(58) $\mathbf{0}_C$ is an initial object.

(59) For all morphisms f_1, f_2 from $\mathbf{0}_C$ to a holds $f_1 = f_2$.

Let us consider C, a. $!^a$ is a morphism from $\mathbf{0}_C$ to a.

We now state a number of propositions:

(60) $\hom(\mathbf{0}_C, a) \neq \emptyset$.

(61) $!^a = |^{\mathbf{0}_C} a$.

(62) $\mathrm{dom}(!^a) = \mathbf{0}_C$ and $\mathrm{cod}(!^a) = a$.

(63) $\hom(\mathbf{0}_C, a) = \{!^a\}$.

(64) $\mathrm{dom}\,\mathrm{in}_1(a+b) = a$ and $\mathrm{cod}\,\mathrm{in}_1(a+b) = a+b$.

(65) $\mathrm{dom}\,\mathrm{in}_2(a+b) = b$ and $\mathrm{cod}\,\mathrm{in}_2(a+b) = a+b$.

(66) $\hom(a, a+b) \neq \emptyset$ and $\hom(b, a+b) \neq \emptyset$.

(67) $a + b$ is a coproduct w.r.t. $\mathrm{in}_1(a + b)$ and $\mathrm{in}_2(a + b)$.

(68) C has finite coproduct.

(69) If $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$, then $\mathrm{in}_1(a + b)$ is coretraction and $\mathrm{in}_2(a + b)$ is coretraction.

Let us consider C, a, b. Then $\mathrm{in}_1(a+b)$ is a morphism from a to $a+b$. Then $\mathrm{in}_2(a+b)$ is a morphism from b to $a+b$. Let us consider C, a, b, c, and let f be a morphism from a to c, and let g be a morphism from b to c. Let us assume that $\hom(a, c) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$. The functor $\langle f, g\rangle$ yielding a morphism from $a + b$ to c is defined as follows:

(Def.26) $\langle f, g\rangle \cdot \mathrm{in}_1(a + b) = f$ and $\langle f, g\rangle \cdot \mathrm{in}_2(a + b) = g$.

Next we state several propositions:

(70) If $\hom(a, c) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$, then $\hom(a + b, c) \neq \emptyset$.

(71) $\langle \mathrm{in}_1(a + b), \mathrm{in}_2(a + b)\rangle = \mathrm{id}_{(a+b)}$.

(72) For every morphism f from a to c and for every morphism g from b to c and for every morphism h from c to d such that $\hom(a, c) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$ and $\hom(c, d) \neq \emptyset$ holds $\langle h \cdot f, h \cdot g\rangle = h \cdot \langle f, g\rangle$.

(73) For all morphisms f, k from a to c and for all morphisms g, h from b to c such that $\hom(a, c) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$ and $\langle f, g\rangle = \langle k, h\rangle$ holds $f = k$ and $g = h$.

(74) For every morphism f from a to c and for every morphism g from b to c such that $\hom(a, c) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$ and also f is epi or g is epi holds $\langle f, g\rangle$ is epi.

(75) a and $a + \mathbf{0}_C$ are isomorphic and a and $\mathbf{0}_C + a$ are isomorphic.

(76) $a + b$ and $b + a$ are isomorphic.

(77) $(a + b) + c$ and $a + (b + c)$ are isomorphic.

We now define two new functors. Let us consider C, a. The functor ∇_a yields a morphism from $a + a$ to a and is defined by:

(Def.27) $\nabla_a = \langle \mathrm{id}_a, \mathrm{id}_a\rangle$.

Let us consider C, a, b, c, d, and let f be a morphism from a to c, and let g be a morphism from b to d. The functor $f + g$ yielding a morphism from $a + b$ to $c + d$ is defined as follows:

(Def.28) $f + g = \langle \mathrm{in}_1(c + d) \cdot f, \mathrm{in}_2(c + d) \cdot g\rangle$.

The following propositions are true:

(78) If $\hom(a, c) \neq \emptyset$ and $\hom(b, d) \neq \emptyset$, then $\hom(a + b, c + d) \neq \emptyset$.

(79) $\mathrm{id}_a + \mathrm{id}_b = \mathrm{id}_{(a+b)}$.

(80) Let f be a morphism from a to c. Let h be a morphism from b to d. Then for every morphism g from c to e and for every morphism k from d to e such that $\hom(a, c) \neq \emptyset$ and $\hom(b, d) \neq \emptyset$ and $\hom(c, e) \neq \emptyset$ and $\hom(d, e) \neq \emptyset$ holds $\langle g, k\rangle \cdot (f + h) = \langle g \cdot f, k \cdot h\rangle$.

(81) For every morphism f from a to c and for every morphism g from b to c such that $\hom(a, c) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$ holds $\nabla_c \cdot (f + g) = \langle f, g\rangle$.

(82) Let f be a morphism from a to c. Let h be a morphism from b to d. Then for every morphism g from c to e and for every morphism k from d to s such that $\hom(a, c) \neq \emptyset$ and $\hom(b, d) \neq \emptyset$ and $\hom(c, e) \neq \emptyset$ and $\hom(d, s) \neq \emptyset$ holds $(g + k) \cdot (f + h) = g \cdot f + k \cdot h$.

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Algebra of Vector Functions

Hiroshi Yamazaki
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. We develop the algebra of partial vector functions, with domains being algebra of vector functions.

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The terminology and notation used in this paper have been introduced in the following papers: [10], [5], [2], [3], [1], [12], [9], [4], [6], [11], [8], and [7]. For simplicity we adopt the following rules: X, Y will denote sets, C will denote a non-empty set, c will denote an element of C, V will denote a real normed space, f, f_1, f_2, f_3 will denote partial functions from C to the carrier of V, and r, p will denote real numbers. We now define several new functors. Let us consider C, V, f_1, f_2. The functor $f_1 + f_2$ yielding a partial function from C to the carrier of V is defined as follows:

(Def.1) $\mathrm{dom}(f_1 + f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 + f_2)$ holds $(f_1 + f_2)(c) = f_1(c) + f_2(c)$.

The functor $f_1 - f_2$ yields a partial function from C to the carrier of V and is defined as follows:

(Def.2) $\mathrm{dom}(f_1 - f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 - f_2)$ holds $(f_1 - f_2)(c) = f_1(c) - f_2(c)$.

Let us consider C, and let us consider V, and let f_1 be a partial function from C to $\mathbb{R}$, and let us consider f_2. The functor $f_1\, f_2$ yielding a partial function from C to the carrier of V is defined by:

(Def.3) $\mathrm{dom}(f_1\, f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1\, f_2)$ holds $(f_1\, f_2)(c) = f_1(c) \cdot f_2(c)$.

Let us consider C, V, f, r. The functor $r\, f$ yielding a partial function from C to the carrier of V is defined as follows:

(Def.4) $\mathrm{dom}(r\, f) = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}(r\, f)$ holds $(r\, f)(c) = r \cdot f(c)$.

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Let us consider C, V, f. The functor $\|f\|$ yields a partial function from C to $\mathbb{R}$ and is defined by:

(Def.5) $\operatorname{dom}\|f\| = \operatorname{dom} f$ and for every c such that $c \in \operatorname{dom}\|f\|$ holds $\|f\|(c) = \|f(c)\|$.

The functor $-f$ yielding a partial function from C to the carrier of V is defined as follows:

(Def.6) $\operatorname{dom}(-f) = \operatorname{dom} f$ and for every c such that $c \in \operatorname{dom}(-f)$ holds $(-f)(c) = -f(c)$.

Next we state a number of propositions:

(1) $f = f_1 + f_2$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) + f_2(c)$.

(2) $f = f_1 - f_2$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) - f_2(c)$.

(3) For every partial function f_1 from C to $\mathbb{R}$ holds $f = f_1\, f_2$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) \cdot f_2(c)$.

(4) $f = r\, f_1$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = r \cdot f_1(c)$.

(5) For every partial function f from C to $\mathbb{R}$ holds $f = \|f_1\|$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = \|f_1(c)\|$.

(6) $f = -f_1$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = -f_1(c)$.

(7) For every partial function f_1 from C to $\mathbb{R}$ holds $\operatorname{dom}(f_1\, f_2) \setminus (f_1\, f_2)^{-1}\, \{0_V\} = (\operatorname{dom} f_1 \setminus f_1^{-1}\, \{0\}) \cap (\operatorname{dom} f_2 \setminus f_2^{-1}\, \{0_V\})$.

(8) $\|f\|^{-1}\, \{0\} = f^{-1}\, \{0_V\}$ and $(-f)^{-1}\, \{0_V\} = f^{-1}\, \{0_V\}$.

(9) If $r \neq 0$, then $(r\, f)^{-1}\, \{0_V\} = f^{-1}\, \{0_V\}$.

(10) $f_1 + f_2 = f_2 + f_1$.

(11) $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(12) For all partial functions f_1, f_2 from C to $\mathbb{R}$ and for every partial function f_3 from C to the carrier of V holds $(f_1\, f_2)\, f_3 = f_1\, (f_2\, f_3)$.

(13) For all partial functions f_1, f_2 from C to $\mathbb{R}$ holds $(f_1 + f_2)\, f_3 = f_1\, f_3 + f_2\, f_3$.

(14) For every partial function f_3 from C to $\mathbb{R}$ holds $f_3\, (f_1 + f_2) = f_3\, f_1 + f_3\, f_2$.

(15) For every partial function f_1 from C to $\mathbb{R}$ holds $r\, (f_1\, f_2) = (r\, f_1)\, f_2$.

(16) For every partial function f_1 from C to $\mathbb{R}$ holds $r\, (f_1\, f_2) = f_1\, (r\, f_2)$.

(17) For all partial functions f_1, f_2 from C to $\mathbb{R}$ holds $(f_1 - f_2)\, f_3 = f_1\, f_3 - f_2\, f_3$.

(18) For every partial function f_3 from C to $\mathbb{R}$ holds $f_3\, f_1 - f_3\, f_2 = f_3\, (f_1 - f_2)$.

(19) $r\, (f_1 + f_2) = r\, f_1 + r\, f_2$.

(20) $(r \cdot p) f = r (p f)$.

(21) $r (f_1 - f_2) = r f_1 - r f_2$.

(22) $f_1 - f_2 = (-1) (f_2 - f_1)$.

(23) $f_1 - (f_2 + f_3) = f_1 - f_2 - f_3$.

(24) $1 f = f$.

(25) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(26) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(27) For every partial function f_1 from C to $\mathbb{R}$ holds $\|f_1 f_2\| = |f_1| \, \|f_2\|$.

(28) $\|r f\| = |r| \, \|f\|$.

(29) $-f = (-1) f$.

(30) $--f = f$.

(31) $f_1 - f_2 = f_1 + -f_2$.

We now state a number of propositions:

(32) $f_1 - -f_2 = f_1 + f_2$.

(33) $(f_1 + f_2) \restriction X = f_1 \restriction X + f_2 \restriction X$ and $(f_1 + f_2) \restriction X = f_1 \restriction X + f_2$ and $(f_1 + f_2) \restriction X = f_1 + f_2 \restriction X$.

(34) For every partial function f_1 from C to $\mathbb{R}$ holds $(f_1 f_2) \restriction X = (f_1 \restriction X) (f_2 \restriction X)$ and $(f_1 f_2) \restriction X = (f_1 \restriction X) f_2$ and $(f_1 f_2) \restriction X = f_1 (f_2 \restriction X)$.

(35) $(-f) \restriction X = -f \restriction X$ and $\|f\| \restriction X = \|f \restriction X\|$.

(36) $(f_1 - f_2) \restriction X = f_1 \restriction X - f_2 \restriction X$ and $(f_1 - f_2) \restriction X = f_1 \restriction X - f_2$ and $(f_1 - f_2) \restriction X = f_1 - f_2 \restriction X$.

(37) $(r f) \restriction X = r (f \restriction X)$.

(38) f_1 is total and f_2 is total if and only if $f_1 + f_2$ is total and also f_1 is total and f_2 is total if and only if $f_1 - f_2$ is total.

(39) For every partial function f_1 from C to $\mathbb{R}$ holds f_1 is total and f_2 is total if and only if $f_1 f_2$ is total.

(40) f is total if and only if $r f$ is total.

(41) f is total if and only if $-f$ is total.

(42) f is total if and only if $\|f\|$ is total.

(43) If f_1 is total and f_2 is total, then $(f_1 + f_2)(c) = f_1(c) + f_2(c)$ and $(f_1 - f_2)(c) = f_1(c) - f_2(c)$.

(44) For every partial function f_1 from C to $\mathbb{R}$ such that f_1 is total and f_2 is total holds $(f_1 f_2)(c) = f_1(c) \cdot f_2(c)$.

(45) If f is total, then $(r f)(c) = r \cdot f(c)$.

(46) If f is total, then $(-f)(c) = -f(c)$ and $\|f\|(c) = \|f(c)\|$.

Let us consider C, V, f, Y. We say that f is bounded on Y if and only if:

(Def.7) there exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $\|f(c)\| \leq r$.

Next we state a number of propositions:

(47) f is bounded on Y if and only if there exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $\|f(c)\| \leq r$.

(48) If $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(49) If $X \cap \operatorname{dom} f = \emptyset$, then f is bounded on X.

(50) If $0 = r$, then $r\, f$ is bounded on Y.

(51) If f is bounded on Y, then $r\, f$ is bounded on Y.

(52) If f is bounded on Y, then $\|f\|$ is bounded on Y and $-f$ is bounded on Y.

(53) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(54) For every partial function f_1 from C to $\mathbb{R}$ such that f_1 is bounded on X and f_2 is bounded on Y holds $f_1\, f_2$ is bounded on $X \cap Y$.

(55) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 - f_2$ is bounded on $X \cap Y$.

(56) If f is bounded on X and f is bounded on Y, then f is bounded on $X \cup Y$.

(57) If f_1 is a constant on X and f_2 is a constant on Y, then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$.

(58) For every partial function f_1 from C to $\mathbb{R}$ such that f_1 is a constant on X and f_2 is a constant on Y holds $f_1\, f_2$ is a constant on $X \cap Y$.

(59) If f is a constant on Y, then $p\, f$ is a constant on Y.

(60) If f is a constant on Y, then $\|f\|$ is a constant on Y and $-f$ is a constant on Y.

(61) If f is a constant on Y, then f is bounded on Y.

(62) If f is a constant on Y, then for every r holds $r\, f$ is bounded on Y and $-f$ is bounded on Y and $\|f\|$ is bounded on Y.

(63) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(64) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$.

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On a Duality Between Weakly Separated Subspaces of Topological Spaces

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X be a topological space and let X_1 and X_2 be subspaces of X with the carriers A_1 and A_2, respectively. Recall that X_1 and X_2 are *weakly separated* if $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are separated (see [2] and also [1] for applications). Our purpose is to list a number of properties of such subspaces, supplementary to those given in [2]. Note that in the Mizar formalism the carrier of any topological space (hence the carrier of any its subspace) is always non–empty, therefore for convenience we list beforehand analogous properties of weakly separated subsets without any additional conditions.

To present the main results we first formulate a useful definition. We say that X_1 and X_2 *constitute a decomposition* of X if A_1 and A_2 are disjoint and the union of A_1 and A_2 covers the carrier of X (comp. [3]). We are ready now to present the following duality property between pairs of weakly separated subspaces : *If each pair of subspaces X_1, Y_1 and X_2, Y_2 of X constitutes a decomposition of X, then X_1 and X_2 are weakly separated iff Y_1 and Y_2 are weakly separated.* From this theorem we get immediately that under the same hypothesis, *X_1 and X_2 are separated iff X_1 misses X_2 and Y_1 and Y_2 are weakly separated.* Moreover, we show the following enlargement theorem : *If X_i and Y_i are subspaces of X such that Y_i is a subspace of X_i and $Y_1 \cup Y_2 = X_1 \cup X_2$ and if Y_1 and Y_2 are weakly separated, then X_1 and X_2 are weakly separated.* We show also the following dual extenuation theorem : *If X_i and Y_i are subspaces of X such that Y_i is a subspace of X_i and $Y_1 \cap Y_2 = X_1 \cap X_2$ and if X_1 and X_2 are weakly separated, then Y_1 and Y_2 are weakly separated.* At the end we give a few properties of weakly separated subspaces in subspaces.

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The papers [6], [7], [4], [8], [5], and [2] provide the notation and terminology for this paper.

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1. Certain Set–Decompositions of a Topological Space

In the sequel X denotes a topological space. Next we state the proposition

(1) For all subsets A, B of X holds $A^{\rm c} \setminus B^{\rm c} = B \setminus A$.

Let X be a topological space, and let A_1, A_2 be subsets of X. We say that A_1 and A_2 constitute a decomposition if and only if:

(Def.1) $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 =$ the carrier of X.

In the sequel A, A_1, A_2, B_1, B_2 are subsets of X. We now state a number of propositions:

(2) A_1 and A_2 constitute a decomposition if and only if $A_1 \cap A_2 = \emptyset_X$ and $A_1 \cup A_2 = \Omega_X$.

(3) If A_1 and A_2 constitute a decomposition, then A_2 and A_1 constitute a decomposition.

(4) If A_1 and A_2 constitute a decomposition, then $A_1 = {A_2}^{\rm c}$ and $A_2 = {A_1}^{\rm c}$.

(5) If $A_1 = {A_2}^{\rm c}$ or $A_2 = {A_1}^{\rm c}$, then A_1 and A_2 constitute a decomposition.

(6) A and $A^{\rm c}$ constitute a decomposition and $A^{\rm c}$ and A constitute a decomposition.

(7) $\emptyset_X$ and Ω_X constitute a decomposition and Ω_X and $\emptyset_X$ constitute a decomposition.

(8) If A is non-empty, then A and A do not constitute a decomposition.

(9) If A_1 and A constitute a decomposition and A and A_2 constitute a decomposition, then $A_1 = A_2$.

(10) If A_1 and A_2 constitute a decomposition, then $\overline{A_1}$ and $\mathrm{Int}\, A_2$ constitute a decomposition and $\mathrm{Int}\, A_1$ and $\overline{A_2}$ constitute a decomposition.

(11) $\overline{A}$ and $\mathrm{Int}(A^{\rm c})$ constitute a decomposition and $\overline{A^{\rm c}}$ and $\mathrm{Int}\, A$ constitute a decomposition and $\mathrm{Int}\, A$ and $\overline{A^{\rm c}}$ constitute a decomposition and $\mathrm{Int}(A^{\rm c})$ and $\overline{A}$ constitute a decomposition.

(12) If A_1 and A_2 constitute a decomposition, then A_1 is open if and only if A_2 is closed.

(13) If A_1 and A_2 constitute a decomposition, then A_1 is closed if and only if A_2 is open.

(14) If A_1 and A_2 constitute a decomposition and B_1 and B_2 constitute a decomposition, then $A_1 \cap B_1$ and $A_2 \cup B_2$ constitute a decomposition.

(15) If A_1 and A_2 constitute a decomposition and B_1 and B_2 constitute a decomposition, then $A_1 \cup B_1$ and $A_2 \cap B_2$ constitute a decomposition.

2. Duality between Pairs of Weakly Separated Subsets

In the sequel X will denote a topological space and A_1, A_2 will denote subsets of X. Next we state a number of propositions:

(16) For all subsets A_1, A_2, C_1, C_2 of X such that A_1 and C_1 constitute a decomposition and A_2 and C_2 constitute a decomposition holds A_1 and A_2 are weakly separated if and only if C_1 and C_2 are weakly separated.

(17) A_1 and A_2 are weakly separated if and only if ${A_1}^{\mathrm{c}}$ and ${A_2}^{\mathrm{c}}$ are weakly separated.

(18) For all subsets A_1, A_2, C_1, C_2 of X such that A_1 and C_1 constitute a decomposition and A_2 and C_2 constitute a decomposition holds if A_1 and A_2 are separated, then C_1 and C_2 are weakly separated.

(19) For all subsets A_1, A_2, C_1, C_2 of X such that A_1 and C_1 constitute a decomposition and A_2 and C_2 constitute a decomposition holds if $A_1 \cap A_2 = \emptyset$ and C_1 and C_2 are weakly separated, then A_1 and A_2 are separated.

(20) For all subsets A_1, A_2, C_1, C_2 of X such that A_1 and C_1 constitute a decomposition and A_2 and C_2 constitute a decomposition holds if $C_1 \cup C_2 =$ the carrier of X and C_1 and C_2 are weakly separated, then A_1 and A_2 are separated.

(21) If A_1 and A_2 constitute a decomposition, then A_1 and A_2 are weakly separated if and only if A_1 and A_2 are separated.

(22) A_1 and A_2 are weakly separated if and only if $(A_1 \cup A_2) \setminus A_1$ and $(A_1 \cup A_2) \setminus A_2$ are separated.

(23) For all subsets A_1, A_2, C_1, C_2 of X such that $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C_1 \cup C_2 = A_1 \cup A_2$ holds if C_1 and C_2 are weakly separated, then A_1 and A_2 are weakly separated.

(24) A_1 and A_2 are weakly separated if and only if $A_1 \setminus A_1 \cap A_2$ and $A_2 \setminus A_1 \cap A_2$ are separated.

(25) For all subsets A_1, A_2, C_1, C_2 of X such that $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C_1 \cap C_2 = A_1 \cap A_2$ holds if A_1 and A_2 are weakly separated, then C_1 and C_2 are weakly separated.

In the sequel X_0 will denote a subspace of X and B_1, B_2 will denote subsets of X_0. One can prove the following propositions:

(26) If $B_1 = A_1$ and $B_2 = A_2$, then A_1 and A_2 are separated if and only if B_1 and B_2 are separated.

(27) If $B_1 =$ (the carrier of X_0) $\cap A_1$ and $B_2 =$ (the carrier of X_0) $\cap A_2$, then if A_1 and A_2 are separated, then B_1 and B_2 are separated.

(28) If $B_1 = A_1$ and $B_2 = A_2$, then A_1 and A_2 are weakly separated if and only if B_1 and B_2 are weakly separated.

(29) If $B_1 =$ (the carrier of X_0) $\cap A_1$ and $B_2 =$ (the carrier of X_0) $\cap A_2$, then if A_1 and A_2 are weakly separated, then B_1 and B_2 are weakly separated.

3. Certain Subspace–Decompositions of a Topological Space

Let X be a topological space, and let X_1, X_2 be subspaces of X. We say that X_1 and X_2 constitute a decomposition if and only if:

(Def.2) for all subsets A_1, A_2 of X such that $A_1 =$ the carrier of X_1 and $A_2 =$ the carrier of X_2 holds A_1 and A_2 constitute a decomposition.

In the sequel X_0, X_1, X_2, Y_1, Y_2 denote subspaces of X. The following propositions are true:

(30) X_1 and X_2 constitute a decomposition if and only if X_1 misses X_2 and the topological structure of $X = X_1 \cup X_2$.

(31) If X_1 and X_2 constitute a decomposition, then X_2 and X_1 constitute a decomposition.

(32) X_0 and X_0 do not constitute a decomposition.

(33) If X_1 and X_0 constitute a decomposition and X_0 and X_2 constitute a decomposition, then the topological structure of $X_1 =$ the topological structure of X_2.

(34) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds $Y_1 \cup Y_2 =$ the topological structure of X if and only if X_1 misses X_2.

(35) If X_1 and X_2 constitute a decomposition, then X_1 is open if and only if X_2 is closed.

(36) If X_1 and X_2 constitute a decomposition, then X_1 is closed if and only if X_2 is open.

(37) If X_1 meets Y_1 and X_1 and X_2 constitute a decomposition and Y_1 and Y_2 constitute a decomposition, then $X_1 \cap Y_1$ and $X_2 \cup Y_2$ constitute a decomposition.

(38) If X_2 meets Y_2 and X_1 and X_2 constitute a decomposition and Y_1 and Y_2 constitute a decomposition, then $X_1 \cup Y_1$ and $X_2 \cap Y_2$ constitute a decomposition.

4. Duality between Pairs of Weakly Separated Subspaces

In the sequel X is a topological space. We now state several propositions:

(39) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds X_1 and X_2 are weakly separated if and only if Y_1 and Y_2 are weakly separated.

(40) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds if X_1 and X_2 are separated, then Y_1 and Y_2 are weakly separated.

(41) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds if X_1

misses X_2 and Y_1 and Y_2 are weakly separated, then X_1 and X_2 are separated.

(42) For all subspaces X_1, X_2, Y_1, Y_2 of X such that X_1 and Y_1 constitute a decomposition and X_2 and Y_2 constitute a decomposition holds if $Y_1 \cup Y_2 =$ the topological structure of X and Y_1 and Y_2 are weakly separated, then X_1 and X_2 are separated.

(43) For all subspaces X_1, X_2 of X such that X_1 and X_2 constitute a decomposition holds X_1 and X_2 are weakly separated if and only if X_1 and X_2 are separated.

(44) For all subspaces X_1, X_2, Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 and $Y_1 \cup Y_2 = X_1 \cup X_2$ holds if Y_1 and Y_2 are weakly separated, then X_1 and X_2 are weakly separated.

(45) For all subspaces X_1, X_2, Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 and Y_1 meets Y_2 and $Y_1 \cap Y_2 = X_1 \cap X_2$ holds if X_1 and X_2 are weakly separated, then Y_1 and Y_2 are weakly separated.

In the sequel X_0 will denote a subspace of X. Next we state four propositions:

(46) For all subspaces X_1, X_2 of X and for all subspaces Y_1, Y_2 of X_0 such that the carrier of $X_1 =$ the carrier of Y_1 and the carrier of $X_2 =$ the carrier of Y_2 holds X_1 and X_2 are separated if and only if Y_1 and Y_2 are separated.

(47) For all subspaces X_1, X_2 of X such that X_1 meets X_0 and X_2 meets X_0 and for all subspaces Y_1, Y_2 of X_0 such that $Y_1 = X_1 \cap X_0$ and $Y_2 = X_2 \cap X_0$ holds if X_1 and X_2 are separated, then Y_1 and Y_2 are separated.

(48) For all subspaces X_1, X_2 of X and for all subspaces Y_1, Y_2 of X_0 such that the carrier of $X_1 =$ the carrier of Y_1 and the carrier of $X_2 =$ the carrier of Y_2 holds X_1 and X_2 are weakly separated if and only if Y_1 and Y_2 are weakly separated.

(49) For all subspaces X_1, X_2 of X such that X_1 meets X_0 and X_2 meets X_0 and for all subspaces Y_1, Y_2 of X_0 such that $Y_1 = X_1 \cap X_0$ and $Y_2 = X_2 \cap X_0$ holds if X_1 and X_2 are weakly separated, then Y_1 and Y_2 are weakly separated.

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Basic Petri Net Concepts

Pauline N. Kawamoto
Shinshu University
Nagano

Yasushi Fuwa
Shinshu University
Nagano

Yatsuka Nakamura
Shinshu University
Nagano

Summary. This article presents the basic place/transition net structure definition for building various types of Petri nets. The basic net structure fields include places, transitions, and arcs (place-transition, transition-place) which may be supplemented with other fields (e.g., capacity, weight, marking, etc.) as needed. The theorems included in this article are divided into the following categories: deadlocks, traps, and dual net theorems. Here, a dual net is taken as the result of inverting all arcs (place-transition arcs to transition-place arcs and vice-versa) in the original net.

MML Identifier: `PETRI`.

The papers [3], [5], [6], [7], [1], [4], and [2] provide the terminology and notation for this paper.

1. Basic Place/Transition Net Structure Definition

Let A, B be non-empty sets. Observe that there exists a non-empty relation between A and B.

Let A, B be non-empty sets, and let r be a non-empty relation between A and B. We see that the element of r is an element of $[: A, B :]$.

We consider place/transitions net structures which are systems

$\langle$places, transitions, S-T arcs, T-S arcs$\rangle$,

where the places, the transitions constitute non-empty sets, the S-T arcs constitute a non-empty relation between the places and the transitions, and the T-S arcs constitute a non-empty relation between the transitions and the places.

In the sequel P_1 will denote a place/transitions net structure. We now define several new modes. Let us consider P_1. A place of P_1 is an element of the places of P_1.

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A transition of P_1 is an element of the transitions of P_1.

An S-T arc of P_1 is an element of the S-T arcs of P_1.

A T-S arc of P_1 is an element of the T-S arcs of P_1.

Let us consider P_1, and let x be an S-T arc of P_1. Then $x_{\mathbf{1}}$ is a place of P_1. Then $x_{\mathbf{2}}$ is a transition of P_1. Let us consider P_1, and let x be a T-S arc of P_1. Then $x_{\mathbf{1}}$ is a transition of P_1. Then $x_{\mathbf{2}}$ is a place of P_1.

The scheme *Set_of_Elements* deals with a non-empty set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\{x : \mathcal{P}[x]\}$, where x ranges over elements of $\mathcal{A}$, is a subset of $\mathcal{A}$

for all values of the parameters.

In the sequel S_0 will denote a set of places of P_1. We now define two new functors. Let us consider P_1, S_0. The functor ${}^*S_0$ yielding a set of transitions of P_1 is defined as follows:

(Def.1) ${}^*S_0 = \{t : \bigvee_f \bigvee_s [s \in S_0 \wedge f = \langle t, s\rangle]\}$, where t ranges over transitions of P_1, and f ranges over T-S arcs of P_1, and s ranges over places of P_1.

The functor ${S_0}^*$ yielding a set of transitions of P_1 is defined as follows:

(Def.2) ${S_0}^* = \{t : \bigvee_f \bigvee_s [s \in S_0 \wedge f = \langle s, t\rangle]\}$, where t ranges over transitions of P_1, and f ranges over S-T arcs of P_1, and s ranges over places of P_1.

Next we state four propositions:

(1) ${}^*S_0 = \{f_{\mathbf{1}} : f_{\mathbf{2}} \in S_0\}$, where f ranges over T-S arcs of P_1.

(2) For an arbitrary x holds $x \in {}^*S_0$ if and only if there exists a T-S arc f of P_1 and there exists a place s of P_1 such that $s \in S_0$ and $f = \langle x, s\rangle$.

(3) ${S_0}^* = \{f_{\mathbf{2}} : f_{\mathbf{1}} \in S_0\}$, where f ranges over S-T arcs of P_1.

(4) For an arbitrary x holds $x \in {S_0}^*$ if and only if there exists an S-T arc f of P_1 and there exists a place s of P_1 such that $s \in S_0$ and $f = \langle s, x\rangle$.

In the sequel T_0 is a set of transitions of P_1. We now define two new functors. Let us consider P_1, T_0. The functor ${}^*T_0$ yields a set of places of P_1 and is defined by:

(Def.3) ${}^*T_0 = \{s : \bigvee_f \bigvee_t [t \in T_0 \wedge f = \langle s, t\rangle]\}$, where s ranges over places of P_1, and f ranges over S-T arcs of P_1, and t ranges over transitions of P_1.

The functor ${T_0}^*$ yielding a set of places of P_1 is defined by:

(Def.4) ${T_0}^* = \{s : \bigvee_f \bigvee_t [t \in T_0 \wedge f = \langle t, s\rangle]\}$, where s ranges over places of P_1, and f ranges over T-S arcs of P_1, and t ranges over transitions of P_1.

Next we state several propositions:

(5) ${}^*T_0 = \{f_{\mathbf{1}} : f_{\mathbf{2}} \in T_0\}$, where f ranges over S-T arcs of P_1.

(6) For an arbitrary x holds $x \in {}^*T_0$ if and only if there exists an S-T arc f of P_1 and there exists a transition t of P_1 such that $t \in T_0$ and $f = \langle x, t\rangle$.

(7) ${T_0}^* = \{f_{\mathbf{2}} : f_{\mathbf{1}} \in T_0\}$, where f ranges over T-S arcs of P_1.

(8) For an arbitrary x holds $x \in {T_0}^*$ if and only if there exists a T-S arc f of P_1 and there exists a transition t of P_1 such that $t \in T_0$ and $f = \langle t, x\rangle$.

(9) ${}^*(\emptyset_{\text{the places of } P_1}) = \emptyset$.

(10) $(\emptyset_{\text{the places of } P_1})^* = \emptyset$.

(11) ${}^*(\emptyset_{\text{the transitions of } P_1}) = \emptyset$.

(12) $(\emptyset_{\text{the transitions of } P_1})^* = \emptyset$.

2. Deadlocks

We now define two new attributes. Let us consider P_1. A set of places of P_1 is deadlock-like if:

(Def.5) *it is a subset of it*.

A place/transitions net structure has deadlocks if:

(Def.6) there exists a set of places of it which is deadlock-like.

3. Traps

We now define two new attributes. Let us consider P_1. A set of places of P_1 is trap-like if:

(Def.7) it* is a subset of *it.

A place/transitions net structure has traps if:

(Def.8) there exists a set of places of it which is trap-like.

Let A, B be non-empty sets, and let r be a non-empty relation between A and B. Then $r^\smile$ is a non-empty relation between B and A.

4. Duality Theorems for Place/Transition Nets

Let us consider P_1. The functor ${P_1}^\circ$ yields a strict place/transitions net structure and is defined by:

(Def.9) ${P_1}^\circ = \langle$the places of P_1, the transitions of P_1, (the T-S arcs of $P_1)^\smile$, (the S-T arcs of $P_1)^\smile\rangle$.

One can prove the following propositions:

(13) $({P_1}^\circ)^\circ =$ the place/transitions net structure of P_1.

(14) The places of $P_1 =$ the places of ${P_1}^\circ$ and the transitions of $P_1 =$ the transitions of ${P_1}^\circ$ and (the S-T arcs of $P_1)^\smile =$ the T-S arcs of ${P_1}^\circ$ and (the T-S arcs of $P_1)^\smile =$ the S-T arcs of ${P_1}^\circ$.

We now define several new functors. Let us consider P_1, and let S_0 be a set of places of P_1. The functor ${S_0}^\circ$ yields a set of places of ${P_1}^\circ$ and is defined as follows:

(Def.10) ${S_0}^\circ = S_0$.

Let us consider P_1, and let s be a place of P_1. The functor s° yields a place of ${P_1}^\circ$ and is defined by:

(Def.11) $s^\circ = s$.

Let us consider P_1, and let S_0 be a set of places of ${P_1}^\circ$. The functor ${}^\circ S_0$ yields a set of places of P_1 and is defined by:

(Def.12) ${}^\circ S_0 = S_0$.

Let us consider P_1, and let s be a place of ${P_1}^\circ$. The functor ${}^\circ s$ yields a place of P_1 and is defined by:

(Def.13) ${}^\circ s = s$.

Let us consider P_1, and let T_0 be a set of transitions of P_1. The functor ${T_0}^\circ$ yielding a set of transitions of ${P_1}^\circ$ is defined by:

(Def.14) ${T_0}^\circ = T_0$.

Let us consider P_1, and let t be a transition of P_1. The functor t° yields a transition of ${P_1}^\circ$ and is defined as follows:

(Def.15) $t^\circ = t$.

Let us consider P_1, and let T_0 be a set of transitions of ${P_1}^\circ$. The functor ${}^\circ T_0$ yielding a set of transitions of P_1 is defined by:

(Def.16) ${}^\circ T_0 = T_0$.

Let us consider P_1, and let t be a transition of ${P_1}^\circ$. The functor ${}^\circ t$ yielding a transition of P_1 is defined by:

(Def.17) ${}^\circ t = t$.

In the sequel S will denote a set of places of P_1. Next we state several propositions:

(15) $(S^\circ)^* = {}^*S$.

(16) ${}^*(S^\circ) = S^*$.

(17) S is deadlock-like if and only if S° is trap-like.

(18) S is trap-like if and only if S° is deadlock-like.

(19) For every P_1 being a place/transitions net structure and for every transition t of P_1 and for every S_0 being a set of places of P_1 holds $t \in {S_0}^*$ if and only if ${}^*\{t\} \cap S_0 \neq \emptyset$.

(20) For every P_1 being a place/transitions net structure and for every transition t of P_1 and for every S_0 being a set of places of P_1 holds $t \in {}^*S_0$ if and only if $\{t\}^* \cap S_0 \neq \emptyset$.

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Finite Topological Spaces

Hiroshi Imura
Shinshu University
Nagano

Masayoshi Eguchi
Shinshu University
Nagano

Summary. By borrowing the concept of neighborhood from the theory of topological space in continuous cases and extending it to a discrete case such as a space of lattice points we have defined such concepts as boundaries, closures, interiors, isolated points, and connected points as in the case of continuity. We have proved various properties which are satisfied by these concepts.

MML Identifier: `FIN_TOPO`.

The articles [15], [8], [2], [5], [16], [6], [14], [19], [10], [12], [17], [9], [11], [3], [4], [13], [7], [18], and [1] provide the notation and terminology for this paper. The scheme *Set_of_Elements* deals with a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(x) : \mathcal{P}[x]\}$, where x ranges over elements of $\mathcal{A}$, is a subset of $\mathcal{A}$

for all values of the parameters.

One can prove the following propositions:

(1) Let A be a set. Let f be a finite sequence of elements of 2^A. Then if for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $\pi_i f \subseteq \pi_{i+1} f$, then for all natural numbers i, j such that $i \leq j$ and $1 \leq i$ and $j \leq \operatorname{len} f$ holds $\pi_i f \subseteq \pi_j f$.

(2) Let A be a set. Let f be a finite sequence of elements of 2^A. Suppose for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $\pi_i f \subseteq \pi_{i+1} f$. Then for all natural numbers i, j such that $i < j$ and $1 \leq i$ and $j \leq \operatorname{len} f$ and $\pi_j f \subseteq \pi_i f$ and for every natural number k such that $i \leq k$ and $k \leq j$ holds $\pi_j f = \pi_k f$.

(3) For every set F such that F is finite and $F \neq \emptyset$ and for all sets B, C such that $B \in F$ and $C \in F$ holds $B \subseteq C$ or $C \subseteq B$ there exists a set m such that $m \in F$ and for every set C such that $C \in F$ holds $C \subseteq m$.

(4) For all sets x, A holds $x \subseteq A$ if and only if $x \in 2^A$.

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(5) For every function f if for every natural number i holds $f(i) \subseteq f(i+1)$, and for all natural numbers i, j such that $i \leq j$ holds $f(i) \subseteq f(j)$.

The scheme *MaxFinSeqEx* deals with a non-empty set $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, a subset $\mathcal{C}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$ and states that:

there exists a finite sequence f of elements of $2^{\mathcal{A}}$ such that $\operatorname{len} f > 0$ and $\pi_1 f = \mathcal{C}$ and for every natural number i such that $i > 0$ and $i < \operatorname{len} f$ holds $\pi_{i+1} f = \mathcal{F}(\pi_i f)$ and $\mathcal{F}(\pi_{\operatorname{len} f} f) = \pi_{\operatorname{len} f} f$ and for all natural numbers i, j such that $i > 0$ and $i < j$ and $j \leq \operatorname{len} f$ holds $\pi_i f \subseteq \mathcal{B}$ and $\pi_i f \subseteq \pi_j f$ and $\pi_i f \neq \pi_j f$ provided the parameters meet the following requirements:

- $\mathcal{B}$ is finite,
- $\mathcal{C} \subseteq \mathcal{B}$,
- for every subset A of $\mathcal{A}$ such that $A \subseteq \mathcal{B}$ holds $A \subseteq \mathcal{F}(A)$ and $\mathcal{F}(A) \subseteq \mathcal{B}$.

We consider finite topology spaces which are extension of a 1-sorted structure and are systems

$\langle$a carrier, a neighbour-map$\rangle$,

where the carrier is a non-empty set and the neighbour-map is a function from the carrier into $2^{\text{the carrier}}$.

In the sequel F_1 denotes a finite topology space. We now define two new modes. Let F_1 be a 1-sorted structure. An element of F_1 is an element of the carrier of F_1.

A subset of F_1 is a subset of the carrier of F_1.

In the sequel x, y are elements of F_1. Let F_1 be a finite topology space, and let x be an element of F_1. The functor $U(x)$ yields a subset of F_1 and is defined as follows:

(Def.1) $U(x) = (\text{the neighbour-map of } F_1)(x)$.

One can prove the following proposition

(6) For every F_1 being a finite topology space and for every element x of F_1 holds $U(x) = (\text{the neighbour-map of } F_1)(x)$.

We now define three new constructions. Let x be arbitrary, and let y be a subset of $\{x\}$. Then $x \dot{\longmapsto} y$ is a function from $\{x\}$ into $2^{\{x\}}$. The strict finite topology space $\mathrm{FT}_{\{0\}}$ is defined as follows:

(Def.2) $\mathrm{FT}_{\{0\}} = \langle \{0\,\mathbf{qua}\,\text{any}\}, 0 \dot{\longmapsto} \Omega_{\{0\,\mathbf{qua}\,any\}} \rangle$.

A finite topology space is filled if:

(Def.3) for every element x of it holds $x \in U(x)$.

A 1-sorted structure is finite if:

(Def.4) the carrier of it is finite.

One can prove the following two propositions:

(7) $\mathrm{FT}_{\{0\}}$ is filled.

(8) $\mathrm{FT}_{\{0\}}$ is finite.

Let us observe that there exists a finite filled strict finite topology space.

Let T be a 1-sorted structure, and let F be a set. We say that F is a cover of T if and only if:

(Def.5) the carrier of $T \subseteq \bigcup F$.

Next we state the proposition

(9) For every F_1 being a filled finite topology space holds $\{U(x)\}$, where x ranges over elements of F_1, is a cover of F_1.

In the sequel A is a subset of F_1. Let us consider F_1, and let A be a subset of F_1. The functor A^δ yielding a subset of F_1 is defined as follows:

(Def.6) $A^\delta = \{x : U(x) \cap A \neq \emptyset \wedge U(x) \cap A^{\mathrm{c}} \neq \emptyset\}$.

The following proposition is true

(10) $x \in A^\delta$ if and only if $U(x) \cap A \neq \emptyset$ and $U(x) \cap A^{\mathrm{c}} \neq \emptyset$.

We now define two new functors. Let us consider F_1, and let A be a subset of F_1. The functor A^{δ_i} yielding a subset of F_1 is defined as follows:

(Def.7) $A^{\delta_i} = A \cap A^\delta$.

The functor A^{δ_o} yields a subset of F_1 and is defined as follows:

(Def.8) $A^{\delta_o} = A^{\mathrm{c}} \cap A^\delta$.

Next we state the proposition

(11) $A^\delta = A^{\delta_i} \cup A^{\delta_o}$.

We now define several new constructions. Let us consider F_1, and let A be a subset of F_1. The functor A^i yielding a subset of F_1 is defined by:

(Def.9) $A^i = \{x : U(x) \subseteq A\}$.

The functor A^b yielding a subset of F_1 is defined as follows:

(Def.10) $A^b = \{x : U(x) \cap A \neq \emptyset\}$.

The functor A^s yielding a subset of F_1 is defined by:

(Def.11) $A^s = \{x : x \in A \wedge (U(x) \setminus \{x\}) \cap A = \emptyset\}$.

Let us consider F_1, and let A be a subset of F_1. The functor A^n yielding a subset of F_1 is defined as follows:

(Def.12) $A^n = A \setminus A^s$.

The functor A^f yields a subset of F_1 and is defined as follows:

(Def.13) $A^f = \{x : \bigvee_y [y \in A \wedge x \in U(y)]\}$.

A finite topology space is symmetric if:

(Def.14) for all elements x, y of the carrier of it such that $y \in U(x)$ holds $x \in U(y)$.

The following propositions are true:

(12) $x \in A^i$ if and only if $U(x) \subseteq A$.

(13) $x \in A^b$ if and only if $U(x) \cap A \neq \emptyset$.

(14) $x \in A^s$ if and only if $x \in A$ and $(U(x) \setminus \{x\}) \cap A = \emptyset$.

(15) $x \in A^n$ if and only if $x \in A$ and $(U(x) \setminus \{x\}) \cap A \neq \emptyset$.

(16) $x \in A^f$ if and only if there exists y such that $y \in A$ and $x \in U(y)$.

(17) F_1 is symmetric if and only if for every A holds $A^b = A^f$.

In the sequel F will be a subset of F_1. We now define five new constructions. Let us consider F_1. A subset of F_1 is open if:

(Def.15) $\text{it} = \text{it}^i$.

A subset of F_1 is closed if:

(Def.16) $\text{it} = \text{it}^b$.

A subset of F_1 is connected if:

(Def.17) for all subsets B, C of F_1 such that $\text{it} = B \cup C$ and $B \neq \emptyset$ and $C \neq \emptyset$ and $B \cap C = \emptyset$ holds $B^b \cap C \neq \emptyset$.

Let us consider F_1, and let A be a subset of F_1. The functor A^{f_b} yields a subset of F_1 and is defined as follows:

(Def.18) $A^{f_b} = \bigcap\{F : A \subseteq F \wedge F \text{ is closed}\}$.

The functor A^{f_i} yielding a subset of F_1 is defined by:

(Def.19) $A^{f_i} = \bigcup\{F : A \subseteq F \wedge F \text{ is open}\}$.

Next we state a number of propositions:

(18) For every F_1 being a filled finite topology space and for every subset A of F_1 holds $A \subseteq A^b$.

(19) For every F_1 being a finite topology space and for all subsets A, B of F_1 such that $A \subseteq B$ holds $A^b \subseteq B^b$.

(20) Let F_1 be a filled finite finite topology space. Let A be a subset of F_1. Then A is connected if and only if for every element x of F_1 such that $x \in A$ there exists a finite sequence S of elements of $2^{\text{the carrier of } F_1}$ such that $\operatorname{len} S > 0$ and $\pi_1 S = \{x\}$ and for every natural number i such that $i > 0$ and $i < \operatorname{len} S$ holds $\pi_{i+1} S = (\pi_i S)^b \cap A$ and $A \subseteq \pi_{\operatorname{len} S} S$.

(21) For every non-empty set E and for every subset A of E and for every element x of E holds $x \in A^{\mathrm{c}}$ if and only if $x \notin A$.

(22) $((A^{\mathrm{c}})^i)^{\mathrm{c}} = A^b$.

(23) $((A^{\mathrm{c}})^b)^{\mathrm{c}} = A^i$.

(24) $A^\delta = A^b \cap (A^{\mathrm{c}})^b$.

(25) $(A^{\mathrm{c}})^\delta = A^\delta$.

(26) If $x \in A^s$, then $x \notin (A \setminus \{x\})^b$.

(27) If $A^s \neq \emptyset$ and $\operatorname{card} A > 1$, then A is connected.

(28) For every F_1 being a filled finite topology space and for every subset A of F_1 holds $A^i \subseteq A$.

(29) For every set E and for all subsets A, B of E holds $A = B$ if and only if $A^{\mathrm{c}} = B^{\mathrm{c}}$.

(30) If A is open, then A^{c} is closed.

(31) If A is closed, then A^{c} is open.

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Sets and Functions of Trees and Joining Operations of Trees

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warszawa

Summary. In the article we deal with sets of trees and functions yielding trees. So, we introduce the sets of all trees, all finite trees and of all trees decorated by elements from some set. Next, the functions and the finite sequences yielding (finite, decorated) trees are introduced. There are shown some convenient but technical lemmas and clusters concerning with those concepts. In the fourth section we deal with trees decorated by Cartesian product and we introduce the concept of a tree called a substitution of structure of some finite tree. Finally, we introduce the operations of joining trees, i.e. for the finite sequence of trees we define the tree which is made by joining the trees from the sequence by common root. For one and two trees there are introduced the same operations.

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The notation and terminology used here are introduced in the following papers: [17], [13], [3], [9], [18], [6], [11], [16], [15], [19], [1], [10], [14], [7], [8], [4], [5], [12], and [2].

1. Finite sets

For simplicity we adopt the following rules: x, y will be arbitrary, i, n will be natural numbers, p, q will be finite sequences, X, Y will be sets, and f will be a function. Let X be a set. Observe that there exists a finite subset of X and every finite sequence-like function is finite.

Let X be a non-empty set. One can check that there exists a finite non-empty subset of X.

Let X be a finite set. Observe that every subset of X is finite.

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Let us consider x. Then $\{x\}$ is a finite non-empty set. Let us consider y. Then $\{x, y\}$ is a finite non-empty set. Let us consider n. Then $\operatorname{Seg} n$ is a finite set of natural numbers. Then the elementary tree of n is a finite tree.

2. Sets of trees

We now define five new constructions. The non-empty set Trees is defined by:

(Def.1) Trees is the set of all trees.

The non-empty subset FinTrees of Trees is defined as follows:

(Def.2) FinTrees is the set of all finite trees.

A set is constituted of trees if:

(Def.3) for every x such that $x \in$ it holds x is a tree.

A set is constituted of finite trees if:

(Def.4) for every x such that $x \in$ it holds x is a finite tree.

A set is constituted of decorated trees if:

(Def.5) for every x such that $x \in$ it holds x is a decorated tree.

Next we state a number of propositions:

(1) X is constituted of trees if and only if $X \subseteq$ Trees.

(2) X is constituted of finite trees if and only if $X \subseteq$ FinTrees.

(3) X is constituted of trees and Y is constituted of trees if and only if $X \cup Y$ is constituted of trees.

(4) If X is constituted of trees and Y is constituted of trees, then $X \dot{-} Y$ is constituted of trees.

(5) If X is constituted of trees, then $X \cap Y$ is constituted of trees and $Y \cap X$ is constituted of trees and $X \setminus Y$ is constituted of trees.

(6) X is constituted of finite trees and Y is constituted of finite trees if and only if $X \cup Y$ is constituted of finite trees.

(7) If X is constituted of finite trees and Y is constituted of finite trees, then $X \dot{-} Y$ is constituted of finite trees.

(8) If X is constituted of finite trees, then $X \cap Y$ is constituted of finite trees and $Y \cap X$ is constituted of finite trees and $X \setminus Y$ is constituted of finite trees.

(9) X is constituted of decorated trees and Y is constituted of decorated trees if and only if $X \cup Y$ is constituted of decorated trees.

(10) If X is constituted of decorated trees and Y is constituted of decorated trees, then $X \dot{-} Y$ is constituted of decorated trees.

(11) If X is constituted of decorated trees, then $X \cap Y$ is constituted of decorated trees and $Y \cap X$ is constituted of decorated trees and $X \setminus Y$ is constituted of decorated trees.

(12) $\emptyset$ is constituted of trees, constituted of finite trees and constituted of decorated trees.

(13) $\{x\}$ is constituted of trees if and only if x is a tree.

(14) $\{x\}$ is constituted of finite trees if and only if x is a finite tree.

(15) $\{x\}$ is constituted of decorated trees if and only if x is a decorated tree.

(16) $\{x, y\}$ is constituted of trees if and only if x is a tree and y is a tree.

(17) $\{x, y\}$ is constituted of finite trees if and only if x is a finite tree and y is a finite tree.

(18) $\{x, y\}$ is constituted of decorated trees if and only if x is a decorated tree and y is a decorated tree.

(19) If X is constituted of trees and $Y \subseteq X$, then Y is constituted of trees.

(20) If X is constituted of finite trees and $Y \subseteq X$, then Y is constituted of finite trees.

(21) If X is constituted of decorated trees and $Y \subseteq X$, then Y is constituted of decorated trees.

We now define three new constructions. One can verify the following observations:

* there exists a finite constituted of trees constituted of finite trees non-empty set,
* there exists a finite constituted of decorated trees non-empty set, and
* every constituted of finite trees set is constituted of trees.

Let X be a constituted of trees set. One can check that every subset of X is constituted of trees.

Let X be a constituted of finite trees set. One can check that every subset of X is constituted of finite trees.

Let X be a constituted of decorated trees set. Note that every subset of X is constituted of decorated trees.

Let D be a constituted of trees non-empty set. We see that the element of D is a tree. Let D be a constituted of finite trees non-empty set. We see that the element of D is a finite tree. Let D be a constituted of decorated trees non-empty set. We see that the element of D is a decorated tree. Let us note that it makes sense to consider the following constant. Then Trees is a constituted of trees non-empty set. Let us observe that there exists a constituted of finite trees non-empty subset of Trees.

Let us note that it makes sense to consider the following constant. Then FinTrees is a constituted of finite trees non-empty subset of Trees. Let D be a non-empty set. A set is called a set of trees decorated by D if:

(Def.6) for every x such that $x \in$ it holds x is a tree decorated by D.

Let D be a non-empty set. Note that every set of trees decorated by D is constituted of decorated trees.

Let D be a non-empty set. Note that there exists a set of trees decorated by D which is finite and non-empty.

Let D be a non-empty set, and let E be a non-empty set of trees decorated by D. We see that the element of E is a tree decorated by D. Let T be a tree, and let D be a non-empty set. Then D^T is a non-empty set of trees decorated by D. We see that the function from T into D is a tree decorated by D. Let D be a non-empty set. The functor $\mathrm{Trees}(D)$ yielding a non-empty set of trees decorated by D is defined as follows:

(Def.7) for every tree T decorated by D holds $T \in \mathrm{Trees}(D)$.

Let D be a non-empty set. The functor $\mathrm{FinTrees}(D)$ yielding a non-empty set of trees decorated by D is defined as follows:

(Def.8) for every tree T decorated by D holds $\mathrm{dom}\, T$ is finite if and only if $T \in \mathrm{FinTrees}(D)$.

The following proposition is true

(22) For every non-empty set D holds $\mathrm{FinTrees}(D) \subseteq \mathrm{Trees}(D)$.

3. Functions yielding trees

We now define three new attributes. A function is tree yielding if:

(Def.9) rng it is constituted of trees.

A function is finite tree yielding if:

(Def.10) rng it is constituted of finite trees.

A function is decorated tree yielding if:

(Def.11) rng it is constituted of decorated trees.

One can prove the following propositions:

(23) ε is tree yielding, finite tree yielding and decorated tree yielding.

(24) f is tree yielding if and only if for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a tree.

(25) f is finite tree yielding if and only if for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a finite tree.

(26) f is decorated tree yielding if and only if for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a decorated tree.

(27) p is tree yielding and q is tree yielding if and only if $p \frown q$ is tree yielding.

(28) p is finite tree yielding and q is finite tree yielding if and only if $p \frown q$ is finite tree yielding.

(29) p is decorated tree yielding and q is decorated tree yielding if and only if $p \frown q$ is decorated tree yielding.

(30) $\langle x \rangle$ is tree yielding if and only if x is a tree.

(31) $\langle x \rangle$ is finite tree yielding if and only if x is a finite tree.

(32) $\langle x \rangle$ is decorated tree yielding if and only if x is a decorated tree.

(33) $\langle x, y \rangle$ is tree yielding if and only if x is a tree and y is a tree.

(34) $\langle x, y\rangle$ is finite tree yielding if and only if x is a finite tree and y is a finite tree.

(35) $\langle x, y\rangle$ is decorated tree yielding if and only if x is a decorated tree and y is a decorated tree.

(36) If $i \neq 0$, then $i \longmapsto x$ is tree yielding if and only if x is a tree.

(37) If $i \neq 0$, then $i \longmapsto x$ is finite tree yielding if and only if x is a finite tree.

(38) If $i \neq 0$, then $i \longmapsto x$ is decorated tree yielding if and only if x is a decorated tree.

One can verify the following observations:

* there exists a tree yielding finite tree yielding non-empty finite sequence,
* there exists a decorated tree yielding non-empty finite sequence,
* there exists a tree yielding finite tree yielding non-empty function,
* there exists a decorated tree yielding non-empty function, and
* every function which is finite tree yielding is also tree yielding.

Let D be a constituted of trees non-empty set. Observe that every finite sequence of elements of D is tree yielding.

Let p, q be tree yielding finite sequences. Then $p \frown q$ is a tree yielding finite sequence. Let D be a constituted of finite trees non-empty set. Note that every finite sequence of elements of D is finite tree yielding.

Let p, q be finite tree yielding finite sequences. Then $p \frown q$ is a finite tree yielding finite sequence. Let D be a constituted of decorated trees non-empty set. One can check that every finite sequence of elements of D is decorated tree yielding.

Let p, q be decorated tree yielding finite sequences. Then $p \frown q$ is a decorated tree yielding finite sequence. Let T be a tree. Then $\langle T\rangle$ is a tree yielding non-empty finite sequence. Let S be a tree. Then $\langle T, S\rangle$ is a tree yielding non-empty finite sequence. Let n be a natural number, and let T be a tree. Then $n \longmapsto T$ is a tree yielding finite sequence. Let T be a finite tree. Then $\langle T\rangle$ is a finite tree yielding tree yielding non-empty finite sequence. Let S be a finite tree. Then $\langle T, S\rangle$ is a finite tree yielding non-empty tree yielding finite sequence. Let n be a natural number, and let T be a finite tree. Then $n \longmapsto T$ is a finite tree yielding finite sequence. Let T be a decorated tree. Then $\langle T\rangle$ is a decorated tree yielding non-empty finite sequence. Let S be a decorated tree. Then $\langle T, S\rangle$ is a decorated tree yielding non-empty finite sequence. Let n be a natural number, and let T be a decorated tree. Then $n \longmapsto T$ is a decorated tree yielding finite sequence.

The following proposition is true

(39) For every decorated tree yielding function f holds $\mathrm{dom}(\mathrm{dom}_\kappa f(\kappa)) = \mathrm{dom} f$ and $\mathrm{dom}_\kappa f(\kappa)$ is tree yielding.

Let p be a decorated tree yielding finite sequence. Then $\mathrm{dom}_\kappa p(\kappa)$ is a tree yielding finite sequence.

One can prove the following proposition

(40) For every decorated tree yielding finite sequence p holds $\operatorname{len}(\operatorname{dom}_\kappa p(\kappa)) = \operatorname{len} p$.

4. Trees decorated by Cartesian product and structure of substitution

We now define four new constructions. Let D, E be non-empty sets. A tree decorated by D and E is a tree decorated by $[\!: D, E :\!]$.

A set of trees decorated by D and E is a set of trees decorated by $[\!: D, E :\!]$.

Let T_1, T_2 be decorated trees. Then $\langle T_1, T_2 \rangle$ is a decorated tree. Let D_1, D_2 be non-empty sets, and let T_1 be a tree decorated by D_1, and let T_2 be a tree decorated by D_2. Then $\langle T_1, T_2 \rangle$ is a tree decorated by D_1 and D_2. Let D, E be non-empty sets, and let T be a tree decorated by D, and let f be a function from D into E. Then $f \cdot T$ is a tree decorated by E. Let D_1, D_2 be non-empty sets. Then $\pi_1(D_1 \times D_2)$ is a function from $[\!: D_1, D_2 :\!]$ into D_1. Then $\pi_2(D_1 \times D_2)$ is a function from $[\!: D_1, D_2 :\!]$ into D_2. Let D_1, D_2 be non-empty sets, and let T be a tree decorated by D_1 and D_2. The functor $T_{\mathbf{1}}$ yielding a tree decorated by D_1 is defined by:

(Def.12) $T_{\mathbf{1}} = \pi_1(D_1 \times D_2) \cdot T$.

The functor $T_{\mathbf{2}}$ yielding a tree decorated by D_2 is defined by:

(Def.13) $T_{\mathbf{2}} = \pi_2(D_1 \times D_2) \cdot T$.

The following propositions are true:

(41) For all non-empty sets D_1, D_2 and for every tree T decorated by D_1 and D_2 and for every element t of $\operatorname{dom} T$ holds $T(t)_{\mathbf{1}} = T_{\mathbf{1}}(t)$ and $T_{\mathbf{2}}(t) = T(t)_{\mathbf{2}}$.

(42) For all non-empty sets D_1, D_2 and for every tree T decorated by D_1 and D_2 holds $\langle T_{\mathbf{1}}, T_{\mathbf{2}} \rangle = T$.

We now define two new modes. Let T be a finite tree. Then $\operatorname{Leaves} T$ is a finite non-empty subset of T. Let T be a tree, and let S be a non-empty subset of T. We see that the element of S is an element of T. Let T be a finite tree. We see that the leaf of T is an element of $\operatorname{Leaves} T$. Let T be a finite tree. A tree is called a substitution of structure of T if:

(Def.14) for every element t of it holds $t \in T$ or there exists a leaf l of T such that $l \prec t$.

Let T be a finite tree, and let t be a leaf of T, and let S be a tree. Then $T(t/S)$ is a substitution of structure of T. Let T be a finite tree. Observe that there exists a finite substitution of structure of T.

Let us consider n. A substitution of structure of n is a substitution of structure of the elementary tree of n.

We now state two propositions:

(43) Every tree is a substitution of structure of 0.

(44) For all trees T_1, T_2 such that T_1-level$(1) \subseteq T_2$-level(1) and for every n such that $\langle n \rangle \in T_1$ holds $T_1 \restriction \langle n \rangle = T_2 \restriction \langle n \rangle$ holds $T_1 \subseteq T_2$.

5. Joining of trees

Next we state several propositions:

(45) For all trees T, T' and for every element p of T holds $p \in T(p/T')$.

(46) For all trees T, T' and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in \operatorname{Leaves} T$ holds $T \subseteq T(p/T')$.

(47) For all decorated trees T, T' and for every element p of $\operatorname{dom} T$ holds $T(p/T')(p) = T'(\varepsilon)$.

(48) For all decorated trees T, T' and for all elements p, q of $\operatorname{dom} T$ such that $p \not\preceq q$ holds $T(p/T')(q) = T(q)$.

(49) For all decorated trees T, T' and for every element p of $\operatorname{dom} T$ and for every element q of $\operatorname{dom} T'$ holds $T(p/T')(p \frown q) = T'(q)$.

Let T_1, T_2 be trees. Then $T_1 \cup T_2$ is a tree.

One can prove the following proposition

(50) Let T_1, T_2 be trees. Let p be an element of $T_1 \cup T_2$. Then
 (i) if $p \in T_1$ and $p \in T_2$, then $(T_1 \cup T_2) \restriction p = T_1 \restriction p \cup T_2 \restriction p$,
 (ii) if $p \notin T_1$, then $(T_1 \cup T_2) \restriction p = T_2 \restriction p$,
 (iii) if $p \notin T_2$, then $(T_1 \cup T_2) \restriction p = T_1 \restriction p$.

We now define three new functors. Let us consider p satisfying the condition: p is tree yielding. The functor $\overbrace{p}$ yielding a tree is defined as follows:

(Def.15) $x \in \overbrace{p}$ if and only if $x = \varepsilon$ or there exist n, q such that $n < \operatorname{len} p$ and $q \in p(n+1)$ and $x = \langle n \rangle \frown q$.

Let T be a tree. The functor $\overbrace{T}$ yields a tree and is defined by:

(Def.16) $\overbrace{T} = \overbrace{\langle T \rangle}$.

Let T_1, T_2 be trees. The functor $\overbrace{T_1, T_2}$ yields a tree and is defined by:

(Def.17) $\overbrace{T_1, T_2} = \overbrace{\langle T_1, T_2 \rangle}$.

One can prove the following propositions:

(51) If p is tree yielding, then $\langle n \rangle \frown q \in \overbrace{p}$ if and only if $n < \operatorname{len} p$ and $q \in p(n+1)$.

(52) If p is tree yielding, then $\overbrace{p}$-level$(1) = \{\langle n \rangle : n < \operatorname{len} p\}$ and for every n such that $n < \operatorname{len} p$ holds $\overbrace{p} \restriction \langle n \rangle = p(n+1)$.

(53) For all tree yielding finite sequences p, q such that $\overbrace{p} = \overbrace{q}$ holds $p = q$.

(54) For all tree yielding finite sequences p_1, p_2 and for every tree T holds $p \in T$ if and only if $\langle \operatorname{len} p_1 \rangle ^\frown p \in \overbrace{p_1 ^\frown \langle T \rangle ^\frown p_2}$.

(55) $\overbrace{\varepsilon} =$ the elementary tree of 0.

(56) If p is tree yielding, then the elementary tree of $\operatorname{len} p \subseteq \overbrace{p}$.

(57) The elementary tree of $i = \overbrace{i \longmapsto \text{the elementary tree of } 0}$.

(58) For every tree T and for every tree yielding finite sequence p holds $\overbrace{p ^\frown \langle T \rangle} = (\overbrace{p} \cup \text{the elementary tree of } \operatorname{len} p + 1)(\langle \operatorname{len} p \rangle / T)$.

(59) For every tree yielding finite sequence p holds $\overbrace{p ^\frown \langle \text{the elementary tree of } 0 \rangle} = \overbrace{p} \cup$ the elementary tree of $\operatorname{len} p + 1$.

(60) For all tree yielding finite sequences p, q and for all trees T_1, T_2 holds $\overbrace{p ^\frown \langle T_1 \rangle ^\frown q} = \overbrace{p ^\frown \langle T_2 \rangle ^\frown q}(\langle \operatorname{len} p \rangle / T_1)$.

(61) For every tree T holds $\overbrace{T} =$ (the elementary tree of $1)(\langle 0 \rangle / T)$.

(62) For all trees T_1, T_2 holds $\overbrace{T_1, T_2} =$ (the elementary tree of $2)(\langle 0 \rangle / T_1)(\langle 1 \rangle / T_2)$.

Let p be a finite tree yielding finite sequence. Then $\overbrace{p}$ is a finite tree. Let T be a finite tree. Then $\overbrace{T}$ is a finite tree. Let T_1, T_2 be finite trees. Then $\overbrace{T_1, T_2}$ is a finite tree.

One can prove the following propositions:

(63) For every tree T and for an arbitrary x holds $x \in \overbrace{T}$ if and only if $x = \varepsilon$ or there exists p such that $p \in T$ and $x = \langle 0 \rangle ^\frown p$.

(64) For every tree T and for every finite sequence p holds $p \in T$ if and only if $\langle 0 \rangle ^\frown p \in \overbrace{T}$.

(65) For every tree T holds the elementary tree of $1 \subseteq \overbrace{T}$.

(66) For all trees T_1, T_2 such that $T_1 \subseteq T_2$ holds $\overbrace{T_1} \subseteq \overbrace{T_2}$.

(67) For all trees T_1, T_2 such that $\overbrace{T_1} = \overbrace{T_2}$ holds $T_1 = T_2$.

(68) For every tree T holds $\overbrace{T} \restriction \langle 0 \rangle = T$.

(69) For all trees T_1, T_2 holds $\overbrace{T_1}(\langle 0 \rangle / T_2) = \overbrace{T_2}$.

(70) $\overbrace{\text{the elementary tree of } 0} =$ the elementary tree of 1.

(71) For all trees T_1, T_2 and for an arbitrary x holds $x \in \overbrace{T_1, T_2}$ if and only if $x = \varepsilon$ or there exists p such that $p \in T_1$ and $x = \langle 0 \rangle ^\frown p$ or $p \in T_2$ and $x = \langle 1 \rangle ^\frown p$.

(72) For all trees T_1, T_2 and for every finite sequence p holds $p \in T_1$ if and only if $\langle 0 \rangle \mathbin{^\frown} p \in \overbrace{T_1, T_2}$.

(73) For all trees T_1, T_2 and for every finite sequence p holds $p \in T_2$ if and only if $\langle 1 \rangle \mathbin{^\frown} p \in \overbrace{T_1, T_2}$.

(74) For all trees T_1, T_2 holds the elementary tree of $2 \subseteq \overbrace{T_1, T_2}$.

(75) For all trees T_1, T_2, W_1, W_2 such that $T_1 \subseteq W_1$ and $T_2 \subseteq W_2$ holds $\overbrace{T_1, T_2} \subseteq \overbrace{W_1, W_2}$.

(76) For all trees T_1, T_2, W_1, W_2 such that $\overbrace{T_1, T_2} = \overbrace{W_1, W_2}$ holds $T_1 = W_1$ and $T_2 = W_2$.

(77) For all trees T_1, T_2 holds $\overbrace{T_1, T_2} \restriction \langle 0 \rangle = T_1$ and $\overbrace{T_1, T_2} \restriction \langle 1 \rangle = T_2$.

(78) For all trees T, T_1, T_2 holds $\overbrace{T_1, T_2}(\langle 0 \rangle / T) = \overbrace{T, T_2}$ and $\overbrace{T_1, T_2}(\langle 1 \rangle / T) = \overbrace{T_1, T}$.

(79) $\overbrace{\text{the elementary tree of } 0, \text{the elementary tree of } 0}$ = the elementary tree of 2.

In the sequel w is a finite tree yielding finite sequence. One can prove the following propositions:

(80) For every w if for every finite tree t such that $t \in \operatorname{rng} w$ holds height $t \leq n$, then height $\overbrace{w} \leq n + 1$.

(81) For every finite tree t such that $t \in \operatorname{rng} w$ holds height $\overbrace{w} >$ height t.

(82) For every finite tree t such that $t \in \operatorname{rng} w$ and for every finite tree t' such that $t' \in \operatorname{rng} w$ holds height $t' \leq$ height t holds height $\overbrace{w} =$ height $t + 1$.

(83) For every finite tree T holds height $\overbrace{T} =$ height $T + 1$.

(84) For all finite trees T_1, T_2 holds height $\overbrace{T_1, T_2} = \max(\text{height } T_1, \text{height } T_2) + 1$.

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Sum and Product of Finite Sequences of Elements of a Field

Katarzyna Zawadzka
Warsaw University
Białystok

Summary. This article is concerned with a generalization of concepts introduced in [10], i.e., there are introduced the sum and the product of finite number of elements of any field. Moreover, the product of vectors which yields a vector is introduced. According to [10], some operations on i-tuples of elements of field are introduced: addition, subtraction, and complement. Some properties on the sum and the product of finite number of elements of a field are present.

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The articles [17], [2], [18], [7], [8], [3], [4], [14], [13], [15], [19], [16], [6], [5], [9], [1], [20], [22], [21], [11], and [12] provide the notation and terminology for this paper.

1. Auxiliary theorems

For simplicity we adopt the following convention: i, j, k will denote natural numbers, K will denote a field, a, a', a_1, a_2, a_3 will denote elements of the carrier of K, p, p_1, p_2, q will denote finite sequences of elements of the carrier of K, and R, R_1, R_2, R_3 will denote elements of (the carrier of $K)^i$. We now state a number of propositions:

(1) $-0_K = 0_K$.

(2) The addition of K is commutative.

(3) The addition of K is associative.

(4) The multiplication of K is commutative.

(5) The multiplication of K is associative.

(6) 1_K is a unity w.r.t. the multiplication of K.

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(7) $\mathbf{1}_{\text{the multiplication of } K} = 1_K$.
(8) 0_K is a unity w.r.t. the addition of K.
(9) $\mathbf{1}_{\text{the addition of } K} = 0_K$.
(10) The addition of K has a unity.
(11) The multiplication of K has a unity.
(12) The multiplication of K is distributive w.r.t. the addition of K.

We now define two new functors. Let us consider K, and let a be an element of the carrier of K. The functor $\cdot^a$ yields a unary operation on the carrier of K and is defined by:

(Def.1) $\cdot^a = (\text{the multiplication of } K)^\circ(a, \mathrm{id}_{(\text{the carrier of } K)})$.

Let us consider K. The functor $-_K$ yields a binary operation on the carrier of K and is defined as follows:

(Def.2) $-_K = (\text{the addition of } K) \circ (\mathrm{id}_{(\text{the carrier of } K)}, \text{the reverse-map of } K)$.

We now state several propositions:

(13) $-_K = (\text{the addition of } K) \circ (\mathrm{id}_{(\text{the carrier of } K)}, \text{the reverse-map of } K)$.
(14) $-_K(a_1, a_2) = a_1 - a_2$.
(15) $\cdot^a$ is distributive w.r.t. the addition of K.
(16) The reverse-map of K is an inverse operation w.r.t. the addition of K.
(17) The addition of K has an inverse operation.
(18) The inverse operation w.r.t. the addition of K = the reverse-map of K.
(19) The reverse-map of K is distributive w.r.t. the addition of K.

Let us consider K, p_1, p_2. The functor $p_1 + p_2$ yielding a finite sequence of elements of the carrier of K is defined as follows:

(Def.3) $p_1 + p_2 = (\text{the addition of } K)^\circ(p_1, p_2)$.

Next we state two propositions:

(20) $p_1 + p_2 = (\text{the addition of } K)^\circ(p_1, p_2)$.
(21) If $i \in \mathrm{Seg}\,\mathrm{len}(p_1 + p_2)$ and $a_1 = p_1(i)$ and $a_2 = p_2(i)$, then $(p_1 + p_2)(i) = a_1 + a_2$.

Let us consider i, and let us consider K, and let R_1, R_2 be elements of (the carrier of $K)^i$. Then $R_1 + R_2$ is an element of (the carrier of $K)^i$.

Next we state several propositions:

(22) If $j \in \mathrm{Seg}\, i$ and $a_1 = R_1(j)$ and $a_2 = R_2(j)$, then $(R_1 + R_2)(j) = a_1 + a_2$.
(23) $\varepsilon_{(\text{the carrier of } K)} + p = \varepsilon_{(\text{the carrier of } K)}$ and $p + \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.
(24) $\langle a_1 \rangle + \langle a_2 \rangle = \langle a_1 + a_2 \rangle$.
(25) $(i \longmapsto a_1) + (i \longmapsto a_2) = i \longmapsto a_1 + a_2$.
(26) $R_1 + R_2 = R_2 + R_1$.
(27) $R_1 + (R_2 + R_3) = (R_1 + R_2) + R_3$.
(28) $R + (i \longmapsto 0_K) = R$ and $R = (i \longmapsto 0_K) + R$.

Let us consider K, p. The functor $-p$ yields a finite sequence of elements of the carrier of K and is defined as follows:

(Def.4) $-p = (\text{the reverse-map of } K) \cdot p$.

The following two propositions are true:

(29) $-p = (\text{the reverse-map of } K) \cdot p$.

(30) If $i \in \mathrm{Seg}\,\mathrm{len}(-p)$ and $a = p(i)$, then $(-p)(i) = -a$.

Let us consider i, K, R. Then $-R$ is an element of (the carrier of $K)^i$.

One can prove the following propositions:

(31) If $j \in \mathrm{Seg}\,i$ and $a = R(j)$, then $(-R)(j) = -a$.

(32) $-\varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(33) $-\langle a \rangle = \langle -a \rangle$.

(34) $-(i \longmapsto a) = i \longmapsto -a$.

(35) $R + -R = i \longmapsto 0_K$ and $-R + R = i \longmapsto 0_K$.

(36) If $R_1 + R_2 = i \longmapsto 0_K$, then $R_1 = -R_2$ and $R_2 = -R_1$.

(37) $--R = R$.

(38) If $-R_1 = -R_2$, then $R_1 = R_2$.

(39) If $R_1 + R = R_2 + R$ or $R_1 + R = R + R_2$, then $R_1 = R_2$.

(40) $-(R_1 + R_2) = -R_1 + -R_2$.

Let us consider K, p_1, p_2. The functor $p_1 - p_2$ yielding a finite sequence of elements of the carrier of K is defined as follows:

(Def.5) $p_1 - p_2 = (-_K)^\circ(p_1, p_2)$.

Next we state two propositions:

(41) $p_1 - p_2 = (-_K)^\circ(p_1, p_2)$.

(42) If $i \in \mathrm{Seg}\,\mathrm{len}(p_1 - p_2)$ and $a_1 = p_1(i)$ and $a_2 = p_2(i)$, then $(p_1 - p_2)(i) = a_1 - a_2$.

Let us consider i, K, R_1, R_2. Then $R_1 - R_2$ is an element of (the carrier of $K)^i$.

The following propositions are true:

(43) If $j \in \mathrm{Seg}\,i$ and $a_1 = R_1(j)$ and $a_2 = R_2(j)$, then $(R_1 - R_2)(j) = a_1 - a_2$.

(44) $\varepsilon_{(\text{the carrier of } K)} - p = \varepsilon_{(\text{the carrier of } K)}$ and $p - \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(45) $\langle a_1 \rangle - \langle a_2 \rangle = \langle a_1 - a_2 \rangle$.

(46) $(i \longmapsto a_1) - (i \longmapsto a_2) = i \longmapsto a_1 - a_2$.

(47) $R_1 - R_2 = R_1 + -R_2$.

(48) $R - (i \longmapsto 0_K) = R$.

(49) $(i \longmapsto 0_K) - R = -R$.

(50) $R_1 - -R_2 = R_1 + R_2$.

(51) $-(R_1 - R_2) = R_2 - R_1$.

(52) $-(R_1 - R_2) = -R_1 + R_2$.

(53) $R - R = i \longmapsto 0_K$.

(54) If $R_1 - R_2 = i \longmapsto 0_K$, then $R_1 = R_2$.

(55) $R_1 - R_2 - R_3 = R_1 - (R_2 + R_3)$.

(56) $R_1 + (R_2 - R_3) = (R_1 + R_2) - R_3$.

(57) $R_1 - (R_2 - R_3) = (R_1 - R_2) + R_3$.

(58) $R_1 = (R_1 + R) - R$.

(59) $R_1 = (R_1 - R) + R$.

(60) For all elements a, b of the carrier of K holds ((the multiplication of $K)^{\circ}(a, \mathrm{id}_{(\text{the carrier of } K)}))(b) = a \cdot b$.

(61) For all elements a, b of the carrier of K holds $\cdot^{a}(b) = a \cdot b$.

Let us consider K, and let p be a finite sequence of elements of the carrier of K, and let a be an element of the carrier of K. The functor $a \cdot p$ yielding a finite sequence of elements of the carrier of K is defined as follows:

(Def.6) $a \cdot p = \cdot^{a} \cdot p$.

Next we state the proposition

(62) If $i \in \operatorname{Seg} \operatorname{len}(a \cdot p)$ and $a' = p(i)$, then $(a \cdot p)(i) = a \cdot a'$.

Let us consider i, K, R, a. Then $a \cdot R$ is an element of (the carrier of $K)^{i}$.

The following propositions are true:

(63) If $j \in \operatorname{Seg} i$ and $a' = R(j)$, then $(a \cdot R)(j) = a \cdot a'$.

(64) $a \cdot \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(65) $a \cdot \langle a_1 \rangle = \langle a \cdot a_1 \rangle$.

(66) $a_1 \cdot (i \longmapsto a_2) = i \longmapsto a_1 \cdot a_2$.

(67) $(a_1 \cdot a_2) \cdot R = a_1 \cdot (a_2 \cdot R)$.

(68) $(a_1 + a_2) \cdot R = a_1 \cdot R + a_2 \cdot R$.

(69) $a \cdot (R_1 + R_2) = a \cdot R_1 + a \cdot R_2$.

(70) $1_K \cdot R = R$.

(71) $0_K \cdot R = i \longmapsto 0_K$.

(72) $(-1_K) \cdot R = -R$.

Let us consider K, p_1, p_2. The functor $p_1 \bullet p_2$ yields a finite sequence of elements of the carrier of K and is defined as follows:

(Def.7) $p_1 \bullet p_2 =$ (the multiplication of $K)^{\circ}(p_1, p_2)$.

One can prove the following proposition

(73) If $i \in \operatorname{Seg} \operatorname{len}(p_1 \bullet p_2)$ and $a_1 = p_1(i)$ and $a_2 = p_2(i)$, then $(p_1 \bullet p_2)(i) = a_1 \cdot a_2$.

Let us consider i, K, R_1, R_2. Then $R_1 \bullet R_2$ is an element of (the carrier of $K)^{i}$.

We now state a number of propositions:

(74) If $j \in \operatorname{Seg} i$ and $a_1 = R_1(j)$ and $a_2 = R_2(j)$, then $(R_1 \bullet R_2)(j) = a_1 \cdot a_2$.

(75) $\varepsilon_{(\text{the carrier of } K)} \bullet p = \varepsilon_{(\text{the carrier of } K)}$ and $p \bullet \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(76) $\langle a_1 \rangle \bullet \langle a_2 \rangle = \langle a_1 \cdot a_2 \rangle$.

(77) $R_1 \bullet R_2 = R_2 \bullet R_1$.

(78) $p \bullet q = q \bullet p$.

(79) $R_1 \bullet (R_2 \bullet R_3) = (R_1 \bullet R_2) \bullet R_3$.

(80) $(i \longmapsto a) \bullet R = a \cdot R$ and $R \bullet (i \longmapsto a) = a \cdot R$.

(81) $(i \longmapsto a_1) \bullet (i \longmapsto a_2) = i \longmapsto a_1 \cdot a_2$.

(82) $a \cdot (R_1 \bullet R_2) = a \cdot R_1 \bullet R_2$.

(83) $a \cdot (R_1 \bullet R_2) = a \cdot R_1 \bullet R_2$ and $a \cdot (R_1 \bullet R_2) = R_1 \bullet a \cdot R_2$.

(84) $a \cdot R = (i \longmapsto a) \bullet R$.

Let us consider K, and let p be a finite sequence of elements of the carrier of K. The functor $\sum p$ yielding an element of the carrier of K is defined as follows:

(Def.8) $\sum p =$ the addition of $K \circledast p$.

The following propositions are true:

(85) $\sum(\varepsilon_{(\text{the carrier of } K)}) = 0_K$.

(86) $\sum \langle a \rangle = a$.

(87) $\sum(p \frown \langle a \rangle) = \sum p + a$.

(88) $\sum(p_1 \frown p_2) = \sum p_1 + \sum p_2$.

(89) $\sum(\langle a \rangle \frown p) = a + \sum p$.

(90) $\sum \langle a_1, a_2 \rangle = a_1 + a_2$.

(91) $\sum \langle a_1, a_2, a_3 \rangle = a_1 + a_2 + a_3$.

(92) $\sum(a \cdot p) = a \cdot \sum p$.

(93) For every element R of (the carrier of $K)^0$ holds $\sum R = 0_K$.

(94) $\sum(-p) = -\sum p$.

(95) $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$.

(96) $\sum(R_1 - R_2) = \sum R_1 - \sum R_2$.

Let us consider K, and let p be a finite sequence of elements of the carrier of K. The functor $\prod p$ yielding an element of the carrier of K is defined by:

(Def.9) $\prod p =$ the multiplication of $K \circledast p$.

The following propositions are true:

(97) $\prod p =$ the multiplication of $K \circledast p$.

(98) $\prod(\varepsilon_{(\text{the carrier of } K)}) = 1_K$.

(99) $\prod \langle a \rangle = a$.

(100) $\prod(p \frown \langle a \rangle) = \prod p \cdot a$.

(101) $\prod(p_1 \frown p_2) = \prod p_1 \cdot \prod p_2$.

(102) $\prod(\langle a \rangle \frown p) = a \cdot \prod p$.

(103) $\prod \langle a_1, a_2 \rangle = a_1 \cdot a_2$.

(104) $\prod \langle a_1, a_2, a_3 \rangle = a_1 \cdot a_2 \cdot a_3$.

(105) For every element R of (the carrier of $K)^0$ holds $\prod R = 1_K$.

(106) $\prod(i \longmapsto 1_K) = 1_K$.

(107) There exists k such that $k \in \mathrm{Seg}\,\mathrm{len}\,p$ and $p(k) = 0_K$ if and only if $\prod p = 0_K$.

(108) $\prod(i + j \longmapsto a) = \prod(i \longmapsto a) \cdot \prod(j \longmapsto a)$.

(109) $\prod(i \cdot j \longmapsto a) = \prod(j \longmapsto \prod(i \longmapsto a))$.

(110) $\prod(i \longmapsto a_1 \cdot a_2) = \prod(i \longmapsto a_1) \cdot \prod(i \longmapsto a_2)$.

(111) $\prod(R_1 \bullet R_2) = \prod R_1 \cdot \prod R_2$.

(112) $\prod(a \cdot R) = \prod(i \longmapsto a) \cdot \prod R$.

Let us consider K, and let p, q be finite sequences of elements of the carrier of K. The functor $p \cdot q$ yielding an element of the carrier of K is defined by:

(Def.10) $p \cdot q = \sum(p \bullet q)$.

One can prove the following propositions:

(113) For all elements a, b of the carrier of K holds $\langle a \rangle \cdot \langle b \rangle = a \cdot b$.

(114) For all elements a_1, a_2, b_1, b_2 of the carrier of K holds $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 \cdot b_1 + a_2 \cdot b_2$.

(115) For all finite sequences p, q of elements of the carrier of K holds $p \cdot q = q \cdot p$.

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Monoids

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Summary. The goal of the article is to define the concept of monoid. In the preliminary section we introduce the notion of some properties of binary operations. The second section is concerning with structures with a set and a binary operation on this set: there is introduced the notion corresponding to the notion of some properties of binary operations and there are shown some useful clusters. Next, we are concerning with the structure with a set, a binary operation on the set and with an element of the set. Such a structure is called monoid iff the operation is associative and the element is a unity of the operation. In the fourth section the concept of subsystems of monoid (group) is introduced. Subsystems are submonoids (subgroups) or other parts of monoid (group) with are closed w.r.t. the operation. The are present facts on inheritness of some properties by subsystems. Finally, there are construct the examples of groups and monoids: the group $\rangle\mathbb{R},+\langle$ of real numbers with addition, the group $\mathbb{Z}^+$ of integers as the subsystem of the group $\rangle\mathbb{R},+\langle$, the semigroup $\rangle\mathbb{N},+\langle$ of natural numbers as the subsystem of $\mathbb{Z}^+$, and the monoid $\rangle\mathbb{N},+,0\langle$ of natural numbers with addition and zero as monoidal extension of the semigroup $\rangle\mathbb{N},+\langle$. The semigroups of real and natural numbers with multiplication are also introduced. The monoid of finite sequences over some set with concatenation as binary operation and with empty sequence as neutral element is defined in sixth section. Last section deals with monoids with the composition of functions as the operation, i.e. with the monoid of partial and total functions and the monoid of permutations.

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The papers [15], [16], [13], [1], [6], [2], [7], [3], [5], [9], [17], [10], [14], [12], [4], [18], [8], and [11] provide the terminology and notation for this paper.

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1. Binary operations preliminary

In the sequel x is arbitrary and X, Y denote sets. We now define several new constructions. Let G be a 1-sorted structure. An element of G is an element of the carrier of G.

A finite sequence of elements of G is a finite sequence of elements of the carrier of G.

A binary operation on G is a binary operation on the carrier of G.

A subset of G is a subset of the carrier of G.

A 1-sorted structure is constituted functions if:

(Def.1) every element of it is a function.

A 1-sorted structure is constituted finite sequences if:

(Def.2) every element of it is a finite sequence.

Let X be a constituted functions 1-sorted structure. One can check the following observations:

- $*$ every element of X is function-like,
- $*$ every constituted finite sequences 1-sorted structure is constituted functions, and
- $*$ every constituted finite sequences half group structure is constituted functions.

Let X be a constituted finite sequences 1-sorted structure. Note that every element of X is finite sequence-like.

Let D be a non-empty set, and let p, q be finite sequences of elements of D. Then $p \frown q$ is an element of D^*. Let g, f be functions. We introduce the functor $f \circ g$ as a synonym of $f \cdot g$. Let X be a set, and let g, f be functions from X into X. Then $f \cdot g$ is a function from X into X. Let X be a set, and let g, f be permutations of X. Then $f \cdot g$ is a permutation of X. Let A be a set, and let B, C be non-empty sets, and let g be a function from A into B, and let f be a function from B into C. Then $f \cdot g$ is a function from A into C. Let A, B, C be sets, and let g be a partial function from A to B, and let f be a partial function from B to C. Then $f \cdot g$ is a partial function from A to C. Let D be a non-empty set. A binary operation on D is left invertible if:

(Def.3) for every elements a, b of D there exists an element l of D such that $\mathrm{it}(l, a) = b$.

A binary operation on D is right invertible if:

(Def.4) for every elements a, b of D there exists an element r of D such that $\mathrm{it}(a, r) = b$.

A binary operation on D is invertible if:

(Def.5) for every elements a, b of D there exist elements r, l of D such that $\mathrm{it}(a, r) = b$ and $\mathrm{it}(l, a) = b$.

A binary operation on D is left cancelable if:

(Def.6) for all elements a, b, c of D such that $\mathrm{it}(a, b) = \mathrm{it}(a, c)$ holds $b = c$.

A binary operation on D is right cancelable if:

(Def.7) for all elements a, b, c of D such that $\mathrm{it}(b, a) = \mathrm{it}(c, a)$ holds $b = c$.

A binary operation on D is cancelable if:

(Def.8) for all elements a, b, c of D such that $\mathrm{it}(a, b) = \mathrm{it}(a, c)$ or $\mathrm{it}(b, a) = \mathrm{it}(c, a)$ holds $b = c$.

A binary operation on D has uniquely decomposable unity if:

(Def.9) it has a unity and for all elements a, b of D such that $\mathrm{it}(a, b) = \mathbf{1}_{\mathrm{it}}$ holds $a = b$ and $b = \mathbf{1}_{\mathrm{it}}$.

We now state three propositions:

(1) For every non-empty set D and for every binary operation f on D holds f is invertible if and only if f is left invertible and right invertible.

(2) For every non-empty set D and for every binary operation f on D holds f is cancelable if and only if f is left cancelable and right cancelable.

(3) For every binary operation f on $\{x\}$ holds $f = \{\langle x, x\rangle\} \longmapsto x$ and f has a unity and f is commutative and f is associative and f is idempotent and f is invertible and cancelable and has uniquely decomposable unity.

2. Semigroups

We adopt the following convention: G denotes a half group structure, D denotes a non-empty set, and a, b, c, r, l denote elements of G. We now define several new attributes. A half group structure is unital if:

(Def.10) the operation of it has a unity.

A half group structure is commutative if:

(Def.11) the operation of it is commutative.

A half group structure is associative if:

(Def.12) the operation of it is associative.

A half group structure is idempotent if:

(Def.13) the operation of it is idempotent.

A half group structure is left invertible if:

(Def.14) the operation of it is left invertible.

A half group structure is right invertible if:

(Def.15) the operation of it is right invertible.

A half group structure is invertible if:

(Def.16) the operation of it is invertible.

A half group structure is left cancelable if:

(Def.17) the operation of it is left cancelable.

A half group structure is right cancelable if:

(Def.18) the operation of it is right cancelable.

A half group structure is cancelable if:

(Def.19) the operation of it is cancelable.

A half group structure has uniquely decomposable unity if:

(Def.20) the operation of it has uniquely decomposable unity.

One can verify that there exists a unital commutative associative cancelable idempotent invertible with uniquely decomposable unity constituted functions constituted finite sequences strict half group structure.

We now state a number of propositions:

(4) If G is unital, then $\mathbf{1}_{\text{the operation of } G}$ is a unity w.r.t. the operation of G.

(5) G is unital if and only if for every a holds $\mathbf{1}_{\text{the operation of } G} \cdot a = a$ and $a \cdot \mathbf{1}_{\text{the operation of } G} = a$.

(6) G is unital if and only if there exists a such that for every b holds $a \cdot b = b$ and $b \cdot a = b$.

(7) G is commutative if and only if for all a, b holds $a \cdot b = b \cdot a$.

(8) G is associative if and only if for all a, b, c holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(9) G is idempotent if and only if for every a holds $a \cdot a = a$.

(10) G is left invertible if and only if for every a, b there exists l such that $l \cdot a = b$.

(11) G is right invertible if and only if for every a, b there exists r such that $a \cdot r = b$.

(12) G is invertible if and only if for every a, b there exist r, l such that $a \cdot r = b$ and $l \cdot a = b$.

(13) G is left cancelable if and only if for all a, b, c such that $a \cdot b = a \cdot c$ holds $b = c$.

(14) G is right cancelable if and only if for all a, b, c such that $b \cdot a = c \cdot a$ holds $b = c$.

(15) G is cancelable if and only if for all a, b, c such that $a \cdot b = a \cdot c$ or $b \cdot a = c \cdot a$ holds $b = c$.

(16) G has uniquely decomposable unity if and only if the operation of G has a unity and for all elements a, b of G such that $a \cdot b = \mathbf{1}_{\text{the operation of } G}$ holds $a = b$ and $b = \mathbf{1}_{\text{the operation of } G}$.

(17) If G is associative, then G is invertible if and only if G is unital and the operation of G has an inverse operation.

One can check the following observations:

* every group-like half group structure is associative and invertible,
* every associative invertible half group structure is group-like,
* every half group structure which is invertible is also left invertible and right invertible,
* every half group structure which is left invertible and right invertible is also invertible,

* every cancelable half group structure is left cancelable and right cancelable,
* every left cancelable right cancelable half group structure is cancelable,
* every half group structure which is associative and invertible is also unital and cancelable,
* every Abelian group is commutative, and
* every commutative group is Abelian.

3. Monoids

We consider monoid structures which are extension of a half group structure and are systems

⟨a carrier, an operation, a unity⟩,

where the carrier is a non-empty set, the operation is a binary operation on the carrier, and the unity is an element of the carrier.

In the sequel M will be a monoid structure. A monoid structure is well unital if:

(Def.21) the unity of it is a unity w.r.t. the operation of it.

Next we state the proposition

(18) M is well unital if and only if for every element a of M holds (the unity of M) $\cdot\, a = a$ and $a \cdot$ the unity of $M = a$.

Let us mention that every monoid structure which is well unital is also unital.

We now state the proposition

(19) For every M being a monoid structure such that M is well unital holds the unity of $M = \mathbf{1}_{\text{the operation of } M}$.

We now define two new modes. Let us note that there exists a well unital commutative associative cancelable idempotent invertible with uniquely decomposable unity unital constituted functions constituted finite sequences strict monoid structure.

A monoid is a well unital associative monoid structure.

Let G be a half group structure. A monoid structure is called a monoidal extension of G if:

(Def.22) the half group structure of it = the half group structure of G.

One can prove the following proposition

(20) For every monoidal extension M of G holds the carrier of M = the carrier of G and the operation of M = the operation of G and for all elements a, b of M and for all elements a', b' of G such that $a = a'$ and $b = b'$ holds $a \cdot b = a' \cdot b'$.

Let G be a half group structure. Note that there exists a strict monoidal extension of G.

The following proposition is true

(21) Let G be a half group structure. Let M be a monoidal extension of G. Then if G is unital, then M is unital and also if G is commutative, then M is commutative and also if G is associative, then M is associative and also if G is invertible, then M is invertible and also if G has uniquely decomposable unity, then M has uniquely decomposable unity and also if G is cancelable, then M is cancelable.

Let G be a constituted functions half group structure. One can check that every monoidal extension of G is constituted functions.

Let G be a constituted finite sequences half group structure. Note that every monoidal extension of G is constituted finite sequences.

Let G be a unital half group structure. Observe that every monoidal extension of G is unital.

Let G be an associative half group structure. One can verify that every monoidal extension of G is associative.

Let G be a commutative half group structure. One can verify that every monoidal extension of G is commutative.

Let G be an invertible half group structure. Note that every monoidal extension of G is invertible.

Let G be a cancelable half group structure. One can check that every monoidal extension of G is cancelable.

Let G be a half group structure with uniquely decomposable unity. Note that every monoidal extension of G is with uniquely decomposable unity.

Let G be a unital half group structure. Note that there exists a well unital strict monoidal extension of G.

The following proposition is true

(22) For every G being a unital half group structure and for all well unital strict monoidal extensions M_1, M_2 of G holds $M_1 = M_2$.

4. Subsystems

We now define two new modes. Let G be a half group structure. A half group structure is said to be a subsystem of G if:

(Def.23) the operation of it $\leq$ the operation of G.

Let G be a half group structure. One can check that there exists a subsystem of G which is strict.

Let G be a unital half group structure. Observe that there exists a subsystem of G which is unital associative commutative cancelable idempotent invertible with uniquely decomposable unity and strict.

Let G be a half group structure. A monoid structure is called a monoidal subsystem of G if:

(Def.24) the operation of it $\leq$ the operation of G and for every M being a monoid structure such that $G = M$ holds the unity of it $=$ the unity of M.

Let G be a half group structure. Note that there exists a monoidal subsystem of G which is strict.

Let M be a monoid structure. Let us note that the monoidal subsystem of M can be characterized by the following (equivalent) condition:

(Def.25) the operation of it $\leq$ the operation of M and the unity of it $=$ the unity of M.

Let G be a well unital monoid structure. Observe that there exists a well unital associative commutative cancelable idempotent invertible with uniquely decomposable unity strict monoidal subsystem of G.

We now state the proposition

(23) For every G being a half group structure every monoidal subsystem of G is a subsystem of G.

Let G be a half group structure, and let M be a monoidal extension of G. We see that the subsystem of M is a subsystem of G. Let G_1 be a half group structure, and let G_2 be a subsystem of G_1. We see that the subsystem of G_2 is a subsystem of G_1. Let G_1 be a half group structure, and let G_2 be a monoidal subsystem of G_1. We see that the subsystem of G_2 is a subsystem of G_1. Let G be a half group structure, and let M be a monoidal subsystem of G. We see that the monoidal subsystem of M is a monoidal subsystem of G.

We now state the proposition

(24) G is a subsystem of G and M is a monoidal subsystem of M.

In the sequel H is a subsystem of G and N is a monoidal subsystem of G. One can prove the following propositions:

(25) The carrier of $H \subseteq$ the carrier of G and the carrier of $N \subseteq$ the carrier of G.

(26) For every G being a half group structure and for every subsystem H of G holds the operation of $H =$ (the operation of G) $\restriction$ [: the carrier of H, the carrier of H :].

(27) For all elements a, b of H and for all elements a', b' of G such that $a = a'$ and $b = b'$ holds $a \cdot b = a' \cdot b'$.

(28) For all subsystems H_1, H_2 of G such that the carrier of $H_1 =$ the carrier of H_2 holds the half group structure of $H_1 =$ the half group structure of H_2.

(29) For all monoidal subsystems H_1, H_2 of M such that the carrier of $H_1 =$ the carrier of H_2 holds the monoid structure of $H_1 =$ the monoid structure of H_2.

(30) For all subsystems H_1, H_2 of G such that the carrier of $H_1 \subseteq$ the carrier of H_2 holds H_1 is a subsystem of H_2.

(31) For all monoidal subsystems H_1, H_2 of M such that the carrier of $H_1 \subseteq$ the carrier of H_2 holds H_1 is a monoidal subsystem of H_2.

(32) If G is unital and $\mathbf{1}_{\text{the operation of } G} \in$ the carrier of H, then H is unital and $\mathbf{1}_{\text{the operation of } G} = \mathbf{1}_{\text{the operation of } H}$.

(33) For every M being a well unital monoid structure every monoidal subsystem of M is well unital.

(34) If G is commutative, then H is commutative.

(35) If G is associative, then H is associative.

(36) If G is idempotent, then H is idempotent.

(37) If G is cancelable, then H is cancelable.

(38) If $\mathbf{1}_{\text{the operation of } G} \in$ the carrier of H and G has uniquely decomposable unity, then H has uniquely decomposable unity.

(39) For every M being a well unital monoid structure with uniquely decomposable unity every monoidal subsystem of M has uniquely decomposable unity.

Let G be a constituted functions half group structure. Observe that every subsystem of G is constituted functions and every monoidal subsystem of G is constituted functions.

Let G be a constituted finite sequences half group structure. One can verify that every subsystem of G is constituted finite sequences and every monoidal subsystem of G is constituted finite sequences.

Let M be a well unital monoid structure. Note that every monoidal subsystem of M is well unital.

Let G be a commutative half group structure. Observe that every subsystem of G is commutative and every monoidal subsystem of G is commutative.

Let G be an associative half group structure. One can verify that every subsystem of G is associative and every monoidal subsystem of G is associative.

Let G be an idempotent half group structure. Observe that every subsystem of G is idempotent and every monoidal subsystem of G is idempotent.

Let G be a cancelable half group structure. Observe that every subsystem of G is cancelable and every monoidal subsystem of G is cancelable.

Let M be a well unital monoid structure with uniquely decomposable unity. Observe that every monoidal subsystem of M is with uniquely decomposable unity.

In this article we present several logical schemes. The scheme *SubStrEx1* deals with a half group structure $\mathcal{A}$ and a non-empty subset $\mathcal{B}$ of $\mathcal{A}$ and states that:

there exists a strict subsystem H of $\mathcal{A}$ such that the carrier of $H = \mathcal{B}$

provided the following condition is met:

- for all elements x, y of $\mathcal{B}$ holds $x \cdot y \in \mathcal{B}$.

The scheme *SubStrEx2* deals with a half group structure $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a strict subsystem H of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $x \in$ the carrier of H if and only if $\mathcal{P}[x]$

provided the following conditions are met:

- for all elements x, y of $\mathcal{A}$ such that $\mathcal{P}[x]$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x \cdot y]$,
- there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[x]$.

The scheme *MonoidalSubStrEx1* concerns a monoid structure $\mathcal{A}$ and a non-empty subset $\mathcal{B}$ of $\mathcal{A}$ and states that:

there exists a strict monoidal subsystem H of $\mathcal{A}$ such that the carrier of $H = \mathcal{B}$

provided the parameters meet the following requirements:

- for all elements x, y of $\mathcal{B}$ holds $x \cdot y \in \mathcal{B}$,
- the unity of $\mathcal{A} \in \mathcal{B}$.

The scheme *MonoidalSubStrEx2* deals with a monoid structure $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a strict monoidal subsystem M of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $x \in$ the carrier of M if and only if $\mathcal{P}[x]$

provided the following requirements are met:

- for all elements x, y of $\mathcal{A}$ such that $\mathcal{P}[x]$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x \cdot y]$,
- $\mathcal{P}$[the unity of $\mathcal{A}$].

Let us consider G, a, b. Then $a \cdot b$ is an element of G. We introduce the functor $a \otimes b$ as a synonym of $a \cdot b$.

5. The examples of monoids of numbers

The unital associative invertible commutative cancelable strict half group structure $\langle \mathbb{R}, + \rangle$ is defined by:

(Def.26) $\langle \mathbb{R}, + \rangle = \langle \mathbb{R}, +_{\mathbb{R}} \rangle$.

The following propositions are true:

(40) The carrier of $\langle \mathbb{R}, + \rangle = \mathbb{R}$ and the operation of $\langle \mathbb{R}, + \rangle = +_{\mathbb{R}}$ and for all elements a, b of $\langle \mathbb{R}, + \rangle$ and for all real numbers x, y such that $a = x$ and $b = y$ holds $a \cdot b = x + y$.

(41) x is an element of $\langle \mathbb{R}, + \rangle$ if and only if x is a real number.

(42) $\mathbf{1}_{\text{the operation of } \langle \mathbb{R}, + \rangle} = 0$.

(43) For every subsystem N of $\langle \mathbb{R}, + \rangle$ and for all elements a, b of N and for all real numbers x, y such that $a = x$ and $b = y$ holds $a \cdot b = x + y$.

(44) For every unital subsystem N of $\langle \mathbb{R}, + \rangle$ holds $\mathbf{1}_{\text{the operation of } N} = 0$.

(45) For every subsystem N of $\langle \mathbb{R}, + \rangle$ such that 0 is an element of N holds N is unital and $\mathbf{1}_{\text{the operation of } N} = 0$.

Let G be a unital half group structure. Observe that every associative invertible subsystem of G is unital cancelable and group-like.

Let us note that it makes sense to consider the following constant. Then $\mathbb{Z}^+$ is a unital invertible strict subsystem of $\langle \mathbb{R}, + \rangle$.

The following two propositions are true:

(46) For every strict subsystem G of $\langle \mathbb{R}, + \rangle$ holds $G = \mathbb{Z}^+$ if and only if the carrier of $G = \mathbb{Z}$.

(47) x is an element of $\mathbb{Z}^+$ if and only if x is an integer.

We now define three new functors. The unital strict subsystem $\langle\mathbb{N},+\rangle$ of $\mathbb{Z}^+$ with uniquely decomposable unity is defined by:

(Def.27) the carrier of $\langle\mathbb{N},+\rangle = \mathbb{N}$.

$\langle\mathbb{N},+,0\rangle$ is a well unital strict monoidal extension of $\langle\mathbb{N},+\rangle$.

The binary operation $+_{\mathbb{N}}$ on $\mathbb{N}$ is defined by:

(Def.28) $+_{\mathbb{N}}$ = the operation of $\langle\mathbb{N},+\rangle$.

Next we state several propositions:

(48) x is an element of $\langle\mathbb{N},+\rangle$ if and only if x is a natural number.

(49) $\langle\mathbb{N},+\rangle = \langle\mathbb{N},+_{\mathbb{N}}\rangle$.

(50) x is an element of $\langle\mathbb{N},+,0\rangle$ if and only if x is a natural number.

(51) For all natural numbers n_1, n_2 and for all elements m_1, m_2 of $\langle\mathbb{N},+,0\rangle$ such that $n_1 = m_1$ and $n_2 = m_2$ holds $m_1 \cdot m_2 = n_1 + n_2$.

(52) $\langle\mathbb{N},+,0\rangle = \langle\mathbb{N},+_{\mathbb{N}},0\rangle$.

(53) $+_{\mathbb{N}} = +_{\mathbb{R}} \restriction [\mathbb{N}, \mathbb{N}]$ and $+_{\mathbb{N}} = (+_{\mathbb{Z}}) \restriction [\mathbb{N}, \mathbb{N}]$.

(54) 0 is a unity w.r.t. $+_{\mathbb{N}}$ and $+_{\mathbb{N}}$ has a unity and $\mathbf{1}_{+_{\mathbb{N}}} = 0$ and $+_{\mathbb{N}}$ is commutative and $+_{\mathbb{N}}$ is associative and $+_{\mathbb{N}}$ has uniquely decomposable unity.

The unital commutative associative strict half group structure $\langle\mathbb{R},\cdot\rangle$ is defined by:

(Def.29) $\langle\mathbb{R},\cdot\rangle = \langle\mathbb{R},\cdot_{\mathbb{R}}\rangle$.

Next we state several propositions:

(55) The carrier of $\langle\mathbb{R},\cdot\rangle = \mathbb{R}$ and the operation of $\langle\mathbb{R},\cdot\rangle = \cdot_{\mathbb{R}}$ and for all elements a, b of $\langle\mathbb{R},\cdot\rangle$ and for all real numbers x, y such that $a = x$ and $b = y$ holds $a \cdot b = x \cdot y$.

(56) x is an element of $\langle\mathbb{R},\cdot\rangle$ if and only if x is a real number.

(57) $\mathbf{1}_{\text{the operation of } \langle\mathbb{R},\cdot\rangle} = 1$.

(58) For every subsystem N of $\langle\mathbb{R},\cdot\rangle$ and for all elements a, b of N and for all real numbers x, y such that $a = x$ and $b = y$ holds $a \cdot b = x \cdot y$.

(59) For every subsystem N of $\langle\mathbb{R},\cdot\rangle$ such that 1 is an element of N holds N is unital and $\mathbf{1}_{\text{the operation of } N} = 1$.

(60) For every unital subsystem N of $\langle\mathbb{R},\cdot\rangle$ holds $\mathbf{1}_{\text{the operation of } N} = 0$ or $\mathbf{1}_{\text{the operation of } N} = 1$.

We now define three new functors. The unital strict subsystem $\langle\mathbb{N},\cdot\rangle$ of $\langle\mathbb{R},\cdot\rangle$ with uniquely decomposable unity is defined by:

(Def.30) the carrier of $\langle\mathbb{N},\cdot\rangle = \mathbb{N}$.

$\langle\mathbb{N},\cdot,1\rangle$ is a well unital strict monoidal extension of $\langle\mathbb{N},\cdot\rangle$.

The binary operation $\cdot_{\mathbb{N}}$ on $\mathbb{N}$ is defined by:

(Def.31) $\cdot_{\mathbb{N}}$ = the operation of $\langle\mathbb{N},\cdot\rangle$.

One can prove the following propositions:

(61) $\langle\mathbb{N},\cdot\rangle = \langle\mathbb{N},\cdot_{\mathbb{N}}\rangle$.

(62) For all natural numbers n_1, n_2 and for all elements m_1, m_2 of $\langle \mathbb{N}, \cdot \rangle$ such that $n_1 = m_1$ and $n_2 = m_2$ holds $m_1 \cdot m_2 = n_1 \cdot n_2$.

(63) $\mathbf{1}_{\text{the operation of } \langle \mathbb{N}, \cdot \rangle} = 1$.

(64) For all natural numbers n_1, n_2 and for all elements m_1, m_2 of $\langle \mathbb{N}, \cdot, 1 \rangle$ such that $n_1 = m_1$ and $n_2 = m_2$ holds $m_1 \cdot m_2 = n_1 \cdot n_2$.

(65) $\langle \mathbb{N}, \cdot, 1 \rangle = \langle \mathbb{N}, \cdot_{\mathbb{N}}, 1 \rangle$.

(66) $\cdot_{\mathbb{N}} = \cdot_{\mathbb{R}} \restriction [\!: \mathbb{N}, \mathbb{N} :\!]$.

(67) 1 is a unity w.r.t. $\cdot_{\mathbb{N}}$ and $\cdot_{\mathbb{N}}$ has a unity and $\mathbf{1}_{\cdot_{\mathbb{N}}} = 1$ and $\cdot_{\mathbb{N}}$ is commutative and $\cdot_{\mathbb{N}}$ is associative and $\cdot_{\mathbb{N}}$ has uniquely decomposable unity.

6. The monoid of finite sequences over the set

We now define three new functors. Let D be a non-empty set. The functor $\langle D^*, ^\frown \rangle$ yielding a unital associative cancelable constituted finite sequences strict half group structure with uniquely decomposable unity is defined by:

(Def.32) the carrier of $\langle D^*, ^\frown \rangle = D^*$ and for all elements p, q of $\langle D^*, ^\frown \rangle$ holds $p \otimes q = p \mathbin{^\frown} q$.

Let us consider D. $\langle D^*, ^\frown, \varepsilon \rangle$ is a well unital strict monoidal extension of $\langle D^*, ^\frown \rangle$.

The concatenation of D yielding a binary operation on D^* is defined as follows:

(Def.33) the concatenation of $D =$ the operation of $\langle D^*, ^\frown \rangle$.

We now state several propositions:

(68) $\langle D^*, ^\frown \rangle = \langle D^*, \text{the concatenation of } D \rangle$.

(69) $\mathbf{1}_{\text{the operation of } \langle D^*, ^\frown \rangle} = \varepsilon$.

(70) The carrier of $\langle D^*, ^\frown, \varepsilon \rangle = D^*$ and the operation of $\langle D^*, ^\frown, \varepsilon \rangle =$ the concatenation of D and the unity of $\langle D^*, ^\frown, \varepsilon \rangle = \varepsilon$.

(71) For all elements a, b of $\langle D^*, ^\frown, \varepsilon \rangle$ holds $a \otimes b = a \mathbin{^\frown} b$.

(72) For every subsystem F of $\langle D^*, ^\frown \rangle$ and for all elements p, q of F holds $p \otimes q = p \mathbin{^\frown} q$.

(73) For every unital subsystem F of $\langle D^*, ^\frown \rangle$ holds $\mathbf{1}_{\text{the operation of } F} = \varepsilon$.

(74) For every subsystem F of $\langle D^*, ^\frown \rangle$ such that ε is an element of F holds F is unital and $\mathbf{1}_{\text{the operation of } F} = \varepsilon$.

(75) For all non-empty sets A, B such that $A \subseteq B$ holds $\langle A^*, ^\frown \rangle$ is a subsystem of $\langle B^*, ^\frown \rangle$.

(76) The concatenation of D has a unity and $\mathbf{1}_{\text{the concatenation of } D} = \varepsilon$ and the concatenation of D is associative.

7. Monoids of mappings

We now define three new functors. Let X be a set. The semigroup of partial functions onto X yields a unital associative constituted functions strict half group structure and is defined by:

(Def.34) the carrier of the semigroup of partial functions onto $X = X \dot{\rightarrow} X$ and for all elements f, g of the semigroup of partial functions onto X holds $f \otimes g = f \circ g$.

Let X be a set. The monoid of partial functions onto X is a well unital strict monoidal extension of the semigroup of partial functions onto X.

The composition of X yields a binary operation on $X \dot{\rightarrow} X$ and is defined as follows:

(Def.35) the composition of $X =$ the operation of the semigroup of partial functions onto X.

We now state several propositions:

(77) x is an element of the semigroup of partial functions onto X if and only if x is a partial function from X to X.

(78) $\mathbf{1}_{\text{the operation of the semigroup of partial functions onto } X} = \mathrm{id}_X$.

(79) For every subsystem F of the semigroup of partial functions onto X and for all elements f, g of F holds $f \otimes g = f \circ g$.

(80) For every subsystem F of the semigroup of partial functions onto X such that id_X is an element of F holds F is unital and $\mathbf{1}_{\text{the operation of } F} = \mathrm{id}_X$.

(81) If $Y \subseteq X$, then the semigroup of partial functions onto Y is a subsystem of the semigroup of partial functions onto X.

We now define two new functors. Let X be a set. The semigroup of functions onto X yielding a unital strict subsystem of the semigroup of partial functions onto X is defined as follows:

(Def.36) the carrier of the semigroup of functions onto $X = X^X$.

Let X be a set. The monoid of functions onto X is a well unital strict monoidal extension of the semigroup of functions onto X.

The following four propositions are true:

(82) x is an element of the semigroup of functions onto X if and only if x is a function from X into X.

(83) The operation of the semigroup of functions onto $X =$ (the composition of X) $\restriction [\!: X^X, X^X :\!]$.

(84) $\mathbf{1}_{\text{the operation of the semigroup of functions onto } X} = \mathrm{id}_X$.

(85) The carrier of the monoid of functions onto $X = X^X$ and the operation of the monoid of functions onto $X =$ (the composition of X) $\restriction [\!: X^X, X^X :\!]$ and the unity of the monoid of functions onto $X = \mathrm{id}_X$.

Let X be a set. The group of permutations onto X yields a unital invertible strict subsystem of the semigroup of functions onto X and is defined by:

(Def.37) for every element f of the semigroup of functions onto X holds $f \in$ the carrier of the group of permutations onto X if and only if f is a permutation of X.

One can prove the following three propositions:

(86) x is an element of the group of permutations onto X if and only if x is a permutation of X.

(87) $\mathbf{1}_{\text{the operation of the group of permutations onto } X} = \mathrm{id}_X$ and $1_{\text{the group of permutations onto } X} = \mathrm{id}_X$.

(88) For every element f of the group of permutations onto X holds $f^{-1} = (f$ **qua** a function$)^{-1}$.

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Monoid of Multisets and Subsets

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Summary. The monoid of functions yielding elements of a group is introduced. The monoid of multisets over a set is constructed as such monoid where the target group is the group of natural numbers with addition. Moreover, the generalization of group operation onto the operation on subsets is present. That generalization is used to introduce the group 2^G of subsets of a group G.

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The articles [21], [22], [15], [3], [17], [10], [5], [14], [11], [7], [16], [20], [9], [8], [19], [6], [13], [1], [18], [23], [24], [12], [2], and [4] provide the notation and terminology for this paper.

1. Updating

We adopt the following convention: x, y are arbitrary, X, Y, Z are sets, and n is a natural number. We now define two new constructions. Let D be a non-empty set, and let d be an element of D. Then $\{d\}$ is a non-empty subset of D. Let D be a non-empty set, and let X_1, X_2 be subsets of D. Then $X_1 \cup X_2$ is a subset of D. Let D be a non-empty set, and let X_1 be a subset of D, and let X_2 be a non-empty subset of D. Then $X_1 \cup X_2$ is a non-empty subset of D. Let D_1, D_2, D be non-empty sets. A binary function from D_1, D_2 into D is a function from $[: D_1, D_2 :]$ into D.

Let f be a function, and let x_1, x_2, y be arbitrary. The functor $f(x_1, x_2)(y)$ is defined by:

(Def.1) $f(x_1, x_2)(y) = f(\langle x_1, x_2 \rangle)(y)$.

The following proposition is true

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(1) For all functions f, g and for arbitrary x_1, x_2, x such that $\langle x_1, x_2\rangle \in \operatorname{dom} f$ and $g = f(x_1, x_2)$ and $x \in \operatorname{dom} g$ holds $f(x_1,x_2)(x) = g(x)$.

Let A, D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D^A, and let x_1 be an element of D_1, and let x_2 be an element of D_2, and let x be an element of A. Then $f(x_1,x_2)(x)$ is an element of D. Let A be a set, and let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D, and let g_1 be a function from A into D_1, and let g_2 be a function from A into D_2. Then $f^\circ(g_1, g_2)$ is an element of D^A. Let A be a non-empty set, and let n be a natural number, and let x be an element of A. Then $n \longmapsto x$ is a finite sequence of elements of A. We introduce the functor $n \dot{\longmapsto} x$ as a synonym of $n \longmapsto x$. Let D be a non-empty set, and let A be a set, and let d be an element of D. Then $A \longmapsto d$ is an element of D^A. Let A be a set, and let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D, and let d be an element of D_1, and let g be a function from A into D_2. Then $f^\circ(d,g)$ is an element of D^A. Let A be a set, and let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D, and let g be a function from A into D_1, and let d be an element of D_2. Then $f^\circ(g,d)$ is an element of D^A.

We now state the proposition

(2) For all functions f, g and for every set X holds $(f \restriction X) \cdot g = f \cdot (X \restriction g)$.

The scheme *NonUniqFuncDEx* concerns a set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the following condition is met:

- for every x such that $x \in \mathcal{A}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[x, y]$.

2. Monoid of functions into a semigroup

Let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D, and let A be a set. The functor f°_A yields a binary function from ${D_1}^A$, ${D_2}^A$ into D^A and is defined by:

(Def.2) for every element f_1 of ${D_1}^A$ and for every element f_2 of ${D_2}^A$ holds $(f^\circ_A)(f_1, f_2) = f^\circ(f_1, f_2)$.

Next we state the proposition

(3) For all non-empty sets D_1, D_2, D and for every binary function f from D_1, D_2 into D and for every set A and for every function f_1 from A into D_1 and for every function f_2 from A into D_2 and for every x such that $x \in A$ holds $(f^\circ_A)(f_1, f_2)(x) = f(f_1(x), f_2(x))$.

For simplicity we adopt the following convention: A will denote a set, D will denote a non-empty set, a will denote an element of D, o, o' will denote binary operations on D, and f, g, h will denote functions from A into D. The following propositions are true:

(4) If o is commutative, then $o^\circ(f, g) = o^\circ(g, f)$.

(5) If o is associative, then $o^\circ(o^\circ(f, g), h) = o^\circ(f, o^\circ(g, h))$.

(6) If a is a unity w.r.t. o, then $o^\circ(a, f) = f$ and $o^\circ(f, a) = f$.

(7) If o is idempotent, then $o^\circ(f, f) = f$.

(8) If o is commutative, then o°_A is commutative.

(9) If o is associative, then o°_A is associative.

(10) If a is a unity w.r.t. o, then $A \longmapsto a$ is a unity w.r.t. o°_A.

(11) If o has a unity, then $\mathbf{1}_{o^\circ_A} = A \longmapsto \mathbf{1}_o$ and o°_A has a unity.

(12) If o is idempotent, then o°_A is idempotent.

(13) If o is invertible, then o°_A is invertible.

(14) If o is cancelable, then o°_A is cancelable.

(15) If o has uniquely decomposable unity, then o°_A has uniquely decomposable unity.

(16) If o absorbs o', then o°_A absorbs o'°_A.

(17) For all non-empty sets D_1, D_2, D, E_1, E_2, E and for every binary function o_1 from D_1, D_2 into D and for every binary function o_2 from E_1, E_2 into E such that $o_1 \leq o_2$ holds $o_1{}^\circ_A \leq o_2{}^\circ_A$.

Let G be a half group structure, and let A be a set. The functor G^A yielding a half group structure is defined by:

(Def.3) (i) $G^A = \langle(\text{the carrier of } G)^A, (\text{the operation of } G)^\circ_A, A \longmapsto \mathbf{1}_{\text{the operation of } G}$ **qua** an element of (the carrier of $G)^A$ **qua** a non-empty set$\rangle$ if G is unital,

(ii) $G^A = \langle(\text{the carrier of } G)^A, (\text{the operation of } G)^\circ_A\rangle$, otherwise.

In the sequel G denotes a half group structure. We now state two propositions:

(18) The carrier of $G^X =$ (the carrier of $G)^X$ and the operation of $G^X =$ (the operation of $G)^\circ_X$.

(19) x is an element of G^X if and only if x is a function from X into the carrier of G.

Let G be a half group structure, and let A be a set. Then G^A is a constituted functions half group structure.

We now state two propositions:

(20) For every element f of G^X holds $\operatorname{dom} f = X$ and $\operatorname{rng} f \subseteq$ the carrier of G.

(21) For all elements f, g of G^X if for every x such that $x \in X$ holds $f(x) = g(x)$, then $f = g$.

Let G be a half group structure, and let A be a non-empty set, and let f be an element of G^A. Then $\operatorname{rng} f$ is a non-empty subset of G. Let a be an element of A. Then $f(a)$ is an element of G.

We now state the proposition

(22) For all elements f_1, f_2 of G^D and for every element a of D holds $(f_1 \cdot f_2)(a) = f_1(a) \cdot f_2(a)$.

Let G be a unital half group structure, and let A be a set. Then G^A is a well unital constituted functions strict monoid structure.

One can prove the following propositions:

(23) For every G being a unital half group structure holds the unity of $G^X = X \longmapsto \mathbf{1}_{\text{the operation of } G}$.

(24) Let G be a half group structure. Let A be a set. Then

(i) if G is commutative, then G^A is commutative,

(ii) if G is associative, then G^A is associative,

(iii) if G is idempotent, then G^A is idempotent,

(iv) if G is invertible, then G^A is invertible,

(v) if G is cancelable, then G^A is cancelable,

(vi) if G has uniquely decomposable unity, then G^A has uniquely decomposable unity.

(25) For every subsystem H of G holds H^X is a subsystem of G^X.

(26) For every G being a unital half group structure and for every subsystem H of G such that $\mathbf{1}_{\text{the operation of } G} \in$ the carrier of H holds H^X is a monoidal subsystem of G^X.

Let G be a unital associative commutative cancelable half group structure with uniquely decomposable unity, and let A be a set. Then G^A is a commutative cancelable constituted functions strict monoid with uniquely decomposable unity.

3. Monoid of multisets over a set

Let A be a set. The functor $A^{\otimes}_{\omega}$ yields a commutative cancelable constituted functions strict monoid with uniquely decomposable unity and is defined by:

(Def.4) $A^{\otimes}_{\omega} = \langle \mathbb{N}, +, 0 \rangle^A$.

Next we state the proposition

(27) The carrier of $X^{\otimes}_{\omega} = \mathbb{N}^X$ and the operation of $X^{\otimes}_{\omega} = (+_{\mathbb{N}})^{\circ}_X$ and the unity of $X^{\otimes}_{\omega} = X \longmapsto 0$.

Let A be a set. A multiset over A is an element of $A^{\otimes}_{\omega}$.

Next we state two propositions:

(28) x is a multiset over X if and only if x is a function from X into $\mathbb{N}$.

(29) For every multiset m over X holds $\operatorname{dom} m = X$ and $\operatorname{rng} m \subseteq \mathbb{N}$.

Let A be a non-empty set, and let m be a multiset over A. Then $\operatorname{rng} m$ is a non-empty subset of $\mathbb{N}$. Let a be an element of A. Then $m(a)$ is a natural number.

Next we state two propositions:

(30) For all multisets m_1, m_2 over D and for every element a of D holds $(m_1 \otimes m_2)(a) = m_1(a) + m_2(a)$.

(31) $\chi_{Y,X}$ is a multiset over X.

Let us consider Y, X. Then $\chi_{Y,X}$ is a multiset over X. Let us consider X, and let n be a natural number. Then $X \longmapsto n$ is a multiset over X. Let A be a non-empty set, and let a be an element of A. The functor χa yields a multiset over A and is defined as follows:

(Def.5) $\chi a = \chi_{\{a\},A}$.

One can prove the following proposition

(32) For every non-empty set A and for all elements a, b of A holds $(\chi a)(a) = 1$ and also if $b \neq a$, then $(\chi a)(b) = 0$.

For simplicity we follow a convention: A denotes a non-empty set, a denotes an element of A, p, q denote finite sequences of elements of A, and m_1, m_2 denote multisets over A. Next we state the proposition

(33) If for every a holds $m_1(a) = m_2(a)$, then $m_1 = m_2$.

Let A be a set. The functor $A^{\otimes}$ yields a strict monoidal subsystem of $A^{\otimes}_{\omega}$ and is defined as follows:

(Def.6) for every multiset f over A holds $f \in$ the carrier of $A^{\otimes}$ if and only if $f^{-1}(\mathbb{N} \setminus \{0\})$ is finite.

The following three propositions are true:

(34) χa is an element of $A^{\otimes}$.

(35) $\mathrm{dom}(\{x\} \restriction (p \frown \langle x \rangle)) = \mathrm{dom}(\{x\} \restriction p) \cup \{\mathrm{len}\, p + 1\}$.

(36) If $x \neq y$, then $\mathrm{dom}(\{x\} \restriction (p \frown \langle y \rangle)) = \mathrm{dom}(\{x\} \restriction p)$.

Let A be a non-empty set, and let p be a finite sequence of elements of A. The functor $|p|$ yields a multiset over A and is defined as follows:

(Def.7) for every element a of A holds $|p|(a) = \mathrm{card}\,\mathrm{dom}(\{a\} \restriction p)$.

We now state several propositions:

(37) $|\varepsilon_A|(a) = 0$.

(38) $|\varepsilon_A| = A \longmapsto 0$.

(39) $|\langle a \rangle| = \chi a$.

(40) $|p \frown \langle a \rangle| = |p| \otimes \chi a$.

(41) $|p \frown q| = |p| \otimes |q|$.

(42) $|n \dot{\longmapsto} a|(a) = n$ and for every element b of A such that $b \neq a$ holds $|n \dot{\longmapsto} a|(b) = 0$.

Next we state two propositions:

(43) $|p|$ is an element of $A^{\otimes}$.

(44) If x is an element of $A^{\otimes}$, then there exists p such that $x = |p|$.

4. Monoid of subsets of a semigroup

In the sequel a, b will be elements of D. Let D_1, D_2, D be non-empty sets, and let f be a binary function from D_1, D_2 into D. The functor ${}^\circ f$ yields a binary function from 2^{D_1}, 2^{D_2} into 2^D and is defined by:

(Def.8) for every element x of $[\!:2^{D_1}, 2^{D_2}:\!]$ holds $({}^\circ f)(x) = f^\circ [\!: x_{\mathbf{1}}, x_{\mathbf{2}} :\!]$.

One can prove the following propositions:

(45) For all non-empty sets D_1, D_2, D and for every binary function f from D_1, D_2 into D and for every subset X_1 of D_1 and for every subset X_2 of D_2 holds $({}^\circ f)(X_1, X_2) = f^\circ [\!: X_1, X_2 :\!]$.

(46) For all non-empty sets D_1, D_2, D and for every binary function f from D_1, D_2 into D and for every subset X_1 of D_1 and for every subset X_2 of D_2 and for arbitrary x_1, x_2 such that $x_1 \in X_1$ and $x_2 \in X_2$ holds $f(x_1, x_2) \in ({}^\circ f)(X_1, X_2)$.

(47) For all non-empty sets D_1, D_2, D and for every binary function f from D_1, D_2 into D and for every subset X_1 of D_1 and for every subset X_2 of D_2 holds $({}^\circ f)(X_1, X_2) = \{f(a, b) : a \in X_1 \wedge b \in X_2\}$, where a ranges over elements of D_1, and b ranges over elements of D_2.

(48) If o is commutative, then $o^\circ [\!: X, Y :\!] = o^\circ [\!: Y, X :\!]$.

(49) If o is associative, then $o^\circ [\!: o^\circ [\!: X, Y :\!], Z :\!] = o^\circ [\!: X, o^\circ [\!: Y, Z :\!] :\!]$.

(50) If o is commutative, then ${}^\circ o$ is commutative.

(51) If o is associative, then ${}^\circ o$ is associative.

(52) If a is a unity w.r.t. o, then $o^\circ [\!: \{a\}, X :\!] = D \cap X$ and $o^\circ [\!: X, \{a\} :\!] = D \cap X$.

(53) If a is a unity w.r.t. o, then $\{a\}$ is a unity w.r.t. ${}^\circ o$ and ${}^\circ o$ has a unity and $\mathbf{1}_{{}^\circ o} = \{a\}$.

(54) If o has a unity, then ${}^\circ o$ has a unity and $\{\mathbf{1}_o\}$ is a unity w.r.t. ${}^\circ o$ and $\mathbf{1}_{{}^\circ o} = \{\mathbf{1}_o\}$.

(55) If o has uniquely decomposable unity, then ${}^\circ o$ has uniquely decomposable unity.

Let G be a half group structure. The functor 2^G yields a half group structure and is defined by:

(Def.9) (i) $2^G = \langle 2^{\text{the carrier of } G}, {}^\circ(\text{the operation of } G), \{\mathbf{1}_{\text{the operation of } G}\}\rangle$ if G is unital,

(ii) $2^G = \langle 2^{\text{the carrier of } G}, {}^\circ(\text{the operation of } G)\rangle$, otherwise.

Let G be a unital half group structure. Then 2^G is a well unital strict monoid structure.

One can prove the following three propositions:

(56) The carrier of $2^G = 2^{\text{the carrier of } G}$ and the operation of $2^G = {}^\circ(\text{the operation of } G)$.

(57) For every G being a unital half group structure holds the unity of $2^G = \{\mathbf{1}_{\text{the operation of } G}\}$.

(58) For every G being a half group structure holds if G is commutative, then 2^G is commutative and also if G is associative, then 2^G is associative and also if G has uniquely decomposable unity, then 2^G has uniquely decomposable unity.

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Product of Families of Groups and Vector Spaces

Anna Lango
Warsaw University
Białystok

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Summary. In the first section we present properties of fields and Abelian groups in terms of commutativity, associativity, etc. Next, we are concerned with operations on n-tuples on some set which are generalization of operations on this set. It is used in third section to introduce the n-power of a group and the n-power of a field. Besides, we introduce a concept of indexed family of binary (unary) operations over some indexed family of sets and a product of such families which is binary (unary) operation on a product of family sets. We use that product in the last section to introduce the product of a finite sequence of Abelian groups.

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The notation and terminology used in this paper are introduced in the following articles: [16], [9], [10], [13], [3], [17], [2], [5], [6], [12], [4], [8], [7], [14], [1], [11], and [15].

1. Abelian Groups and Fields

In the sequel G will denote an Abelian group. The following propositions are true:

(1) The addition of G is commutative.

(2) The addition of G is associative.

(3) The zero of G is a unity w.r.t. the addition of G.

(4) The reverse-map of G is an inverse operation w.r.t. the addition of G.

In the sequel G_1 will be a group structure. Next we state the proposition

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(5) If the addition of G_1 is commutative and the addition of G_1 is associative and the zero of G_1 is a unity w.r.t. the addition of G_1 and the reverse-map of G_1 is an inverse operation w.r.t. the addition of G_1, then G_1 is an Abelian group.

In the sequel F is a field. We now state several propositions:

(6) The addition of F is commutative.

(7) The multiplication of F is commutative.

(8) The addition of F is associative.

(9) The multiplication of F is associative.

(10) The zero of F is a unity w.r.t. the addition of F.

(11) The unity of F is a unity w.r.t. the multiplication of F.

(12) The reverse-map of F is an inverse operation w.r.t. the addition of F.

(13) The multiplication of F is distributive w.r.t. the addition of F.

One can verify that every field-like field structure is Abelian group-like.

2. The n-Product of a Binary and a Unary Operation

For simplicity we follow a convention: F is a field, n is a natural number, D is a non-empty set, d is an element of D, B is a binary operation on D, and C is a unary operation on D. We now define three new functors. Let us consider D, n, and let F be a binary operation on D, and let x, y be elements of D^n. Then $F^\circ(x, y)$ is an element of D^n. Let D be a non-empty set, and let F be a binary operation on D, and let n be a natural number. The functor $\pi^n F$ yields a binary operation on D^n and is defined by:

(Def.1) for all elements x, y of D^n holds $(\pi^n F)(x, y) = F^\circ(x, y)$.

Let us consider D, and let F be a unary operation on D, and let us consider n. The functor $\pi^n F$ yields a unary operation on D^n and is defined as follows:

(Def.2) for every element x of D^n holds $(\pi^n F)(x) = F \cdot x$.

Let D be a non-empty set, and let us consider n, and let x be an element of D. Then $n \longmapsto x$ is an element of D^n. We introduce the functor $n \dot{\longmapsto} x$ as a synonym of $n \longmapsto x$.

The following four propositions are true:

(14) If B is commutative, then $\pi^n B$ is commutative.

(15) If B is associative, then $\pi^n B$ is associative.

(16) If d is a unity w.r.t. B, then $n \dot{\longmapsto} d$ is a unity w.r.t. $\pi^n B$.

(17) If B has a unity and B is associative and C is an inverse operation w.r.t. B, then $\pi^n C$ is an inverse operation w.r.t. $\pi^n B$.

3. The n-Power of a Group and of a Field

Let F be an Abelian group, and let us consider n. The functor F^n yielding a strict Abelian group is defined as follows:

(Def.3) $F^n = \langle$(the carrier of $F)^n, \pi^n$(the addition of $F), \pi^n$(the reverse-map of $F), n \longmapsto$ the zero of F **qua** an element of (the carrier of $F)^n\rangle$.

We now define two new functors. Let us consider F, n. The functor $\cdot_F^n$ yields a function from $[\!:$ the carrier of F, (the carrier of $F)^n :\!]$ into (the carrier of $F)^n$ and is defined by:

(Def.4) for every element x of F and for every element v of (the carrier of $F)^n$ holds $(\cdot_F^n)(x, v) =$ (the multiplication of $F)^\circ(x, v)$.

Let us consider F, n. The n-dimension vector space over F yielding a strict vector space structure over F is defined as follows:

(Def.5) the group structure of the n-dimension vector space over $F = F^n$ and the multiplication of the n-dimension vector space over $F = \cdot_F^n$.

For simplicity we follow a convention: D will be a non-empty set, H, G will be binary operations on D, d will be an element of D, and t_1, t_2 will be elements of D^n. One can prove the following proposition

(18) If H is distributive w.r.t. G, then $H^\circ(d, G^\circ(t_1, t_2)) = G^\circ(H^\circ(d, t_1), H^\circ(d, t_2))$.

Let D be a non-empty set, and let n be a natural number, and let F be a binary operation on D, and let x be an element of D, and let v be an element of D^n. Then $F^\circ(x, v)$ is an element of D^n. Let us consider F, n. Then the n-dimension vector space over F is a strict vector space over F.

4. Sequences of Non-empty Sets

In the sequel x will be arbitrary. We now define two new attributes. A function is non-empty set yielding if:

(Def.6) $\emptyset \notin \mathrm{rng}\,\mathrm{it}$.

A set is constituted functions if:

(Def.7) if $x \in$ it, then x is a function.

One can check that there exists a non-empty non-empty set yielding finite sequence and there exists a non-empty constituted functions set.

Let F be a constituted functions non-empty set. We see that the element of F is a function. Let f be a non-empty set yielding function. Then $\prod f$ is a constituted functions non-empty set. A sequence of non-empty sets is a non-empty non-empty set yielding finite sequence.

Let a be a non-empty function. Then $\mathrm{dom}\,a$ is a non-empty set.

The scheme *NEFinSeqLambda* concerns a non-empty finite sequence $\mathcal{A}$ and a unary functor $\mathcal{F}$ and states that:

there exists a non-empty finite sequence p such that $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and for every element i of $\operatorname{dom} \mathcal{A}$ holds $p(i) = \mathcal{F}(i)$
for all values of the parameters.

Let a be a non-empty set yielding non-empty function, and let i be an element of $\operatorname{dom} a$. Then $a(i)$ is a non-empty set. Let a be a non-empty set yielding non-empty function, and let f be an element of $\prod a$, and let i be an element of $\operatorname{dom} a$. Then $f(i)$ is an element of $a(i)$.

5. The Product of Families of Operations

In the sequel a will denote a sequence of non-empty sets, i will denote an element of $\operatorname{dom} a$, and p will denote a finite sequence. We now define two new modes. Let a be a non-empty set yielding non-empty function. A function is called a family of binary operations of a if:

(Def.8) $\operatorname{dom} \mathrm{it} = \operatorname{dom} a$ and for every element i of $\operatorname{dom} a$ holds $\mathrm{it}(i)$ is a binary operation on $a(i)$.

A function is said to be a family of unary operations of a if:

(Def.9) $\operatorname{dom} \mathrm{it} = \operatorname{dom} a$ and for every element i of $\operatorname{dom} a$ holds $\mathrm{it}(i)$ is a unary operation on $a(i)$.

Let us consider a. Note that every family of binary operations of a is finite sequence-like and every family of unary operations of a is finite sequence-like.

The following two propositions are true:

(19) p is a family of binary operations of a if and only if $\operatorname{len} p = \operatorname{len} a$ and for every i holds $p(i)$ is a binary operation on $a(i)$.

(20) p is a family of unary operations of a if and only if $\operatorname{len} p = \operatorname{len} a$ and for every i holds $p(i)$ is a unary operation on $a(i)$.

Let us consider a, and let b be a family of binary operations of a, and let us consider i. Then $b(i)$ is a binary operation on $a(i)$. Let us consider a, and let u be a family of unary operations of a, and let us consider i. Then $u(i)$ is a unary operation on $a(i)$. Let F be a constituted functions non-empty set, and let u be a unary operation on F, and let f be an element of F. Then $u(f)$ is an element of F.

In the sequel f is arbitrary. One can prove the following proposition

(21) For all unary operations d, d' on $\prod a$ if for every element f of $\prod a$ and for every element i of $\operatorname{dom} a$ holds $d(f)(i) = d'(f)(i)$, then $d = d'$.

We now state the proposition

(22) For every family u of unary operations of a holds $\operatorname{dom}_\kappa u(\kappa) = a$ and $\prod(\operatorname{rng}_\kappa u(\kappa)) \subseteq \prod a$.

Let us consider a, and let u be a family of unary operations of a. Then $\prod^\circ u$ is a unary operation on $\prod a$.

We now state the proposition

(23) For every family u of unary operations of a and for every element f of $\prod a$ and for every element i of $\mathrm{dom}\, a$ holds $(\prod^{\circ} u)(f)(i) = u(i)(f(i))$.

Let F be a constituted functions non-empty set, and let b be a binary operation on F, and let f, g be elements of F. Then $b(f, g)$ is an element of F.

The following proposition is true

(24) For all binary operations d, d' on $\prod a$ if for all elements f, g of $\prod a$ and for every element i of $\mathrm{dom}\, a$ holds $d(f, g)(i) = d'(f, g)(i)$, then $d = d'$.

In the sequel i will denote an element of $\mathrm{dom}\, a$. Let us consider a, and let b be a family of binary operations of a. The functor $\prod^{\circ} b$ yields a binary operation on $\prod a$ and is defined by:

(Def.10) for all elements f, g of $\prod a$ and for every element i of $\mathrm{dom}\, a$ holds $(\prod^{\circ} b)(f, g)(i) = b(i)(f(i), g(i))$.

The following propositions are true:

(25) For every family b of binary operations of a if for every i holds $b(i)$ is commutative, then $\prod^{\circ} b$ is commutative.

(26) For every family b of binary operations of a if for every i holds $b(i)$ is associative, then $\prod^{\circ} b$ is associative.

(27) For every family b of binary operations of a and for every element f of $\prod a$ if for every i holds $f(i)$ is a unity w.r.t. $b(i)$, then f is a unity w.r.t. $\prod^{\circ} b$.

(28) For every family b of binary operations of a and for every family u of unary operations of a if for every i holds $u(i)$ is an inverse operation w.r.t. $b(i)$ and $b(i)$ has a unity, then $\prod^{\circ} u$ is an inverse operation w.r.t. $\prod^{\circ} b$.

6. The Product of Families of Groups

We now define three new constructions. A function is Abelian group yielding if:

(Def.11) if $x \in \mathrm{rng}\, \mathrm{it}$, then x is an Abelian group.

One can check that there exists a non-empty Abelian group yielding finite sequence.

A sequence of groups is a non-empty Abelian group yielding finite sequence.

Let g be a sequence of groups, and let i be an element of $\mathrm{dom}\, g$. Then $g(i)$ is an Abelian group. Let g be a sequence of groups. The functor $\overline{g}$ yielding a sequence of non-empty sets is defined as follows:

(Def.12) $\mathrm{len}\, \overline{g} = \mathrm{len}\, g$ and for every element j of $\mathrm{dom}\, g$ holds $\overline{g}(j) =$ the carrier of $g(j)$.

In the sequel g is a sequence of groups and i is an element of $\mathrm{dom}\, \overline{g}$. We now define four new functors. Let us consider g, i. Then $g(i)$ is an Abelian group. Let us consider g. The functor $\langle +_{g_i} \rangle_i$ yields a family of binary operations of $\overline{g}$ and is defined by:

(Def.13) $\mathrm{len}(\langle +_{g_i} \rangle_i) = \mathrm{len}\, \overline{g}$ and for every i holds $\langle +_{g_i} \rangle_i(i) =$ the addition of $g(i)$.

The functor $\langle -_{g_i} \rangle_i$ yields a family of unary operations of $\overline{g}$ and is defined by:

(Def.14) $\mathrm{len}(\langle -_{g_i} \rangle_i) = \mathrm{len}\, \overline{g}$ and for every i holds $\langle -_{g_i} \rangle_i(i) =$ the reverse-map of $g(i)$.

The functor $\langle 0_{g_i} \rangle_i$ yields an element of $\prod \overline{g}$ and is defined by:

(Def.15) for every i holds $\langle 0_{g_i} \rangle_i(i) =$ the zero of $g(i)$.

Let G be a sequence of groups. The functor $\prod G$ yields a strict Abelian group and is defined by:

(Def.16) $\prod G = \langle \prod \overline{G}, \prod^{\circ}(\langle +_{G_i} \rangle_i), \prod^{\circ}(\langle -_{G_i} \rangle_i), \langle 0_{G_i} \rangle_i \rangle$.

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On a Mathematical Model of Programs

Yatsuka Nakamura
Shinshu University
Nagano

Andrzej Trybulec
Warsaw University
Białystok

Summary. We continue the work on mathematical modeling of hardware and software started in [17]. The main objective of this paper is the definition of a program. We start with the concept of partial product, i.e. the set of all partial functions f from I to $\bigcup_{i \in I} A_i$, fulfilling the condition $f.i \in A_i$ for $i \in dom f$. The computation and the result of a computation are defined in usual way. A finite partial state is called autonomic if the result of a computation starting with it does not depend on the remaining memory and an AMI is called programmable if it has a non empty autonomic partial finite state. We prove the consistency of the following set of properties of an AMI: data-oriented, halting, steady-programmed, realistic and programmable. For this purpose we define a trivial AMI. It has only the instruction counter and one instruction location. The only instruction of it is the halt instruction. A preprogram is a finite partial state that halts. We conclude with the definition of a program of a partial function F mapping the set of the finite partial states into itself. It is a finite partial state s such that for every finite partial state $s' \in dom F$ the result of any computation starting with $s+s'$ includes $F.s'$.

MML Identifier: `AMI_2`.

The papers [24], [22], [28], [6], [7], [23], [14], [1], [19], [26], [25], [10], [3], [5], [15], [29], [21], [2], [20], [8], [18], [4], [9], [12], [13], [27], [11], [16], and [17] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity we follow the rules: A, B, C will denote sets, f, g, h will denote functions, x, y, z will be arbitrary, and i, j, k will denote natural numbers. The scheme *UniqSet* concerns a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{A} = \mathcal{B}$

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provided the following requirements are met:

- for every x holds $x \in \mathcal{A}$ if and only if $\mathcal{P}[x]$,
- for every x holds $x \in \mathcal{B}$ if and only if $\mathcal{P}[x]$.

The following propositions are true:

(1) A misses $B \setminus C$ if and only if B misses $A \setminus C$.

(2) For every function f holds $\pi_1(\operatorname{dom} f \times \operatorname{rng} f)^\circ f = \operatorname{dom} f$.

(3) If $f \approx g$ and $\langle x, y \rangle \in f$ and $\langle x, z \rangle \in g$, then $y = z$.

(4) If for every x such that $x \in A$ holds x is a function and for all functions f, g such that $f \in A$ and $g \in A$ holds $f \approx g$, then $\bigcup A$ is a function.

(5) If $\operatorname{dom} f \subseteq A \cup B$, then $f \upharpoonright A +\cdot f \upharpoonright B = f$.

(6) $\operatorname{dom} f \subseteq \operatorname{dom}(f +\cdot g)$ and $\operatorname{dom} g \subseteq \operatorname{dom}(f +\cdot g)$.

(7) For arbitrary x_1, x_2, y_1, y_2 holds $[x_1 \longmapsto y_1, x_2 \longmapsto y_2] = (x_1 \dot{\longmapsto} y_1) +\cdot (x_2 \dot{\longmapsto} y_2)$.

(8) For all x, y holds $x \dot{\longmapsto} y = \{\langle x, y \rangle\}$.

(9) For arbitrary a, b, c holds $[a \longmapsto b, a \longmapsto c] = a \dot{\longmapsto} c$.

(10) For every function f holds $\operatorname{dom} f$ is finite if and only if f is finite.

(11) If $x \in \prod f$, then x is a function.

2. Partial products

Let f be a function. The functor $\prod^{\cdot} f$ yields a non-empty set of functions and is defined by:

(Def.1) $x \in \prod^{\cdot} f$ if and only if there exists g such that $x = g$ and $\operatorname{dom} g \subseteq \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} g$ holds $g(x) \in f(x)$.

Next we state a number of propositions:

(12) $x \in \prod^{\cdot} f$ if and only if there exists g such that $x = g$ and $\operatorname{dom} g \subseteq \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} g$ holds $g(x) \in f(x)$.

(13) If $\operatorname{dom} g \subseteq \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} g$ holds $g(x) \in f(x)$, then $g \in \prod^{\cdot} f$.

(14) If $g \in \prod^{\cdot} f$, then $\operatorname{dom} g \subseteq \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} g$ holds $g(x) \in f(x)$.

(15) $\square \in \prod^{\cdot} f$.

(16) $\prod f \subseteq \prod^{\cdot} f$.

(17) If $x \in \prod^{\cdot} f$, then x is a partial function from $\operatorname{dom} f$ to $\bigcup \operatorname{rng} f$.

(18) If $g \in \prod f$ and $h \in \prod^{\cdot} f$, then $g +\cdot h \in \prod f$.

(19) If $\prod f \neq \emptyset$, then $g \in \prod^{\cdot} f$ if and only if there exists h such that $h \in \prod f$ and $g \leq h$.

(20) $\prod^{\cdot} f \subseteq \operatorname{dom} f \dot{\rightarrow} \bigcup \operatorname{rng} f$.

(21) If $f \subseteq g$, then $\prod^{\cdot} f \subseteq \prod^{\cdot} g$.

(22) $\prod^{\cdot} \square = \{\square\}$.

(23) $A \dot{\to} B = \prod^{\cdot}(A \longmapsto B)$.

(24) For all non-empty sets A, B and for every function f from A into B holds $\prod^{\cdot} f = \prod^{\cdot}(f \restriction \{x : f(x) \neq \emptyset\})$, where x ranges over elements of A.

(25) If $x \in \operatorname{dom} f$ and $y \in f(x)$, then $x \dot{\longmapsto} y \in \prod^{\cdot} f$.

(26) $\prod^{\cdot} f = \{\square\}$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = \emptyset$.

(27) If $A \subseteq \prod^{\cdot} f$ and for all functions h_1, h_2 such that $h_1 \in A$ and $h_2 \in A$ holds $h_1 \approx h_2$, then $\bigcup A \in \prod^{\cdot} f$.

(28) If $g \approx h$ and $g \in \prod^{\cdot} f$ and $h \in \prod^{\cdot} f$, then $g \cup h \in \prod^{\cdot} f$.

(29) If $g \subseteq h$ and $h \in \prod^{\cdot} f$, then $g \in \prod^{\cdot} f$.

(30) If $g \in \prod^{\cdot} f$, then $g \restriction A \in \prod^{\cdot} f$.

(31) If $g \in \prod^{\cdot} f$, then $g \restriction A \in \prod^{\cdot}(f \restriction A)$.

(32) If $h \in \prod^{\cdot}(f +\cdot g)$, then there exist functions f', g' such that $f' \in \prod^{\cdot} f$ and $g' \in \prod^{\cdot} g$ and $h = f' +\cdot g'$.

(33) For all functions f', g' such that $\operatorname{dom} g$ misses $\operatorname{dom} f' \setminus \operatorname{dom} g'$ and $f' \in \prod^{\cdot} f$ and $g' \in \prod^{\cdot} g$ holds $f' +\cdot g' \in \prod^{\cdot}(f +\cdot g)$.

(34) For all functions f', g' such that $\operatorname{dom} f'$ misses $\operatorname{dom} g \setminus \operatorname{dom} g'$ and $f' \in \prod^{\cdot} f$ and $g' \in \prod^{\cdot} g$ holds $f' +\cdot g' \in \prod^{\cdot}(f +\cdot g)$.

(35) If $g \in \prod^{\cdot} f$ and $h \in \prod^{\cdot} f$, then $g +\cdot h \in \prod^{\cdot} f$.

(36) For arbitrary x_1, x_2, y_1, y_2 such that $x_1 \in \operatorname{dom} f$ and $y_1 \in f(x_1)$ and $x_2 \in \operatorname{dom} f$ and $y_2 \in f(x_2)$ holds $[x_1 \longmapsto y_1, x_2 \longmapsto y_2] \in \prod^{\cdot} f$.

3. Computations

In the sequel N is a non-empty set with non-empty elements.

We now define five new constructions. Let us consider N, and let S be a von Neumann definite AMI over N, and let s be a state of S. The functor $\operatorname{CurInstr}(s)$ yields an instruction of S and is defined as follows:

(Def.2) $\operatorname{CurInstr}(s) = s(\mathbf{IC}_s)$.

Let us consider N, and let S be a von Neumann definite AMI over N, and let s be a state of S. The functor $\operatorname{Following}(s)$ yielding a state of S is defined by:

(Def.3) $\operatorname{Following}(s) = \operatorname{Exec}(\operatorname{CurInstr}(s), s)$.

Let us consider N, and let S be a von Neumann definite AMI over N, and let s be a state of S. The functor $\operatorname{Computation}(s)$ yielding a function from $\mathbb{N}$ into $\prod$(the object kind of S) **qua** a non-empty set is defined by:

(Def.4) $(\operatorname{Computation}(s))(0) = s$ **qua** an element of $\prod$(the object kind of S) **qua** a non-empty set and for every i and for every element x of $\prod$(the object kind of S) **qua** a non-empty set such that $x = (\operatorname{Computation}(s))(i)$ holds $(\operatorname{Computation}(s))(i+1) = \operatorname{Following}(x)$.

Let us consider N, and let S be a von Neumann definite AMI over N. A state of S is halting if:

(Def.5) there exists k such that CurInstr((Computation(it))(k)) = $\mathbf{halt}_S$.

Let us consider N, and let S be an AMI over N, and let f be a function from $\mathbb{N}$ into $\prod$(the object kind of S) **qua** a non-empty set, and let us consider k. Then $f(k)$ is a state of S. Let us consider N. An AMI over N is realistic if:

(Def.6) the instructions of it $\neq$ the instruction locations of it.

One can prove the following proposition

(37) For every S being a von Neumann definite AMI over N such that S is realistic holds for no instruction-location l of S holds $\mathbf{IC}_S = l$.

In the sequel S denotes a von Neumann definite AMI over N and s denotes a state of S. One can prove the following propositions:

(38) (Computation(s))(0) = s.

(39) (Computation(s))($k+1$) = Following((Computation(s))(k)).

(40) For every k holds
(Computation(s))($i+k$) = (Computation((Computation(s))(i)))(k).

(41) If $i \leq j$, then for every N and for every S being a halting von Neumann definite AMI over N and for every state s of S such that
CurInstr((Computation(s))(i)) = $\mathbf{halt}_S$
holds (Computation(s))(j) = (Computation(s))(i).

Let us consider N, and let S be a halting von Neumann definite AMI over N, and let s be a state of S satisfying the condition: s is halting. The functor Result(s) yields a state of S and is defined as follows:

(Def.7) there exists k such that Result(s) = (Computation(s))(k) and CurInstr(Result(s)) = $\mathbf{halt}_S$.

Next we state the proposition

(42) For every N and for every S being a steady-programmed von Neumann definite AMI over N and for every state s of S and for every instruction-location i of S holds $s(i)$ = (Following(s))(i).

Let us consider N, and let S be a definite AMI over N, and let s be a state of S, and let l be an instruction-location of S. Then $s(l)$ is an instruction of S.

Next we state several propositions:

(43) For every N and for every S being a steady-programmed von Neumann definite AMI over N and for every state s of S and for every instruction-location i of S and for every k holds $s(i)$ = (Computation(s))(k)(i).

(44) For every N and for every S being a steady-programmed von Neumann definite AMI over N and for every state s of S holds (Computation(s))($k+1$) = Exec($s(\mathbf{IC}_{(\mathrm{Computation}(s))(k)})$, (Computation($s$))($k$)).

(45) For every N and for every S being a steady-programmed von Neumann halting definite AMI over N and for every state s of S and for every k such that $s(\mathbf{IC}_{(\mathrm{Computation}(s))(k)}) = \mathbf{halt}_S$
holds Result(s) = (Computation(s))(k).

(46) For every N and for every S being a steady-programmed von Neumann halting definite AMI over N and for every state s of S such that there exists k such that $s(\mathbf{IC}_{(\mathrm{Computation}(s))(k)}) = \mathbf{halt}_S$ and for every i holds $\mathrm{Result}(s) = \mathrm{Result}((\mathrm{Computation}(s))(i))$.

(47) For every S being an AMI over N and for every object o of S holds $\mathrm{ObjectKind}(o)$ is non-empty.

4. Finite partial states

We now define five new constructions. Let us consider N, and let S be an AMI over N. The functor $\mathrm{FinPartSt}(S)$ yielding a subset of $\prod^{\cdot}$ (the object kind of S) is defined by:

(Def.8) $\mathrm{FinPartSt}(S) = \{p : p \text{ is finite}\}$, where p ranges over elements of $\prod^{\cdot}$ (the object kind of S).

Let us consider N, and let S be an AMI over N. An element of $\prod^{\cdot}$ (the object kind of S) is called a finite partial state of S if:

(Def.9) it is finite.

Let us consider N, and let S be a von Neumann definite AMI over N. A finite partial state of S is autonomic if:

(Def.10) for all states s_1, s_2 of S such that it $\subseteq s_1$ and it $\subseteq s_2$ and for every i holds $(\mathrm{Computation}(s_1))(i) \restriction \mathrm{dom}\,\mathrm{it} = (\mathrm{Computation}(s_2))(i) \restriction \mathrm{dom}\,\mathrm{it}$.

A finite partial state of S is halting if:

(Def.11) for every state s of S such that it $\subseteq s$ holds s is halting.

Let us consider N. A von Neumann definite AMI over N is programmable if:

(Def.12) there exists a finite partial state of it which is non-empty and autonomic.

We now state two propositions:

(48) For every S being a von Neumann definite AMI over N and for all non-empty sets A, B and for all objects l_1, l_2 of S such that $\mathrm{ObjectKind}(l_1) = A$ and $\mathrm{ObjectKind}(l_2) = B$ and for every element a of A and for every element b of B holds $[l_1 \longmapsto a, l_2 \longmapsto b]$ is a finite partial state of S.

(49) For every S being a von Neumann definite AMI over N and for every non-empty set A and for every object l_1 of S such that $\mathrm{ObjectKind}(l_1) = A$ and for every element a of A holds $l_1 \dot{\longmapsto} a$ is a finite partial state of S.

Let us consider N, and let S be a von Neumann definite AMI over N, and let l_1 be an object of S, and let a be an element of $\mathrm{ObjectKind}(l_1)$. Then $l_1 \dot{\longmapsto} a$ is a finite partial state of S. Let us consider N, and let S be a von Neumann definite AMI over N, and let l_1, l_2 be objects of S, and let a be an element of $\mathrm{ObjectKind}(l_1)$, and let b be an element of $\mathrm{ObjectKind}(l_2)$. Then $[l_1 \longmapsto a, l_2 \longmapsto b]$ is a finite partial state of S.

5. Trivial AMI

Let us consider N. The functor $\mathbf{AMI}_{\rm t}$ yields a strict AMI over N and is defined by the conditions (Def.13).

(Def.13) (i) The objects of $\mathbf{AMI}_{\rm t} = \{0, 1\}$,

(ii) the instruction counter of $\mathbf{AMI}_{\rm t} = 0$,

(iii) the instruction locations of $\mathbf{AMI}_{\rm t} = \{1\}$,

(iv) the instruction codes of $\mathbf{AMI}_{\rm t} = \{0\}$,

(v) the halt instruction of $\mathbf{AMI}_{\rm t} = 0$,

(vi) the instructions of $\mathbf{AMI}_{\rm t} = \{\langle 0, \varepsilon\rangle\}$,

(vii) the object kind of $\mathbf{AMI}_{\rm t} = [0 \longmapsto \{1\}, 1 \longmapsto \{\langle 0, \varepsilon\rangle\}]$,

(viii) the execution of $\mathbf{AMI}_{\rm t} = \{\langle 0, \varepsilon\rangle\} \longmapsto {\rm id}_{\prod [0 \longmapsto \{1\}, 1 \longmapsto \{\langle 0, \varepsilon\rangle\}]}$.

Next we state several propositions:

(50) $\mathbf{AMI}_{\rm t}$ is von Neumann.

(51) $\mathbf{AMI}_{\rm t}$ is data-oriented.

(52) $\mathbf{AMI}_{\rm t}$ is halting.

(53) For all states s_1, s_2 of $\mathbf{AMI}_{\rm t}$ holds $s_1 = s_2$.

(54) $\mathbf{AMI}_{\rm t}$ is steady-programmed.

(55) $\mathbf{AMI}_{\rm t}$ is definite.

(56) $\mathbf{AMI}_{\rm t}$ is realistic.

Let us consider N. Then $\mathbf{AMI}_{\rm t}$ is a von Neumann definite strict AMI over N.

One can prove the following proposition

(57) $\mathbf{AMI}_{\rm t}$ is programmable.

Let us consider N. Note that there exists a von Neumann definite strict AMI over N which is data-oriented halting steady-programmed realistic and programmable.

One can prove the following two propositions:

(58) For every S being an AMI over N and for every state s of S and for every finite partial state p of S holds $s \restriction \operatorname{dom} p$ is a finite partial state of S.

(59) For every S being an AMI over N holds $\emptyset$ is a finite partial state of S.

Let us consider N, and let S be a von Neumann definite AMI over N. Observe that there exists a non-empty autonomic finite partial state of S.

Let us consider N, and let S be an AMI over N, and let f, g be finite partial states of S. Then $f +\cdot g$ is a finite partial state of S.

6. Autonomic finite partial states

We now state four propositions:

(60) For every S being a realistic von Neumann definite AMI over N and for every instruction-location l_3 of S and for every element l of ObjectKind($\mathbf{IC}_S$) such that $l = l_3$ and for every element h of ObjectKind(l_3) such that $h = \mathbf{halt}_S$ and for every state s of S such that $[\mathbf{IC}_S \longmapsto l, l_3 \longmapsto h] \subseteq s$ holds CurInstr(s) $= \mathbf{halt}_S$.

(61) For every S being a realistic von Neumann definite AMI over N and for every instruction-location l_3 of S and for every element l of ObjectKind($\mathbf{IC}_S$) such that $l = l_3$ and for every element h of ObjectKind(l_3) such that $h = \mathbf{halt}_S$ holds $[\mathbf{IC}_S \longmapsto l, l_3 \longmapsto h]$ is halting.

(62) Let S be a realistic halting von Neumann definite AMI over N. Then for every instruction-location l_3 of S and for every element l of ObjectKind($\mathbf{IC}_S$) such that $l = l_3$ and for every element h of ObjectKind(l_3) such that $h = \mathbf{halt}_S$ and for every state s of S such that $[\mathbf{IC}_S \longmapsto l, l_3 \longmapsto h] \subseteq s$ and for every i holds (Computation(s))(i) $= s$.

(63) For every S being a realistic halting von Neumann definite AMI over N and for every instruction-location l_3 of S and for every element l of ObjectKind($\mathbf{IC}_S$) such that $l = l_3$ and for every element h of ObjectKind(l_3) such that $h = \mathbf{halt}_S$ holds $[\mathbf{IC}_S \longmapsto l, l_3 \longmapsto h]$ is autonomic.

We now define two new constructions. Let us consider N, and let S be a realistic halting von Neumann definite AMI over N. One can check that there exists a finite partial state of S which is autonomic and halting.

Let us consider N, and let S be a realistic halting von Neumann definite AMI over N. A pre-program of S is an autonomic halting finite partial state of S.

Let us consider N, and let S be a realistic halting von Neumann definite AMI over N, and let s be a finite partial state of S. Let us assume that s is a pre-program of S. The functor Result(s) yields a finite partial state of S and is defined as follows:

(Def.14) for every state s' of S such that $s \subseteq s'$ holds Result(s) $=$ Result(s') $\restriction$ dom s.

7. Pre-programs and programs

Let us consider N, and let S be a realistic halting von Neumann definite AMI over N, and let p be a finite partial state of S, and let F be a function. We say that p computes F if and only if:

(Def.15) for an arbitrary x such that $x \in \operatorname{dom} F$ there exists a finite partial state s of S such that $x = s$ and $p +\cdot s$ is a pre-program of S and $F(s) \subseteq$ Result($p +\cdot s$).

The following three propositions are true:

(64) For every S being a realistic halting von Neumann definite AMI over N and for every finite partial state p of S holds p computes $\square$.

(65) For every S being a realistic halting von Neumann definite AMI over N and for every finite partial state p of S holds p is a pre-program of S if and only if p computes $\emptyset \mapsto \mathrm{Result}(p)$.

(66) For every S being a realistic halting von Neumann definite AMI over N and for every finite partial state p of S holds p is a pre-program of S if and only if p computes $\emptyset \mapsto \emptyset$.

Let us consider N, and let S be a realistic halting von Neumann definite AMI over N. A partial function from $\mathrm{FinPartSt}(S)$ to $\mathrm{FinPartSt}(S)$ is computable if:

(Def.16) there exists a finite partial state p of S such that p computes it.

Next we state three propositions:

(67) For every N and for every S being a realistic halting von Neumann definite AMI over N and for every partial function F from $\mathrm{FinPartSt}(S)$ to $\mathrm{FinPartSt}(S)$ such that $F = \square$ holds F is computable.

(68) For every N and for every S being a realistic halting von Neumann definite AMI over N and for every partial function F from $\mathrm{FinPartSt}(S)$ to $\mathrm{FinPartSt}(S)$ such that $F = \emptyset \mapsto \emptyset$ holds F is computable.

(69) For every N and for every S being a realistic halting von Neumann definite AMI over N and for every pre-program p of S and for every partial function F from $\mathrm{FinPartSt}(S)$ to $\mathrm{FinPartSt}(S)$ such that $F = \emptyset \mapsto \mathrm{Result}(p)$ holds F is computable.

Let us consider N, and let S be a realistic halting von Neumann definite AMI over N, and let F be a partial function from $\mathrm{FinPartSt}(S)$ to $\mathrm{FinPartSt}(S)$ satisfying the condition: F is computable. A finite partial state of S is called a program of F if:

(Def.17) it computes F.

The following propositions are true:

(70) For every N and for every S being a realistic halting von Neumann definite AMI over N and for every partial function F from $\mathrm{FinPartSt}(S)$ to $\mathrm{FinPartSt}(S)$ such that $F = \square$ every finite partial state of S is a program of F.

(71) For every N and for every S being a realistic halting von Neumann definite AMI over N and for every partial function F from $\mathrm{FinPartSt}(S)$ to $\mathrm{FinPartSt}(S)$ such that $F = \emptyset \mapsto \emptyset$ every pre-program of S is a program of F.

(72) For every N and for every S being a realistic halting von Neumann definite AMI over N and for every pre-program p of S and for every partial function F from $\mathrm{FinPartSt}(S)$ to $\mathrm{FinPartSt}(S)$ such that $F = \emptyset \mapsto \mathrm{Result}(p)$ holds p is a program of F.

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Basic Notation of Universal Algebra

Jarosław Kotowicz
Warsaw University
Białystok

Beata Madras
Warsaw University
Białystok

Małgorzata Korolkiewicz
Warsaw University
Białystok

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The papers [6], [3], [1], [5], [4], and [2] provide the terminology and notation for this paper. For simplicity we adopt the following convention: A denotes a non-empty set, a denotes an element of A, x, y denote finite sequences of elements of A, h denotes a partial function from A^* to A, and n denotes a natural number. We now define two new attributes. Let us consider A. A partial function from A^* to A is homogeneous if:

(Def.1) for all x, y such that $x \in \mathrm{dom}\,\mathrm{it}$ and $y \in \mathrm{dom}\,\mathrm{it}$ holds $\mathrm{len}\, x = \mathrm{len}\, y$.

Let us consider A. A partial function from A^* to A is quasi total if:

(Def.2) for all x, y such that $\mathrm{len}\, x = \mathrm{len}\, y$ and $x \in \mathrm{dom}\,\mathrm{it}$ holds $y \in \mathrm{dom}\,\mathrm{it}$.

Let us consider A. Note that there exists a homogeneous quasi total non-empty partial function from A^* to A.

We now state three propositions:

(1) h is a non-empty partial function from A^* to A if and only if $\mathrm{dom}\, h \neq \emptyset$.

(2) $\{\varepsilon_A\} \longmapsto a$ is a homogeneous quasi total non-empty partial function from A^* to A.

(3) $\{\varepsilon_A\} \longmapsto a$ is an element of $A^* \dot{\rightarrow} A$.

We now define four new constructions. We consider universal algebra structures which are extension of a 1-sorted structure and are systems

⟨a carrier, a characteristic⟩,

where the carrier is a non-empty set and the characteristic is a finite sequence of elements of $(\text{the carrier})^* \dot{\rightarrow} \text{the carrier}$. Let us consider A. A finite sequence of elements of $A^* \dot{\rightarrow} A$ is homogeneous if:

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(Def.3) for all n, h such that $n \in \mathrm{dom}\,\mathrm{it}$ and $h = \mathrm{it}(n)$ holds h is homogeneous.

Let us consider A. A finite sequence of elements of $A^{*} \dot{\to} A$ is quasi total if:

(Def.4) for all n, h such that $n \in \mathrm{dom}\,\mathrm{it}$ and $h = \mathrm{it}(n)$ holds h is quasi total.

Let us consider A. A finite sequence of elements of $A^{*} \dot{\to} A$ is non-empty if:

(Def.5) for all n, h such that $n \in \mathrm{dom}\,\mathrm{it}$ and $h = \mathrm{it}(n)$ holds h is non-empty.

In the sequel U will be a universal algebra structure. We now define four new constructions. Let us consider U. The functor $\mathrm{Opers}\,U$ yielding a finite sequence of elements of (the carrier of $U)^{*} \dot{\to}$the carrier of U is defined as follows:

(Def.6) $\mathrm{Opers}\,U =$ the characteristic of U.

A universal algebra structure is partial if:

(Def.7) $\mathrm{Opers}\,\mathrm{it}$ is homogeneous.

A universal algebra structure is quasi total if:

(Def.8) $\mathrm{Opers}\,\mathrm{it}$ is quasi total.

A universal algebra structure is non-empty if:

(Def.9) $\mathrm{Opers}\,\mathrm{it} \neq \varepsilon$ and $\mathrm{Opers}\,\mathrm{it}$ is non-empty.

We now state the proposition

(4) For every element x of $A^{*} \dot{\to} A$ such that $x = \{\varepsilon_{A}\} \longmapsto a$ holds $\langle x \rangle$ is homogeneous, quasi total and non-empty.

Let us note that there exists a quasi total partial non-empty strict universal algebra structure.

A universal algebra is a quasi total partial non-empty universal algebra structure.

In the sequel U will be a universal algebra. Let us consider A, and let f be a homogeneous quasi total non-empty partial function from A^{*} to A. The functor $\mathrm{arity}\,f$ yielding a natural number is defined as follows:

(Def.10) if $x \in \mathrm{dom}\,f$, then $\mathrm{arity}\,f = \mathrm{len}\,x$.

The following proposition is true

(5) For every U and for every n such that $n \in \mathrm{dom}\,\mathrm{Opers}\,U$ holds $(\mathrm{Opers}\,U)(n)$ is a homogeneous quasi total non-empty partial function from (the carrier of $U)^{*}$ to the carrier of U.

Let U be a universal algebra. The functor $\mathrm{signature}\,U$ yields a finite sequence of elements of $\mathbb{N}$ and is defined as follows:

(Def.11) $\mathrm{len}\,\mathrm{signature}\,U = \mathrm{len}\,\mathrm{Opers}\,U$ and for every n such that $n \in \mathrm{dom}\,\mathrm{signature}\,U$ and for every homogeneous quasi total non-empty partial function h from (the carrier of $U)^{*}$ to the carrier of U such that $h = (\mathrm{Opers}\,U)(n)$ holds $(\mathrm{signature}\,U)(n) = \mathrm{arity}\,h$.

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Coherent Space

Jarosław Kotowicz
Warsaw University
Białystok

Konrad Raczkowski
Warsaw University
Białystok

Summary. Coherent Space web of coherent space and two categories: category of coherent spaces and category of tolerances on same fixed set.

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The articles [8], [10], [11], [1], [5], [9], [6], [2], [7], [4], and [3] provide the notation and terminology for this paper. We follow a convention: x, y will be arbitrary and a, b, X, A will be sets. Let F be a non-empty set of functions. We see that the element of F is a function.

1. Coherent Space and Web of Coherent Space

We now define three new constructions. A set is down-closed if:

(Def.1) for all a, b such that $a \in$ it and $b \subseteq a$ holds $b \in$ it.

A set is binary complete if:

(Def.2) for every A such that $A \subseteq$ it and for all a, b such that $a \in A$ and $b \in A$ holds $a \cup b \in$ it holds $\bigcup A \in$ it.

Let us observe that there exists a down-closed binary complete non-empty set.

A coherent space is a down-closed binary complete non-empty set.

In the sequel C, D are coherent spaces. Next we state four propositions:

(1) $\emptyset \in C$.

(2) 2^X is a coherent space.

(3) $\{\emptyset\}$ is a coherent space.

(4) If $x \in \bigcup C$, then $\{x\} \in C$.

Let C be a coherent space. The functor Web(C) yields a tolerance of $\bigcup C$ and is defined by:

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(Def.3) for all x, y holds $\langle x, y\rangle \in \mathrm{Web}(C)$ if and only if there exists X such that $X \in C$ and $x \in X$ and $y \in X$.

In the sequel T is a tolerance of $\bigcup C$. One can prove the following propositions:

(5) $T = \mathrm{Web}(C)$ if and only if for all x, y holds $\langle x, y\rangle \in T$ if and only if $\{x, y\} \in C$.

(6) $a \in C$ if and only if for all x, y such that $x \in a$ and $y \in a$ holds $\{x, y\} \in C$.

(7) $a \in C$ if and only if for all x, y such that $x \in a$ and $y \in a$ holds $\langle x, y\rangle \in \mathrm{Web}(C)$.

(8) If for all x, y such that $x \in a$ and $y \in a$ holds $\{x, y\} \in C$, then $a \subseteq \bigcup C$.

(9) If $\mathrm{Web}(C) = \mathrm{Web}(D)$, then $C = D$.

(10) If $\bigcup C \in C$, then $C = 2^{\bigcup C}$.

(11) If $C = 2^{\bigcup C}$, then $\mathrm{Web}(C) = \nabla_{\bigcup C}$.

Let X be a set, and let E be a tolerance of X. The functor $\mathrm{CohSp}(E)$ yielding a coherent space is defined by:

(Def.4) for every a holds $a \in \mathrm{CohSp}(E)$ if and only if for all x, y such that $x \in a$ and $y \in a$ holds $\langle x, y\rangle \in E$.

In the sequel E denotes a tolerance of X. Next we state four propositions:

(12) $\mathrm{Web}(\mathrm{CohSp}(E)) = E$.

(13) $\mathrm{CohSp}(\mathrm{Web}(C)) = C$.

(14) $a \in \mathrm{CohSp}(E)$ if and only if a is a set of mutually elements w.r.t. E.

(15) $\mathrm{CohSp}(E) = \mathrm{TolSets}\, E$.

2. Category of Coherent Spaces

Let us consider X. The functor $\mathrm{CSp}(X)$ yielding a non-empty set is defined as follows:

(Def.5) $\mathrm{CSp}(X) = \{x : x \,\text{is a coherent space}\}$, where x ranges over subsets of 2^X.

In the sequel C, C_1, C_2 denote elements of $\mathrm{CSp}(X)$. Let us consider X, C. The functor ${}^{@}C$ yielding a coherent space is defined as follows:

(Def.6) ${}^{@}C = C$.

The following proposition is true

(16) If $\{x, y\} \in C$, then $x \in \bigcup C$ and $y \in \bigcup C$.

Let us consider X. The functor $\mathrm{Funcs_C} X$ yielding a non-empty set of functions is defined by:

(Def.7) $\mathrm{Funcs_C} X = \bigcup\{(\bigcup y)^{\bigcup x}\}$, where x ranges over elements of $\mathrm{CSp}(X)$, and y ranges over elements of $\mathrm{CSp}(X)$.

In the sequel g is an element of $\mathrm{Funcs}_{\mathrm{C}}X$. The following proposition is true

(17) $x \in \mathrm{Funcs}_{\mathrm{C}}X$ if and only if there exist C_1, C_2 such that if $\bigcup C_2 = \emptyset$, then $\bigcup C_1 = \emptyset$ and also x is a function from $\bigcup C_1$ into $\bigcup C_2$.

Let us consider X. The functor $\mathrm{Maps}_{\mathrm{C}}X$ yielding a non-empty set is defined by:

(Def.8) $\mathrm{Maps}_{\mathrm{C}}X = \{\langle\langle C, C_3\rangle, f\rangle : (\bigcup C_3 = \emptyset \Rightarrow \bigcup C = \emptyset) \wedge f$ is a function from $\bigcup C$ into $\bigcup C_3 \wedge \bigwedge_{x,y}[\{x, y\} \in C \Rightarrow \{f(x), f(y)\} \in C_3]\}$, where C ranges over elements of $\mathrm{CSp}(X)$, and C_3 ranges over elements of $\mathrm{CSp}(X)$, and f ranges over elements of $\mathrm{Funcs}_{\mathrm{C}}X$.

In the sequel l, l_1, l_2, l_3 will be elements of $\mathrm{Maps}_{\mathrm{C}}X$. The following two propositions are true:

(18) There exist g, C_1, C_2 such that $l = \langle\langle C_1, C_2\rangle, g\rangle$ and also if $\bigcup C_2 = \emptyset$, then $\bigcup C_1 = \emptyset$ and g is a function from $\bigcup C_1$ into $\bigcup C_2$ and for all x, y such that $\{x, y\} \in C_1$ holds $\{g(x), g(y)\} \in C_2$.

(19) For every function f from $\bigcup C_1$ into $\bigcup C_2$ such that if $\bigcup C_2 = \emptyset$, then $\bigcup C_1 = \emptyset$ and also for all x, y such that $\{x, y\} \in C_1$ holds $\{f(x), f(y)\} \in C_2$ holds $\langle\langle C_1, C_2\rangle, f\rangle \in \mathrm{Maps}_{\mathrm{C}}X$.

We now define three new functors. Let us consider X, l. The functor $\mathrm{graph}(l)$ yields a function and is defined by:

(Def.9) $\mathrm{graph}(l) = l_{\mathbf{2}}$.

The functor $\mathrm{dom}\, l$ yielding an element of $\mathrm{CSp}(X)$ is defined by:

(Def.10) $\mathrm{dom}\, l = (l_{\mathbf{1}})_{\mathbf{1}}$.

The functor $\mathrm{cod}\, l$ yielding an element of $\mathrm{CSp}(X)$ is defined by:

(Def.11) $\mathrm{cod}\, l = (l_{\mathbf{1}})_{\mathbf{2}}$.

Next we state the proposition

(20) $l = \langle\langle \mathrm{dom}\, l, \mathrm{cod}\, l\rangle, \mathrm{graph}(l)\rangle$.

Let us consider X, C. The functor $\mathrm{id}(C)$ yields an element of $\mathrm{Maps}_{\mathrm{C}}X$ and is defined by:

(Def.12) $\mathrm{id}(C) = \langle\langle C, C\rangle, \mathrm{id}_{\bigcup C}\rangle$.

One can prove the following proposition

(21) $\bigcup \mathrm{cod}\, l \neq \emptyset$ or $\bigcup \mathrm{dom}\, l = \emptyset$ and also $\mathrm{graph}(l)$ is a function from $\bigcup \mathrm{dom}\, l$ into $\bigcup \mathrm{cod}\, l$ and for all x, y such that $\{x, y\} \in \mathrm{dom}\, l$ holds $\{(\mathrm{graph}(l))(x), (\mathrm{graph}(l))(y)\} \in \mathrm{cod}\, l$.

Let us consider X, l_1, l_2. Let us assume that $\mathrm{cod}\, l_1 = \mathrm{dom}\, l_2$. The functor $l_2 \cdot l_1$ yielding an element of $\mathrm{Maps}_{\mathrm{C}}X$ is defined as follows:

(Def.13) $l_2 \cdot l_1 = \langle\langle \mathrm{dom}\, l_1, \mathrm{cod}\, l_2\rangle, \mathrm{graph}(l_2) \cdot \mathrm{graph}(l_1)\rangle$.

We now state four propositions:

(22) If $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$, then $\mathrm{graph}((l_2 \cdot l_1)) = \mathrm{graph}(l_2) \cdot \mathrm{graph}(l_1)$ and $\mathrm{dom}(l_2 \cdot l_1) = \mathrm{dom}\, l_1$ and $\mathrm{cod}(l_2 \cdot l_1) = \mathrm{cod}\, l_2$.

(23) If $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$ and $\mathrm{dom}\, l_3 = \mathrm{cod}\, l_2$, then $l_3 \cdot (l_2 \cdot l_1) = (l_3 \cdot l_2) \cdot l_1$.

(24) $\mathrm{graph}(\mathrm{id}(C)) = \mathrm{id}_{\bigcup C}$ and $\mathrm{dom}\,\mathrm{id}(C) = C$ and $\mathrm{cod}\,\mathrm{id}(C) = C$.

(25) $l \cdot \mathrm{id}(\mathrm{dom}\, l) = l$ and $\mathrm{id}(\mathrm{cod}\, l) \cdot l = l$.

We now define four new functors. Let us consider X. The functor $\mathrm{Dom}_{\mathrm{CSp}}\, X$ yields a function from $\mathrm{Maps}_{\mathrm{C}} X$ into $\mathrm{CSp}(X)$ and is defined as follows:

(Def.14) for every l holds $(\mathrm{Dom}_{\mathrm{CSp}}\, X)(l) = \mathrm{dom}\, l$.

The functor $\mathrm{Cod}_{\mathrm{CSp}}\, X$ yielding a function from $\mathrm{Maps}_{\mathrm{C}} X$ into $\mathrm{CSp}(X)$ is defined by:

(Def.15) for every l holds $(\mathrm{Cod}_{\mathrm{CSp}}\, X)(l) = \mathrm{cod}\, l$.

The functor $\cdot_{\mathrm{CSp}}\, X$ yielding a partial function from $[\!:\mathrm{Maps}_{\mathrm{C}} X, \mathrm{Maps}_{\mathrm{C}} X\,:\!]$ to $\mathrm{Maps}_{\mathrm{C}} X$ is defined by:

(Def.16) for all l_2, l_1 holds $\langle l_2, l_1 \rangle \in \mathrm{dom} \cdot_{\mathrm{CSp}} X$ if and only if $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$ and for all l_2, l_1 such that $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$ holds $(\cdot_{\mathrm{CSp}}\, X)(\langle l_2, l_1 \rangle) = l_2 \cdot l_1$.

The functor $\mathrm{Id}_{\mathrm{CSp}}\, X$ yielding a function from $\mathrm{CSp}(X)$ into $\mathrm{Maps}_{\mathrm{C}} X$ is defined by:

(Def.17) for every C holds $(\mathrm{Id}_{\mathrm{CSp}}\, X)(C) = \mathrm{id}(C)$.

Next we state the proposition

(26) $\langle \mathrm{CSp}(X), \mathrm{Maps}_{\mathrm{C}} X, \mathrm{Dom}_{\mathrm{CSp}}\, X, \mathrm{Cod}_{\mathrm{CSp}}\, X, \cdot_{\mathrm{CSp}}\, X, \mathrm{Id}_{\mathrm{CSp}}\, X \rangle$ is a category.

Let us consider X. The X-coherent space category yields a category and is defined by:

(Def.18) the X-coherent space category $= \langle \mathrm{CSp}(X), \mathrm{Maps}_{\mathrm{C}} X, \mathrm{Dom}_{\mathrm{CSp}}\, X, \mathrm{Cod}_{\mathrm{CSp}}\, X, \cdot_{\mathrm{CSp}}\, X, \mathrm{Id}_{\mathrm{CSp}}\, X \rangle$.

3. Category of Tolerances

We now define two new functors. Let X be a set. The tolerances on X constitute a non-empty set defined by:

(Def.19) the tolerances on X is the set of all tolerances of X.

Let X be a set. The tolerances on subsets of X constitute a non-empty set defined as follows:

(Def.20) the tolerances on subsets of $X = \bigcup\{$the tolerances on $Y\}$, where Y ranges over subsets of X.

In the sequel t denotes an element of the tolerances on subsets of X. The following propositions are true:

(27) $x \in$ the tolerances on subsets of X if and only if there exists A such that $A \subseteq X$ and x is a tolerance of A.

(28) $\nabla_a \in$ the tolerances on a.

(29) $\triangle_a \in$ the tolerances on a.

(30) $\varnothing \in$ the tolerances on subsets of X.

(31) If $a \subseteq X$, then $\nabla_a \in$ the tolerances on subsets of X.

(32) If $a \subseteq X$, then $\triangle_a \in$ the tolerances on subsets of X.

(33) $\nabla_X \in$ the tolerances on subsets of X.

(34) $\triangle_X \in$ the tolerances on subsets of X.

Let us consider X. The functor $\mathrm{TOL}(X)$ yields a non-empty set and is defined by:

(Def.21) $\mathrm{TOL}(X) = \{\langle t, Y\rangle : t \text{ is a tolerance of } Y\}$, where t ranges over elements of the tolerances on subsets of X, and Y ranges over elements of 2^X.

In the sequel T, T_1, T_2 will denote elements of $\mathrm{TOL}(X)$. Next we state several propositions:

(35) $\langle \varnothing, \emptyset\rangle \in \mathrm{TOL}(X)$.

(36) If $a \subseteq X$, then $\langle \triangle_a, a\rangle \in \mathrm{TOL}(X)$.

(37) If $a \subseteq X$, then $\langle \nabla_a, a\rangle \in \mathrm{TOL}(X)$.

(38) $\langle \triangle_X, X\rangle \in \mathrm{TOL}(X)$.

(39) $\langle \nabla_X, X\rangle \in \mathrm{TOL}(X)$.

Let us consider X, T. Then $T_{\mathbf{2}}$ is an element of 2^X. Then $T_{\mathbf{1}}$ is a tolerance of $T_{\mathbf{2}}$. Let us consider X. The functor $\mathrm{Funcs_T}X$ yielding a non-empty set of functions is defined as follows:

(Def.22) $\mathrm{Funcs_T}X = \bigcup\{(T_{3\mathbf{2}})^{T_{\mathbf{2}}}\}$, where T ranges over elements of $\mathrm{TOL}(X)$, and T_3 ranges over elements of $\mathrm{TOL}(X)$.

In the sequel f denotes an element of $\mathrm{Funcs_T}X$. We now state the proposition

(40) $x \in \mathrm{Funcs_T}X$ if and only if there exist T_1, T_2 such that if $T_{2\mathbf{2}} = \emptyset$, then $T_{1\mathbf{2}} = \emptyset$ and also x is a function from $T_{1\mathbf{2}}$ into $T_{2\mathbf{2}}$.

Let us consider X. The functor $\mathrm{Maps_T}X$ yielding a non-empty set is defined by:

(Def.23) $\mathrm{Maps_T}X = \{\langle\langle T, T_3\rangle, f\rangle : (T_{3\mathbf{2}} = \emptyset \Rightarrow T_{\mathbf{2}} = \emptyset) \wedge f \text{ is a function from } T_{\mathbf{2}}$ $\text{into } T_{3\mathbf{2}} \wedge \bigwedge_{x,y}[\langle x, y\rangle \in T_{\mathbf{1}} \Rightarrow \langle f(x), f(y)\rangle \in T_{3\mathbf{1}}]\}$, where T ranges over elements of $\mathrm{TOL}(X)$, and T_3 ranges over elements of $\mathrm{TOL}(X)$, and f ranges over elements of $\mathrm{Funcs_T}X$.

In the sequel m, m_1, m_2, m_3 denote elements of $\mathrm{Maps_T}X$. One can prove the following two propositions:

(41) There exist f, T_1, T_2 such that $m = \langle\langle T_1, T_2\rangle, f\rangle$ and also if $T_{2\mathbf{2}} = \emptyset$, then $T_{1\mathbf{2}} = \emptyset$ and f is a function from $T_{1\mathbf{2}}$ into $T_{2\mathbf{2}}$ and for all x, y such that $\langle x, y\rangle \in T_{1\mathbf{1}}$ holds $\langle f(x), f(y)\rangle \in T_{2\mathbf{1}}$.

(42) For every function f from $T_{1\mathbf{2}}$ into $T_{2\mathbf{2}}$ such that if $T_{2\mathbf{2}} = \emptyset$, then $T_{1\mathbf{2}} = \emptyset$ and also for all x, y such that $\langle x, y\rangle \in T_{1\mathbf{1}}$ holds $\langle f(x), f(y)\rangle \in T_{2\mathbf{1}}$ holds $\langle\langle T_1, T_2\rangle, f\rangle \in \mathrm{Maps_T}X$.

We now define three new functors. Let us consider X, m. The functor $\mathrm{graph}(m)$ yielding a function is defined by:

(Def.24) $\mathrm{graph}(m) = m_{\mathbf{2}}$.

The functor $\mathrm{dom}\, m$ yields an element of $\mathrm{TOL}(X)$ and is defined by:

(Def.25) $\mathrm{dom}\, m = (m_{\mathbf{1}})_{\mathbf{1}}$.

The functor $\mathrm{cod}\, m$ yields an element of $\mathrm{TOL}(X)$ and is defined by:

(Def.26) $\mathrm{cod}\, m = (m_{\mathbf{1}})_{\mathbf{2}}$.

One can prove the following proposition

(43) $m = \langle\langle \mathrm{dom}\, m, \mathrm{cod}\, m\rangle, \mathrm{graph}(m)\rangle$.

Let us consider X, T. The functor $\mathrm{id}(T)$ yields an element of $\mathrm{Maps}_{\mathrm{T}}X$ and is defined by:

(Def.27) $\mathrm{id}(T) = \langle\langle T, T\rangle, \mathrm{id}_{(T_{\mathbf{2}})}\rangle$.

One can prove the following proposition

(44) $(\mathrm{cod}\, m)_{\mathbf{2}} \neq \emptyset$ or $(\mathrm{dom}\, m)_{\mathbf{2}} = \emptyset$ and also $\mathrm{graph}(m)$ is a function from $(\mathrm{dom}\, m)_{\mathbf{2}}$ into $(\mathrm{cod}\, m)_{\mathbf{2}}$ and for all x, y such that $\langle x, y\rangle \in (\mathrm{dom}\, m)_{\mathbf{1}}$ holds $\langle(\mathrm{graph}(m))(x), (\mathrm{graph}(m))(y)\rangle \in (\mathrm{cod}\, m)_{\mathbf{1}}$.

Let us consider X, m_1, m_2. Let us assume that $\mathrm{cod}\, m_1 = \mathrm{dom}\, m_2$. The functor $m_2 \cdot m_1$ yielding an element of $\mathrm{Maps}_{\mathrm{T}}X$ is defined by:

(Def.28) $m_2 \cdot m_1 = \langle\langle \mathrm{dom}\, m_1, \mathrm{cod}\, m_2\rangle, \mathrm{graph}(m_2) \cdot \mathrm{graph}(m_1)\rangle$.

The following propositions are true:

(45) If $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$, then $\mathrm{graph}((m_2 \cdot m_1)) = \mathrm{graph}(m_2) \cdot \mathrm{graph}(m_1)$ and $\mathrm{dom}(m_2 \cdot m_1) = \mathrm{dom}\, m_1$ and $\mathrm{cod}(m_2 \cdot m_1) = \mathrm{cod}\, m_2$.

(46) If $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ and $\mathrm{dom}\, m_3 = \mathrm{cod}\, m_2$, then $m_3 \cdot (m_2 \cdot m_1) = (m_3 \cdot m_2) \cdot m_1$.

(47) $\mathrm{graph}(\mathrm{id}(T)) = \mathrm{id}_{(T_{\mathbf{2}})}$ and $\mathrm{dom}\,\mathrm{id}(T) = T$ and $\mathrm{cod}\,\mathrm{id}(T) = T$.

(48) $m \cdot \mathrm{id}(\mathrm{dom}\, m) = m$ and $\mathrm{id}(\mathrm{cod}\, m) \cdot m = m$.

We now define four new functors. Let us consider X. The functor Dom_X yields a function from $\mathrm{Maps}_{\mathrm{T}}X$ into $\mathrm{TOL}(X)$ and is defined by:

(Def.29) for every m holds $\mathrm{Dom}_X(m) = \mathrm{dom}\, m$.

The functor Cod_X yields a function from $\mathrm{Maps}_{\mathrm{T}}X$ into $\mathrm{TOL}(X)$ and is defined as follows:

(Def.30) for every m holds $\mathrm{Cod}_X(m) = \mathrm{cod}\, m$.

The functor $\cdot_X$ yields a partial function from $[\!:\mathrm{Maps}_{\mathrm{T}}X, \mathrm{Maps}_{\mathrm{T}}X:\!]$ to $\mathrm{Maps}_{\mathrm{T}}X$ and is defined as follows:

(Def.31) for all m_2, m_1 holds $\langle m_2, m_1\rangle \in \mathrm{dom}(\cdot_X)$ if and only if $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ and for all m_2, m_1 such that $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ holds $\cdot_X(\langle m_2, m_1\rangle) = m_2 \cdot m_1$.

The functor Id_X yields a function from $\mathrm{TOL}(X)$ into $\mathrm{Maps}_{\mathrm{T}}X$ and is defined by:

(Def.32) for every T holds $\mathrm{Id}_X(T) = \mathrm{id}(T)$.

Next we state the proposition

(49) $\langle \mathrm{TOL}(X), \mathrm{Maps}_{\mathrm{T}}X, \mathrm{Dom}_X, \mathrm{Cod}_X, \cdot_X, \mathrm{Id}_X\rangle$ is a category.

Let us consider X. The X-tolerance category is a category defined by:

(Def.33) the X-tolerance category $= \langle \mathrm{TOL}(X), \mathrm{Maps}_{\mathrm{T}}X, \mathrm{Dom}_X, \mathrm{Cod}_X, \cdot_X, \mathrm{Id}_X\rangle$.

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Properties of the Intervals of Real Numbers

Białas Józef
Łódź University

Summary. The paper contains definitions and basic properties of the intervals of real numbers.

The article includes the text being a continuation of the paper [5]. Some theorems concerning basic properties of intervals are proved.

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The notation and terminology used here are introduced in the following papers: [16], [15], [11], [12], [9], [10], [1], [14], [2], [13], [4], [6], [8], [7], [3], [5], and [17]. The following propositions are true:

(1) For all *Real numbers* x, y such that $x \neq -\infty$ and $x \neq +\infty$ and $x \leq y$ holds $0_{\overline{\mathbb{R}}} \leq y - x$.

(2) For all *Real numbers* x, y such that it is not true that: $x = -\infty$ and $y = -\infty$ and it is not true that: $x = +\infty$ and $y = +\infty$ and $x \leq y$ holds $0_{\overline{\mathbb{R}}} \leq y - x$.

(3) For all *Real numbers* x, y holds $x \leq y$ or $y \leq x$.

(4) For all *Real numbers* x, y such that $x \neq y$ holds $x < y$ or $y < x$.

(5) For all *Real numbers* x, y holds $x < y$ or $y \leq x$.

(6) For all *Real numbers* x, y holds $x < y$ if and only if $y \not\leq x$.

(7) For all *Real numbers* x, y, z such that $x < y$ and $y < z$ holds $x < z$.

(8) For all *Real numbers* a, b, c such that $b \neq -\infty$ and $b \neq +\infty$ and it is not true that: $a = -\infty$ and $c = -\infty$ and it is not true that: $a = +\infty$ and $c = +\infty$ holds $(c - b) + (b - a) = c - a$.

(9) For all *Real numbers* a_1, a_2 holds $\inf\{a_1, a_2\} \leq a_1$ and $\inf\{a_1, a_2\} \leq a_2$ and $a_1 \leq \sup\{a_1, a_2\}$ and $a_2 \leq \sup\{a_1, a_2\}$.

(10) For all *Real numbers* a, b, c such that $a \leq b$ and $b < c$ or $a < b$ and $b \leq c$ holds $a < c$.

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We now define several new constructions. Let a, b be *Real numbers*. The functor $[a,b]$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def.1) for every *Real number* x holds $x \in [a,b]$ if and only if $a \leq x$ and $x \leq b$ and $x \in \mathbb{R}$.

Let a, b be *Real numbers*. The functor $]a,b[$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def.2) for every *Real number* x holds $x \in]a,b[$ if and only if $a < x$ and $x < b$ and $x \in \mathbb{R}$.

Let a, b be *Real numbers*. The functor $]a,b]$ yielding a subset of $\mathbb{R}$ is defined by:

(Def.3) for every *Real number* x holds $x \in]a,b]$ if and only if $a < x$ and $x \leq b$ and $x \in \mathbb{R}$.

Let a, b be *Real numbers*. The functor $[a,b[$ yields a subset of $\mathbb{R}$ and is defined by:

(Def.4) for every *Real number* x holds $x \in [a,b[$ if and only if $a \leq x$ and $x < b$ and $x \in \mathbb{R}$.

A subset of $\mathbb{R}$ is called an open interval if:

(Def.5) there exist *Real numbers* a, b such that $a \leq b$ and it $=]a,b[$.

A subset of $\mathbb{R}$ is said to be a closed interval if:

(Def.6) there exist *Real numbers* a, b such that $a \leq b$ and it $= [a,b]$.

A subset of $\mathbb{R}$ is said to be a right-open interval if:

(Def.7) there exist *Real numbers* a, b such that $a \leq b$ and it $= [a,b[$.

A subset of $\mathbb{R}$ is called a left-open interval if:

(Def.8) there exist *Real numbers* a, b such that $a \leq b$ and it $=]a,b]$.

A subset of $\mathbb{R}$ is said to be an interval if:

(Def.9) it is an open interval or it is a closed interval or it is a right-open interval or it is a left-open interval.

We see that the open interval is an interval. We see that the closed interval is an interval. We see that the right-open interval is an interval. We see that the left-open interval is an interval.

We now state a number of propositions:

(11) For an arbitrary x and for all *Real numbers* a, b such that $x \in]a,b[$ or $x \in [a,b]$ or $x \in [a,b[$ or $x \in]a,b]$ holds x is a *Real number*.

(12) For all *Real numbers* a, b such that $b < a$ holds $]a,b[= \emptyset$ and $[a,b] = \emptyset$ and $[a,b[= \emptyset$ and $]a,b] = \emptyset$.

(13) For every *Real number* a holds $]a,a[= \emptyset$ and $[a,a[= \emptyset$ and $]a,a] = \emptyset$.

(14) For every *Real number* a holds if $a = -\infty$ or $a = +\infty$, then $[a,a] = \emptyset$ and also if $a \neq -\infty$ and $a \neq +\infty$, then $[a,a] = \{a\}$.

(15) For all *Real numbers* a, b such that $b \leq a$ holds $]a,b[= \emptyset$ and $[a,b[= \emptyset$ and $]a,b] = \emptyset$ and $[a,b] \subseteq \{a\}$ and $[a,b] \subseteq \{b\}$.

(16) For all *Real numbers* a, b, c such that $a < b$ and $b < c$ holds $b \in \mathbb{R}$.

(17) For all *Real numbers* a, b such that $a < b$ there exists a *Real number* x such that $a < x$ and $x < b$ and $x \in \mathbb{R}$.

(18) For all *Real numbers* a, b, c such that $a < b$ and $a < c$ there exists a *Real number* x such that $a < x$ and $x < b$ and $x < c$ and $x \in \mathbb{R}$.

(19) For all *Real numbers* a, b, c such that $a < c$ and $b < c$ there exists a *Real number* x such that $a < x$ and $b < x$ and $x < c$ and $x \in \mathbb{R}$.

(20) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin]a_2, b_2[$ or $x \notin]a_1, b_1[$ and $x \in]a_2, b_2[$.

(21) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin]a_2, b_2[$ or $x \notin]a_1, b_1[$ and $x \in]a_2, b_2[$.

(22) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin]a_2, b_2[$ or $x \notin [a_1, b_1]$ and $x \in]a_2, b_2[$.

(23) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin]a_2, b_2[$ or $x \notin [a_1, b_1]$ and $x \in]a_2, b_2[$.

(24) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin [a_2, b_2]$ or $x \notin]a_1, b_1[$ and $x \in [a_2, b_2]$.

(25) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin [a_2, b_2]$ or $x \notin]a_1, b_1[$ and $x \in [a_2, b_2]$.

(26) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin [a_2, b_2[$ or $x \notin]a_1, b_1[$ and $x \in [a_2, b_2[$.

(27) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin [a_2, b_2[$ or $x \notin]a_1, b_1[$ and $x \in [a_2, b_2[$.

(28) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin]a_2, b_2[$ or $x \notin [a_1, b_1[$ and $x \in]a_2, b_2[$.

(29) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin]a_2, b_2[$ or $x \notin [a_1, b_1[$ and $x \in]a_2, b_2[$.

(30) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin]a_2, b_2]$ or $x \notin]a_1, b_1[$ and $x \in]a_2, b_2]$.

(31) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1[$ and $x \notin]a_2, b_2]$ or $x \notin]a_1, b_1[$ and $x \in]a_2, b_2]$.

(32) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin]a_2, b_2[$ or $x \notin]a_1, b_1]$ and $x \in]a_2, b_2[$.

(33) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin]a_2, b_2[$ or $x \notin]a_1, b_1]$ and $x \in]a_2, b_2[$.

(34) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin [a_2, b_2]$ or $x \notin [a_1, b_1]$ and $x \in [a_2, b_2]$.

(35) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin [a_2, b_2]$ or $x \notin [a_1, b_1]$ and $x \in [a_2, b_2]$.

(36) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin [a_2, b_2[$ or $x \notin [a_1, b_1]$ and $x \in [a_2, b_2[$.

(37) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin [a_2, b_2[$ or $x \notin [a_1, b_1]$ and $x \in [a_2, b_2[$.

(38) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin [a_2, b_2]$ or $x \notin [a_1, b_1[$ and $x \in [a_2, b_2]$.

(39) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin [a_2, b_2]$ or $x \notin [a_1, b_1[$ and $x \in [a_2, b_2]$.

(40) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin]a_2, b_2]$ or $x \notin [a_1, b_1]$ and $x \in]a_2, b_2]$.

(41) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1]$ and $x \notin]a_2, b_2]$ or $x \notin [a_1, b_1]$ and $x \in]a_2, b_2]$.

Next we state a number of propositions:

(42) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin [a_2, b_2]$ or $x \notin]a_1, b_1]$ and $x \in [a_2, b_2]$.

(43) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin [a_2, b_2]$ or $x \notin]a_1, b_1]$ and $x \in [a_2, b_2]$.

(44) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin [a_2, b_2[$ or $x \notin [a_1, b_1[$ and $x \in [a_2, b_2[$.

(45) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin [a_2, b_2[$ or $x \notin [a_1, b_1[$ and $x \in [a_2, b_2[$.

(46) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin]a_2, b_2]$ or $x \notin [a_1, b_1[$ and $x \in]a_2, b_2]$.

(47) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in [a_1, b_1[$ and $x \notin]a_2, b_2]$ or $x \notin [a_1, b_1[$ and $x \in]a_2, b_2]$.

(48) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin [a_2, b_2[$ or $x \notin]a_1, b_1]$ and $x \in [a_2, b_2[$.

(49) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin [a_2, b_2[$ or $x \notin]a_1, b_1]$ and $x \in [a_2, b_2[$.

(50) For all *Real numbers* a_1, a_2, b_1, b_2 such that $a_1 < a_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin]a_2, b_2]$ or $x \notin]a_1, b_1]$ and $x \in]a_2, b_2]$.

(51) For all *Real numbers* a_1, a_2, b_1, b_2 such that $b_1 < b_2$ and also $a_1 < b_1$ or $a_2 < b_2$ there exists a *Real number* x such that $x \in]a_1, b_1]$ and $x \notin]a_2, b_2]$ or $x \notin]a_1, b_1]$ and $x \in]a_2, b_2]$.

(52) Let A be an interval. Let a_1, a_2, b_1, b_2 be *Real numbers.* Suppose that
 (i) $a_1 < b_1$ or $a_2 < b_2$,
 (ii) $A =]a_1, b_1[$ or $A = [a_1, b_1]$ or $A = [a_1, b_1[$ or $A =]a_1, b_1]$ and also $A =]a_2, b_2[$ or $A = [a_2, b_2]$ or $A = [a_2, b_2[$ or $A =]a_2, b_2]$.

 Then $a_1 = a_2$ and $b_1 = b_2$.

Let A be an interval. The functor $\mathrm{vol}(A)$ yielding a *Real number* is defined as follows:

(Def.10) there exist *Real numbers* a, b such that $A =]a, b[$ or $A = [a, b]$ or $A = [a, b[$ or $A =]a, b]$ and also if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

One can prove the following propositions:

(53) For every open interval A and for all *Real numbers* a, b such that $A =]a, b[$ holds if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(54) For every closed interval A and for all *Real numbers* a, b such that $A = [a, b]$ holds if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(55) For every right-open interval A and for all *Real numbers* a, b such that $A = [a, b[$ holds if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(56) For every left-open interval A and for all *Real numbers* a, b such that $A =]a, b]$ holds if $a < b$, then $\mathrm{vol}(A) = b - a$ and also if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(57) Let A be an interval. Let a, b, c be *Real numbers.* Suppose that
 (i) $a = -\infty$,

(ii) $b \in \mathbb{R}$,
(iii) $c = +\infty$,
(iv) $A =]a, b[$ or $A =]b, c[$ or $A = [a, b]$ or $A = [b, c]$ or $A = [a, b[$ or $A = [b, c[$ or $A =]a, b]$ or $A =]b, c]$.
Then $\mathrm{vol}(A) = +\infty$.

(58) For every interval A and for all *Real numbers* a, b such that $a = -\infty$ and $b = +\infty$ and also $A =]a, b[$ or $A = [a, b]$ or $A = [a, b[$ or $A =]a, b]$ holds $\mathrm{vol}(A) = +\infty$.

(59) For every interval A and for every *Real number* a such that $A =]a, a[$ or $A = [a, a]$ or $A = [a, a[$ or $A =]a, a]$ holds $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

Let us note that there exists an empty interval.

Let us note that it makes sense to consider the following constant. Then $\emptyset$ is an empty interval.

Next we state four propositions:

(60) $\mathrm{vol}(\emptyset) = 0_{\overline{\mathbb{R}}}$.

(61) For all intervals A, B and for all *Real numbers* a, b such that $A \subseteq B$ and $B = [a, b]$ and $b \leq a$ holds $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$ and $\mathrm{vol}(B) = 0_{\overline{\mathbb{R}}}$.

(62) For all intervals A, B such that $A \subseteq B$ holds $\mathrm{vol}(A) \leq \mathrm{vol}(B)$.

(63) For every interval A holds $0_{\overline{\mathbb{R}}} \leq \mathrm{vol}(A)$.

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Subspaces of Real Linear Space Generated by One, Two, or Three Vectors and Their Cosets

Wojciech A. Trybulec
Warsaw University

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The articles [7], [2], [1], [3], [4], [11], [10], [5], [6], [9], and [8] provide the notation and terminology for this paper. For simplicity we adopt the following rules: x is arbitrary, a, b, c denote real numbers, V denotes a real linear space, u, v, v_1, v_2, v_3, w, w_1, w_2, w_3 denote vectors of V, and W, W_1, W_2 denote subspaces of V. In this article we present several logical schemes. The scheme *LambdaSep3* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{A}$, an element $\mathcal{E}$ of $\mathcal{A}$, an element $\mathcal{F}$ of $\mathcal{B}$, an element $\mathcal{G}$ of $\mathcal{B}$, an element $\mathcal{H}$ of $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that $f(\mathcal{C}) = \mathcal{F}$ and $f(\mathcal{D}) = \mathcal{G}$ and $f(\mathcal{E}) = \mathcal{H}$ and for every element C of $\mathcal{A}$ such that $C \neq \mathcal{C}$ and $C \neq \mathcal{D}$ and $C \neq \mathcal{E}$ holds $f(C) = \mathcal{F}(C)$

provided the parameters have the following properties:

- $\mathcal{C} \neq \mathcal{D}$,
- $\mathcal{C} \neq \mathcal{E}$,
- $\mathcal{D} \neq \mathcal{E}$.

The scheme *LinCEx1* deals with a real linear space $\mathcal{A}$, a vector $\mathcal{B}$ of $\mathcal{A}$, and a real number $\mathcal{C}$ and states that:

there exists a linear combination l of $\{\mathcal{B}\}$ such that $l(\mathcal{B}) = \mathcal{C}$

for all values of the parameters.

The scheme *LinCEx2* deals with a real linear space $\mathcal{A}$, a vector $\mathcal{B}$ of $\mathcal{A}$, a vector $\mathcal{C}$ of $\mathcal{A}$, a real number $\mathcal{D}$, and a real number $\mathcal{E}$ and states that:

there exists a linear combination l of $\{\mathcal{B}, \mathcal{C}\}$ such that $l(\mathcal{B}) = \mathcal{D}$ and $l(\mathcal{C}) = \mathcal{E}$

provided the following condition is satisfied:

- $\mathcal{B} \neq \mathcal{C}$.

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The scheme *LinCEx3* deals with a real linear space $\mathcal{A}$, a vector $\mathcal{B}$ of $\mathcal{A}$, a vector $\mathcal{C}$ of $\mathcal{A}$, a vector $\mathcal{D}$ of $\mathcal{A}$, a real number $\mathcal{E}$, a real number $\mathcal{F}$, and a real number $\mathcal{G}$ and states that:

there exists a linear combination l of $\{\mathcal{B}, \mathcal{C}, \mathcal{D}\}$ such that $l(\mathcal{B}) = \mathcal{E}$ and $l(\mathcal{C}) = \mathcal{F}$ and $l(\mathcal{D}) = \mathcal{G}$

provided the parameters meet the following conditions:

- $\mathcal{B} \neq \mathcal{C}$,
- $\mathcal{B} \neq \mathcal{D}$,
- $\mathcal{C} \neq \mathcal{D}$.

We now state a number of propositions:

(1) $(v + w) - v = w$ and $(w + v) - v = w$ and $(v - v) + w = w$ and $(w-v)+v = w$ and $v+(w-v) = w$ and $w+(v-v) = w$ and $v-(v-w) = w$.

(2) $(v + u) - w = (v - w) + u$.

(3) If $v_1 + w = v_2 + w$, then $v_1 = v_2$.

(4) If $v_1 - w = v_2 - w$, then $v_1 = v_2$.

(5) $v = v_1 + v_2$ if and only if $v_2 = v - v_1$.

(6) $-a \cdot v = (-a) \cdot v$.

(7) If W_1 is a subspace of W_2, then $v + W_1 \subseteq v + W_2$.

(8) If $u \in v + W$, then $v + W = u + W$.

(9) For every linear combination l of $\{u, v, w\}$ such that $u \neq v$ and $u \neq w$ and $v \neq w$ holds $\sum l = l(u) \cdot u + l(v) \cdot v + l(w) \cdot w$.

(10) $u \neq v$ and $u \neq w$ and $v \neq w$ and $\{u, v, w\}$ is linearly independent if and only if for all a, b, c such that $a \cdot u + b \cdot v + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$.

(11) $x \in \mathrm{Lin}(\{v\})$ if and only if there exists a such that $x = a \cdot v$.

(12) $v \in \mathrm{Lin}(\{v\})$.

(13) $x \in v + \mathrm{Lin}(\{w\})$ if and only if there exists a such that $x = v + a \cdot w$.

(14) $x \in \mathrm{Lin}(\{w_1, w_2\})$ if and only if there exist a, b such that $x = a \cdot w_1 + b \cdot w_2$.

(15) $w_1 \in \mathrm{Lin}(\{w_1, w_2\})$ and $w_2 \in \mathrm{Lin}(\{w_1, w_2\})$.

(16) $x \in v + \mathrm{Lin}(\{w_1, w_2\})$ if and only if there exist a, b such that $x = v + a \cdot w_1 + b \cdot w_2$.

(17) $x \in \mathrm{Lin}(\{v_1, v_2, v_3\})$ if and only if there exist a, b, c such that $x = a \cdot v_1 + b \cdot v_2 + c \cdot v_3$.

(18) $w_1 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$ and $w_2 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$ and $w_3 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$.

(19) $x \in v + \mathrm{Lin}(\{w_1, w_2, w_3\})$ if and only if there exist a, b, c such that $x = v + a \cdot w_1 + b \cdot w_2 + c \cdot w_3$.

(20) If $\{u, v\}$ is linearly independent and $u \neq v$, then $\{u, v - u\}$ is linearly independent.

(21) If $\{u, v\}$ is linearly independent and $u \neq v$, then $\{u, v + u\}$ is linearly independent.

(22) If $\{u, v\}$ is linearly independent and $u \neq v$ and $a \neq 0$, then $\{u, a \cdot v\}$ is linearly independent.

(23) If $\{u, v\}$ is linearly independent and $u \neq v$, then $\{u, -v\}$ is linearly independent.

(24) If $a \neq b$, then $\{a \cdot v, b \cdot v\}$ is linearly dependent.

(25) If $a \neq 1$, then $\{v, a \cdot v\}$ is linearly dependent.

(26) If $\{u, w, v\}$ is linearly independent and $u \neq v$ and $u \neq w$ and $v \neq w$, then $\{u, w, v - u\}$ is linearly independent.

(27) If $\{u, w, v\}$ is linearly independent and $u \neq v$ and $u \neq w$ and $v \neq w$, then $\{u, w - u, v - u\}$ is linearly independent.

(28) If $\{u, w, v\}$ is linearly independent and $u \neq v$ and $u \neq w$ and $v \neq w$, then $\{u, w, v + u\}$ is linearly independent.

(29) If $\{u, w, v\}$ is linearly independent and $u \neq v$ and $u \neq w$ and $v \neq w$, then $\{u, w + u, v + u\}$ is linearly independent.

(30) If $\{u, w, v\}$ is linearly independent and $u \neq v$ and $u \neq w$ and $v \neq w$ and $a \neq 0$, then $\{u, w, a \cdot v\}$ is linearly independent.

(31) If $\{u, w, v\}$ is linearly independent and $u \neq v$ and $u \neq w$ and $v \neq w$ and $a \neq 0$ and $b \neq 0$, then $\{u, a \cdot w, b \cdot v\}$ is linearly independent.

The following propositions are true:

(32) If $\{u, w, v\}$ is linearly independent and $u \neq v$ and $u \neq w$ and $v \neq w$, then $\{u, w, -v\}$ is linearly independent.

(33) If $\{u, w, v\}$ is linearly independent and $u \neq v$ and $u \neq w$ and $v \neq w$, then $\{u, -w, -v\}$ is linearly independent.

(34) If $a \neq b$, then $\{a \cdot v, b \cdot v, w\}$ is linearly dependent.

(35) If $a \neq 1$, then $\{v, a \cdot v, w\}$ is linearly dependent.

(36) If $v \in \mathrm{Lin}(\{w\})$ and $v \neq 0_V$, then $\mathrm{Lin}(\{v\}) = \mathrm{Lin}(\{w\})$.

(37) If $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent and $v_1 \in \mathrm{Lin}(\{w_1, w_2\})$ and $v_2 \in \mathrm{Lin}(\{w_1, w_2\})$, then $\mathrm{Lin}(\{w_1, w_2\}) = \mathrm{Lin}(\{v_1, v_2\})$ and $\{w_1, w_2\}$ is linearly independent and $w_1 \neq w_2$.

(38) If $w \neq 0_V$ and $\{v, w\}$ is linearly dependent, then there exists a such that $v = a \cdot w$.

(39) If $v \neq w$ and $\{v, w\}$ is linearly independent and $\{u, v, w\}$ is linearly dependent, then there exist a, b such that $u = a \cdot v + b \cdot w$.

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Functions and Finite Sequences of Real Numbers

Jarosław Kotowicz
Warsaw University
Białystok

Summary. We define notions of fiberwise equipotent functions, non-increasing finite sequences of real numbers and new operations on finite sequences. Equivalent conditions for fiberwise equivalent functions and basic facts about new constructions are shown.

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The articles [11], [4], [5], [3], [1], [8], [10], [2], [12], [6], [7], and [9] provide the notation and terminology for this paper. In the sequel n will be a natural number. Let F, G be functions. We say that F and G are fiwerwise equipotent if and only if:

(Def.1) for an arbitrary x holds $\overline{\overline{F^{-1}\{x\}}} = \overline{\overline{G^{-1}\{x\}}}$.

Let us observe that the predicate defined above is reflexive and symmetric.

One can prove the following propositions:

(1) For all functions F, G such that F and G are fiwerwise equipotent holds $\operatorname{rng} F = \operatorname{rng} G$.

(2) For all functions F, G, H such that F and G are fiwerwise equipotent and F and H are fiwerwise equipotent holds G and H are fiwerwise equipotent.

(3) For all functions F, G holds F and G are fiwerwise equipotent if and only if there exists a function H such that $\operatorname{dom} H = \operatorname{dom} F$ and $\operatorname{rng} H = \operatorname{dom} G$ and H is one-to-one and $F = G \cdot H$.

(4) For all functions F, G holds F and G are fiwerwise equipotent if and only if for every set X holds $\overline{\overline{F^{-1}X}} = \overline{\overline{G^{-1}X}}$.

(5) For every non-empty set D and for all functions F, G such that $\operatorname{rng} F \subseteq D$ and $\operatorname{rng} G \subseteq D$ holds F and G are fiwerwise equipotent if and only if for every element d of D holds $\overline{\overline{F^{-1}\{d\}}} = \overline{\overline{G^{-1}\{d\}}}$.

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(6) For all functions F, G such that $\operatorname{dom} F = \operatorname{dom} G$ holds F and G are fiwerwise equipotent if and only if there exists a permutation P of $\operatorname{dom} F$ such that $F = G \cdot P$.

(7) For all functions F, G such that F and G are fiwerwise equipotent holds $\overline{\overline{\operatorname{dom} F}} = \overline{\overline{\operatorname{dom} G}}$.

(8) For all functions F, G such that $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite holds F and G are fiwerwise equipotent if and only if for an arbitrary x holds $\operatorname{card}(F^{-1}\{x\}) = \operatorname{card}(G^{-1}\{x\})$.

(9) For all functions F, G such that $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite holds F and G are fiwerwise equipotent if and only if for every set X holds $\operatorname{card}(F^{-1} X) = \operatorname{card}(G^{-1} X)$.

(10) For all functions F, G such that $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite and F and G are fiwerwise equipotent holds $\operatorname{card} \operatorname{dom} F = \operatorname{card} \operatorname{dom} G$.

(11) For every non-empty set D and for all functions F, G such that $\operatorname{rng} F \subseteq D$ and $\operatorname{rng} G \subseteq D$ and $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite holds F and G are fiwerwise equipotent if and only if for every element d of D holds $\operatorname{card}(F^{-1}\{d\}) = \operatorname{card}(G^{-1}\{d\})$.

(12) For all finite sequences f, g holds f and g are fiwerwise equipotent if and only if for an arbitrary x holds $\operatorname{card}(f^{-1}\{x\}) = \operatorname{card}(g^{-1}\{x\})$.

(13) For all finite sequences f, g holds f and g are fiwerwise equipotent if and only if for every set X holds $\operatorname{card}(f^{-1} X) = \operatorname{card}(g^{-1} X)$.

(14) For all finite sequences f, g, h holds f and g are fiwerwise equipotent if and only if $f \frown h$ and $g \frown h$ are fiwerwise equipotent.

(15) For all finite sequences f, g holds $f \frown g$ and $g \frown f$ are fiwerwise equipotent.

(16) For all finite sequences f, g such that f and g are fiwerwise equipotent holds $\operatorname{len} f = \operatorname{len} g$ and $\operatorname{dom} f = \operatorname{dom} g$.

(17) For all finite sequences f, g holds f and g are fiwerwise equipotent if and only if there exists a permutation P of $\operatorname{dom} g$ such that $f = g \cdot P$.

(18) For every function F and for every finite set X there exists a finite sequence f such that $F \restriction X$ and f are fiwerwise equipotent.

Let D be a non-empty set, and let f be a finite sequence of elements of D, and let n be a natural number. The functor $f_{\lfloor n}$ yields a finite sequence of elements of D and is defined as follows:

(Def.2) (i) $\operatorname{len}(f_{\lfloor n}) = \operatorname{len} f - n$ and for every natural number m such that $m \in \operatorname{dom}(f_{\lfloor n})$ holds $f_{\lfloor n}(m) = f(m+n)$ if $n \leq \operatorname{len} f$,

(ii) $f_{\lfloor n} = \varepsilon_D$, otherwise.

The following propositions are true:

(19) For every non-empty set D and for every finite sequence f of elements of D and for all natural numbers n, m such that $n \in \operatorname{dom} f$ and $m \in \operatorname{Seg} n$ holds $(f \restriction n)(m) = f(m)$ and $m \in \operatorname{dom} f$.

(20) For every non-empty set D and for every finite sequence f of elements of D and for every natural number n and for an arbitrary x such that

$\operatorname{len} f = n+1$ and $x = f(n+1)$ holds $f = (f \restriction n) \frown \langle x \rangle$.

(21) For every non-empty set D and for every finite sequence f of elements of D and for every natural number n holds $(f \restriction n) \frown (f_{\downharpoonright n}) = f$.

(22) For all finite sequences R_1, R_2 of elements of $\mathbb{R}$ such that R_1 and R_2 are fiwerwise equipotent holds $\sum R_1 = \sum R_2$.

Let R be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{MIM}(R)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by the conditions (Def.3).

(Def.3) (i) $\operatorname{len} \mathrm{MIM}(R) = \operatorname{len} R$,

(ii) $(\mathrm{MIM}(R))(\operatorname{len} \mathrm{MIM}(R)) = R(\operatorname{len} R)$,

(iii) for every natural number n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{MIM}(R) - 1$ and for all real numbers r, s such that $R(n) = r$ and $R(n+1) = s$ holds $(\mathrm{MIM}(R))(n) = r - s$.

Next we state several propositions:

(23) For every finite sequence R of elements of $\mathbb{R}$ and for every real number r and for every natural number n such that $\operatorname{len} R = n+2$ and $R(n+1) = r$ holds $\mathrm{MIM}(R \restriction (n+1)) = (\mathrm{MIM}(R) \restriction n) \frown \langle r \rangle$.

(24) For every finite sequence R of elements of $\mathbb{R}$ and for all real numbers r, s and for every natural number n such that $\operatorname{len} R = n+2$ and $R(n+1) = r$ and $R(n+2) = s$ holds $\mathrm{MIM}(R) = (\mathrm{MIM}(R) \restriction n) \frown \langle r - s, s \rangle$.

(25) $\mathrm{MIM}(\varepsilon_{\mathbb{R}}) = \varepsilon_{\mathbb{R}}$.

(26) For every real number r holds $\mathrm{MIM}(\langle r \rangle) = \langle r \rangle$.

(27) For all real numbers r, s holds $\mathrm{MIM}(\langle r, s \rangle) = \langle r - s, s \rangle$.

(28) For every finite sequence R of elements of $\mathbb{R}$ and for every natural number n holds $(\mathrm{MIM}(R))_{\downharpoonright n} = \mathrm{MIM}(R_{\downharpoonright n})$.

(29) For every finite sequence R of elements of $\mathbb{R}$ such that $\operatorname{len} R \neq 0$ holds $\sum \mathrm{MIM}(R) = R(1)$.

(30) For every finite sequence R of elements of $\mathbb{R}$ and for every natural number n such that $1 \leq n$ and $n < \operatorname{len} R$ holds $\sum \mathrm{MIM}(R_{\downharpoonright n}) = R(n+1)$.

A finite sequence of elements of $\mathbb{R}$ is non-increasing if:

(Def.4) for every natural number n such that $n \in \operatorname{dom} \mathrm{it}$ and $n+1 \in \operatorname{dom} \mathrm{it}$ and for all real numbers r, s such that $r = \mathrm{it}(n)$ and $s = \mathrm{it}(n+1)$ holds $r \geq s$.

One can check that there exists a non-increasing finite sequence of elements of $\mathbb{R}$.

We now state several propositions:

(31) For every finite sequence R of elements of $\mathbb{R}$ such that $\operatorname{len} R = 0$ or $\operatorname{len} R = 1$ holds R is non-increasing.

(32) For every finite sequence R of elements of $\mathbb{R}$ holds R is non-increasing if and only if for all natural numbers n, m such that $n \in \operatorname{dom} R$ and $m \in \operatorname{dom} R$ and $n < m$ and for all real numbers r, s such that $R(n) = r$ and $R(m) = s$ holds $r \geq s$.

(33) For every non-increasing finite sequence R of elements of $\mathbb{R}$ and for every natural number n holds $R \restriction n$ is a non-increasing finite sequence of elements of $\mathbb{R}$.

(34) For every non-increasing finite sequence R of elements of $\mathbb{R}$ and for every natural number n holds $R_{\restriction n}$ is a non-increasing finite sequence of elements of $\mathbb{R}$.

(35) For every finite sequence R of elements of $\mathbb{R}$ there exists a non-increasing finite sequence R_1 of elements of $\mathbb{R}$ such that R and R_1 are fiwerwise equipotent.

(36) For all non-increasing finite sequences R_1, R_2 of elements of $\mathbb{R}$ such that R_1 and R_2 are fiwerwise equipotent holds $R_1 = R_2$.

(37) For every finite sequence R of elements of $\mathbb{R}$ and for all real numbers r, s such that $r \neq 0$ holds $R^{-1}\{\frac{s}{r}\} = (r \cdot R)^{-1}\{s\}$.

(38) For every finite sequence R of elements of $\mathbb{R}$ holds $(0 \cdot R)^{-1}\{0\} = \mathrm{dom}\, R$.

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Properties of Partial Functions from a Domain to the Set of Real Numbers

Jarosław Kotowicz
Warsaw University
Białystok

Yuji Sakai
Shinshu University
Nagano

Summary. The article consists of two parties. In the first one we consider notion of nonnegative and nonpositive part of a real numbers. In the second we consider partial function from a domain to the set of real numbers (or more general to a domain). We define a few new operations for these functions and show connections between finite sequences of real numbers and functions which domain is finite. We introduce *integrations* for finite domain real valued functions.

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The articles [23], [25], [7], [21], [3], [4], [1], [11], [13], [2], [18], [20], [22], [6], [24], [8], [5], [9], [10], [19], [16], [17], [15], [12], and [14] provide the notation and terminology for this paper.

1. Nonnegative and Nonpositive Part of a Real Number

In the sequel n is a natural number and r is a real number. We now define two new functors. Let n, m be natural numbers. Then $\min(n, m)$ is a natural number. Let r be a real number. The functor $\max_+(r)$ yielding a real number is defined as follows:

(Def.1) $\max_+(r) = \max(r, 0)$.

The functor $\max_-(r)$ yielding a real number is defined as follows:

(Def.2) $\max_-(r) = \max(-r, 0)$.

We now state several propositions:

(1) For every real number r holds $r = \max_+(r) - \max_-(r)$.

(2) For every real number r holds $|r| = \max_+(r) + \max_-(r)$.

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(3) For every real number r holds $2 \cdot \max_+(r) = r + |r|$.

(4) For all real numbers r, s such that $0 \leq r$ holds $\max_+(r \cdot s) = r \cdot \max_+(s)$.

(5) For all real numbers r, s holds $\max_+(r + s) \leq \max_+(r) + \max_+(s)$.

(6) For every real number r holds $0 \leq \max_+(r)$ and $0 \leq \max_-(r)$.

(7) For all real numbers r_1, r_2, s_1, s_2 such that $r_1 \leq s_1$ and $r_2 \leq s_2$ holds $\max(r_1, r_2) \leq \max(s_1, s_2)$.

2. Properties of Real Function

One can prove the following propositions:

(8) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all real numbers r, s such that $r \neq 0$ holds $F^{-1}\{\frac{s}{r}\} = (r\,F)^{-1}\{s\}$.

(9) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $(0\,F)^{-1}\{0\} = \operatorname{dom} F$.

(10) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r such that $0 < r$ holds $|F|^{-1}\{r\} = F^{-1}\{-r, r\}$.

(11) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $|F|^{-1}\{0\} = F^{-1}\{0\}$.

(12) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r such that $r < 0$ holds $|F|^{-1}\{r\} = \emptyset$.

(13) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ and for every real number r such that $r \neq 0$ holds F and G are fiwerwise equipotent if and only if $r\,F$ and $r\,G$ are fiwerwise equipotent.

(14) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ holds F and G are fiwerwise equipotent if and only if $-F$ and $-G$ are fiwerwise equipotent.

(15) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ such that F and G are fiwerwise equipotent holds $|F|$ and $|G|$ are fiwerwise equipotent.

We now define two new constructions. Let X, Y be sets. A non-empty set of functions is said to be a non empty set of partial functions from X to Y if:

(Def.3) every element of it is a partial function from X to Y.

Let X, Y be sets. Then $X \dot{\to} Y$ is a non empty set of partial functions from X to Y. Let P be a non empty set of partial functions from X to Y. We see that the element of P is a partial function from X to Y. Let D, C be non-empty sets, and let X be a subset of D, and let c be an element of C. Then $X \longmapsto c$ is an element of $D \dot{\to} C$. Let D be a non-empty set, and let F_1, F_2 be elements of $D \dot{\to} \mathbb{R}$. Then $F_1 + F_2$ is an element of $D \dot{\to} \mathbb{R}$. Then $F_1 - F_2$ is an element of $D \dot{\to} \mathbb{R}$. Then $F_1\,F_2$ is an element of $D \dot{\to} \mathbb{R}$. Then $\frac{F_1}{F_2}$ is an element of $D \dot{\to} \mathbb{R}$. Let D be a non-empty set, and let F be an element of $D \dot{\to} \mathbb{R}$. Then $|F|$ is an element of $D \dot{\rightarrow} \mathbb{R}$. Then $-F$ is an element of $D \dot{\rightarrow} \mathbb{R}$. Then $\frac{1}{F}$ is an element of $D \dot{\rightarrow} \mathbb{R}$. Let D be a non-empty set, and let F be an element of $D \dot{\rightarrow} \mathbb{R}$, and let r be a real number. Then $r\,F$ is an element of $D \dot{\rightarrow} \mathbb{R}$. Let D be a non-empty set. The functor $+_{D \dot{\rightarrow} \mathbb{R}}$ yielding a binary operation on $D \dot{\rightarrow} \mathbb{R}$ is defined as follows:

(Def.4) for all elements F_1, F_2 of $D \dot{\rightarrow} \mathbb{R}$ holds $+_{D \dot{\rightarrow} \mathbb{R}}(F_1, F_2) = F_1 + F_2$.

The following propositions are true:

(16) For every non-empty set D holds $+_{D \dot{\rightarrow} \mathbb{R}}$ is commutative.

(17) For every non-empty set D holds $+_{D \dot{\rightarrow} \mathbb{R}}$ is associative.

(18) For every non-empty set D holds $\Omega_D \longmapsto 0$ **qua** a real number is a unity w.r.t. $+_{D \dot{\rightarrow} \mathbb{R}}$.

(19) For every non-empty set D holds $\mathbf{1}_{+_{D \dot{\rightarrow} \mathbb{R}}} = \Omega_D \longmapsto 0$ **qua** a real number.

(20) For every non-empty set D holds $+_{D \dot{\rightarrow} \mathbb{R}}$ has a unity.

Let D be a non-empty set, and let f be a finite sequence of elements of $D \dot{\rightarrow} \mathbb{R}$. The functor $\sum f$ yielding an element of $D \dot{\rightarrow} \mathbb{R}$ is defined as follows:

(Def.5) $\sum f = +_{D \dot{\rightarrow} \mathbb{R}} \circledast f$.

Next we state several propositions:

(21) For every non-empty set D holds $\sum(\varepsilon_{(D \dot{\rightarrow} \mathbb{R})}) = \Omega_D \longmapsto 0$ **qua** a real number.

(22) For every non-empty set D and for every element G of $D \dot{\rightarrow} \mathbb{R}$ holds $\sum\langle G \rangle = G$.

(23) For every non-empty set D and for every finite sequence f of elements of $D \dot{\rightarrow} \mathbb{R}$ and for every element G of $D \dot{\rightarrow} \mathbb{R}$ holds $\sum(f \frown \langle G \rangle) = \sum f + G$.

(24) For every non-empty set D and for all finite sequences f_1, f_2 of elements of $D \dot{\rightarrow} \mathbb{R}$ holds $\sum(f_1 \frown f_2) = \sum f_1 + \sum f_2$.

(25) For every non-empty set D and for every finite sequence f of elements of $D \dot{\rightarrow} \mathbb{R}$ and for every element G of $D \dot{\rightarrow} \mathbb{R}$ holds $\sum(\langle G \rangle \frown f) = G + \sum f$.

(26) For every non-empty set D and for all elements G_1, G_2 of $D \dot{\rightarrow} \mathbb{R}$ holds $\sum\langle G_1, G_2 \rangle = G_1 + G_2$.

(27) For every non-empty set D and for all elements G_1, G_2, G_3 of $D \dot{\rightarrow} \mathbb{R}$ holds $\sum\langle G_1, G_2, G_3 \rangle = G_1 + G_2 + G_3$.

(28) For every non-empty set D and for all finite sequences f, g of elements of $D \dot{\rightarrow} \mathbb{R}$ such that f and g are fiwerwise equipotent holds $\sum f = \sum g$.

We now define four new constructions. Let D be a non-empty set, and let f be a finite sequence. The functor $\mathrm{CHI}(f, D)$ yielding a finite sequence of elements of $D \dot{\rightarrow} \mathbb{R}$ is defined by:

(Def.6) $\operatorname{len} \mathrm{CHI}(f, D) = \operatorname{len} f$ and for every n such that $n \in \operatorname{dom} \mathrm{CHI}(f, D)$ holds $(\mathrm{CHI}(f, D))(n) = \chi_{f(n), D}$.

Let D be a non-empty set, and let f be a finite sequence of elements of $D \dot{\rightarrow} \mathbb{R}$, and let R be a finite sequence of elements of $\mathbb{R}$. The functor $R\,f$ yields a finite sequence of elements of $D \dot{\rightarrow} \mathbb{R}$ and is defined as follows:

(Def.7) $\mathrm{len}(R\,f) = \min(\mathrm{len}\,R, \mathrm{len}\,f)$ and for every n such that $n \in \mathrm{dom}(R\,f)$ and for every partial function F from D to $\mathbb{R}$ and for every r such that $r = R(n)$ and $F = f(n)$ holds $(R\,f)(n) = r\,F$.

Let D, C be non-empty sets, and let f be a finite sequence of elements of $D \dot{\rightarrow} C$, and let d be an element of D. The functor $f\#d$ yields a finite sequence of elements of C and is defined as follows:

(Def.8) $\mathrm{len}(f\#d) = \mathrm{len}\,f$ and for every natural number n and for every element G of $D \dot{\rightarrow} C$ such that $n \in \mathrm{dom}(f\#d)$ and $f(n) = G$ holds $(f\#d)(n) = G(d)$.

Let D, C be non-empty sets, and let f be a finite sequence of elements of $D \dot{\rightarrow} C$, and let d be an element of D. We say that d is common for dom f if and only if:

(Def.9) for every element G of $D \dot{\rightarrow} C$ and for every natural number n such that $n \in \mathrm{dom}\,f$ and $f(n) = G$ holds $d \in \mathrm{dom}\,G$.

One can prove the following propositions:

(29) For all non-empty sets D, C and for every finite sequence f of elements of $D \dot{\rightarrow} C$ and for every element d of D and for every natural number n such that d is common for dom f and $n \neq 0$ holds d is common for dom $f \restriction n$.

(30) For all non-empty sets D, C and for every finite sequence f of elements of $D \dot{\rightarrow} C$ and for every element d of D and for every natural number n such that d is common for dom f holds d is common for dom $f_{\lfloor n}$.

(31) For every non-empty set D and for every element d of D and for every finite sequence f of elements of $D \dot{\rightarrow} \mathbb{R}$ such that $\mathrm{len}\,f \neq 0$ holds d is common for dom f if and only if $d \in \mathrm{dom} \sum f$.

(32) For all non-empty sets D, C and for every finite sequence f of elements of $D \dot{\rightarrow} C$ and for every element d of D and for every natural number n holds $(f \restriction n)\#d = (f\#d) \restriction n$.

(33) For every non-empty set D and for every finite sequence f and for every element d of D holds d is common for dom $\mathrm{CHI}(f, D)$.

(34) For every non-empty set D and for every element d of D and for every finite sequence f of elements of $D \dot{\rightarrow} \mathbb{R}$ and for every finite sequence R of elements of $\mathbb{R}$ such that d is common for dom f holds d is common for dom $R\,f$.

(35) For every non-empty set D and for every finite sequence f and for every finite sequence R of elements of $\mathbb{R}$ and for every element d of D holds d is common for dom $R\,\mathrm{CHI}(f, D)$.

(36) For every non-empty set D and for every element d of D and for every finite sequence f of elements of $D \dot{\rightarrow} \mathbb{R}$ such that d is common for dom f holds $(\sum f)(d) = \sum(f\#d)$.

We now define two new functors. Let D be a non-empty set, and let F be a partial function from D to $\mathbb{R}$. The functor $\max_{+}(F)$ yielding a partial function from D to $\mathbb{R}$ is defined as follows:

(Def.10) $\operatorname{dom} \max_+(F) = \operatorname{dom} F$ and for every element d of D such that $d \in \operatorname{dom} \max_+(F)$ holds $(\max_+(F))(d) = \max_+(F(d))$.

The functor $\max_-(F)$ yielding a partial function from D to $\mathbb{R}$ is defined as follows:

(Def.11) $\operatorname{dom} \max_-(F) = \operatorname{dom} F$ and for every element d of D such that $d \in \operatorname{dom} \max_-(F)$ holds $(\max_-(F))(d) = \max_-(F(d))$.

The following propositions are true:

(37) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $F = \max_+(F) - \max_-(F)$ and $|F| = \max_+(F) + \max_-(F)$ and $2 \max_+(F) = F + |F|$.

(38) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r such that $0 < r$ holds $F^{-1} \{r\} = (\max_+(F))^{-1} \{r\}$.

(39) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $F^{-1}]-\infty, 0] = (\max_+(F))^{-1} \{0\}$.

(40) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every element d of D such that $d \in \operatorname{dom} F$ holds $0 \leq (\max_+(F))(d)$.

(41) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r such that $0 < r$ holds $F^{-1} \{-r\} = (\max_-(F))^{-1} \{r\}$.

(42) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $F^{-1} [0, +\infty[= (\max_-(F))^{-1} \{0\}$.

(43) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every element d of D such that $d \in \operatorname{dom} F$ holds $0 \leq (\max_-(F))(d)$.

(44) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ such that F and G are fiwerwise equipotent holds $\max_+(F)$ and $\max_+(G)$ are fiwerwise equipotent.

(45) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ such that F and G are fiwerwise equipotent holds $\max_-(F)$ and $\max_-(G)$ are fiwerwise equipotent.

(46) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ such that $\operatorname{dom} F$ is finite and $\operatorname{dom} G$ is finite and $\max_+(F)$ and $\max_+(G)$ are fiwerwise equipotent and $\max_-(F)$ and $\max_-(G)$ are fiwerwise equipotent holds F and G are fiwerwise equipotent.

(47) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X holds $\max_+(F) \upharpoonright X = \max_+(F \upharpoonright X)$.

(48) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X holds $\max_-(F) \restriction X = \max_-(F \restriction X)$.

(49) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ if for every element d of D such that $d \in \operatorname{dom} F$ holds $F(d) \geq 0$, then $\max_+(F) = F$.

(50) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ if for every element d of D such that $d \in \operatorname{dom} F$ holds $F(d) \leq 0$, then $\max_-(F) = -F$.

Let D be a non-empty set, and let F be a partial function from D to $\mathbb{R}$, and let r be a real number. The functor $F - r$ yields a partial function from D to $\mathbb{R}$ and is defined as follows:

(Def.12) $\operatorname{dom}(F - r) = \operatorname{dom} F$ and for every element d of D such that $d \in \operatorname{dom}(F - r)$ holds $(F - r)(d) = F(d) - r$.

We now state four propositions:

(51) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $F - 0 = F$.

(52) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r and for every set X holds $F \restriction X - r = (F - r) \restriction X$.

(53) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all real numbers r, s holds $F^{-1} \{s + r\} = (F - r)^{-1} \{s\}$.

(54) For all non-empty sets D, C and for every partial function F from D to $\mathbb{R}$ and for every partial function G from C to $\mathbb{R}$ and for every real number r holds F and G are fiwerwise equipotent if and only if $F - r$ and $G - r$ are fiwerwise equipotent.

Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and let X be a set. We say that F is convex on X if and only if the conditions (Def.13) is satisfied.

(Def.13) (i) $X \subseteq \operatorname{dom} F$,

(ii) for every real number p such that $0 \leq p$ and $p \leq 1$ and for all real numbers r, s such that $r \in X$ and $s \in X$ and $p \cdot r + (1 - p) \cdot s \in X$ holds $F(p \cdot r + (1 - p) \cdot s) \leq p \cdot F(r) + (1 - p) \cdot F(s)$.

The following propositions are true:

(55) Let a, b be real numbers. Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then F is convex on $[a, b]$ if and only if the following conditions are satisfied:

(i) $[a, b] \subseteq \operatorname{dom} F$,

(ii) for every real number p such that $0 \leq p$ and $p \leq 1$ and for all real numbers r, s such that $r \in [a, b]$ and $s \in [a, b]$ holds $F(p \cdot r + (1 - p) \cdot s) \leq p \cdot F(r) + (1 - p) \cdot F(s)$.

(56) Let a, b be real numbers. Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Then F is convex on $[a, b]$ if and only if the following conditions are satisfied:

(i) $[a, b] \subseteq \operatorname{dom} F$,

(ii) for all real numbers x_1, x_2, x_3 such that $x_1 \in [a,b]$ and $x_2 \in [a,b]$ and $x_3 \in [a,b]$ and $x_1 < x_2$ and $x_2 < x_3$ holds $\frac{F(x_1)-F(x_2)}{x_1-x_2} \leq \frac{F(x_2)-F(x_3)}{x_2-x_3}$.

(57) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for all sets X, Y such that F is convex on X and $Y \subseteq X$ holds F is convex on Y.

(58) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X and for every real number r holds F is convex on X if and only if $F - r$ is convex on X.

(59) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X and for every real number r such that $0 < r$ holds F is convex on X if and only if $r\,F$ is convex on X.

(60) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X such that $X \subseteq \operatorname{dom} F$ holds $0\,F$ is convex on X.

(61) For all partial functions F, G from $\mathbb{R}$ to $\mathbb{R}$ and for every set X such that F is convex on X and G is convex on X holds $F + G$ is convex on X.

(62) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X and for every real number r such that F is convex on X holds $\max_{+}(F - r)$ is convex on X.

(63) For every partial function F from $\mathbb{R}$ to $\mathbb{R}$ and for every set X such that F is convex on X holds $\max_{+}(F)$ is convex on X.

(64) $\mathrm{id}_{(\Omega_{\mathbb{R}})}$ is convex on $\mathbb{R}$.

(65) For every real number r holds $\max_{+}(\mathrm{id}_{(\Omega_{\mathbb{R}})} - r)$ is convex on $\mathbb{R}$.

Let D be a non-empty set, and let F be a partial function from D to $\mathbb{R}$, and let X be a set. Let us assume that $\operatorname{dom}(F \restriction X)$ is finite. The functor $\operatorname{FinS}(F, X)$ yields a non-increasing finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def.14) $F \restriction X$ and $\operatorname{FinS}(F, X)$ are fiwerwise equipotent.

Next we state a number of propositions:

(66) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X such that $\operatorname{dom}(F \restriction X)$ is finite holds $\operatorname{FinS}(F, \operatorname{dom}(F \restriction X)) = \operatorname{FinS}(F, X)$.

(67) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X such that $\operatorname{dom}(F \restriction X)$ is finite holds $\operatorname{FinS}(F \restriction X, X) = \operatorname{FinS}(F, X)$.

(68) For every non-empty set D and for every element d of D and for every set X and for every partial function F from D to $\mathbb{R}$ such that X is finite and $d \in \operatorname{dom}(F \restriction X)$ holds $(\operatorname{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$ and $F \restriction X$ are fiwerwise equipotent.

(69) For every non-empty set D and for every element d of D and for every set X and for every partial function F from D to $\mathbb{R}$ such that $\operatorname{dom}(F \restriction X)$ is finite and $d \in \operatorname{dom}(F \restriction X)$ holds $(\operatorname{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$ and $F \restriction X$ are fiwerwise equipotent.

(70) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X such that $\operatorname{dom}(F \restriction X)$ is finite holds $\operatorname{len} \operatorname{FinS}(F, X) =$

card dom$(F \restriction X)$.

(71) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $\mathrm{FinS}(F, \emptyset) = \varepsilon_{\mathbb{R}}$.

(72) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every element d of D such that $d \in \operatorname{dom} F$ holds $\mathrm{FinS}(F, \{d\}) = \langle F(d) \rangle$.

(73) Let D be a non-empty set. Let F be a partial function from D to $\mathbb{R}$. Then for every set X and for every element d of D such that $\operatorname{dom}(F \restriction X)$ is finite and $d \in \operatorname{dom}(F \restriction X)$ and $(\mathrm{FinS}(F, X))(\operatorname{len} \mathrm{FinS}(F, X)) = F(d)$ holds $\mathrm{FinS}(F, X) = (\mathrm{FinS}(F, X \setminus \{d\})) \frown \langle F(d) \rangle$.

(74) Let D be a non-empty set. Let F be a partial function from D to $\mathbb{R}$. Let X, Y be sets. Suppose $\operatorname{dom}(F \restriction X)$ is finite and $Y \subseteq X$ and for all elements d_1, d_2 of D such that $d_1 \in \operatorname{dom}(F \restriction Y)$ and $d_2 \in \operatorname{dom}(F \restriction (X \setminus Y))$ holds $F(d_1) \geq F(d_2)$. Then $\mathrm{FinS}(F, X) = (\mathrm{FinS}(F, Y)) \frown \mathrm{FinS}(F, X \setminus Y)$.

(75) Let D be a non-empty set. Let F be a partial function from D to $\mathbb{R}$. Let r be a real number. Let X be a set. Then for every element d of D such that $\operatorname{dom}(F \restriction X)$ is finite and $d \in \operatorname{dom}(F \restriction X)$ holds
$(\mathrm{FinS}(F - r, X))(\operatorname{len} \mathrm{FinS}(F - r, X)) = (F - r)(d)$
if and only if $(\mathrm{FinS}(F, X))(\operatorname{len} \mathrm{FinS}(F, X)) = F(d)$.

(76) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every real number r and for every set X such that $\operatorname{dom}(F \restriction X)$ is finite holds $\mathrm{FinS}(F - r, X) = \mathrm{FinS}(F, X) - (\operatorname{card} \operatorname{dom}(F \restriction X) \longmapsto r)$.

(77) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X such that $\operatorname{dom}(F \restriction X)$ is finite and for every element d of D such that $d \in \operatorname{dom}(F \restriction X)$ holds $F(d) \geq 0$ holds $\mathrm{FinS}(\max_{+}(F), X) = \mathrm{FinS}(F, X)$.

(78) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X and for every real number r such that $\operatorname{dom}(F \restriction X)$ is finite and $\operatorname{rng}(F \restriction X) = \{r\}$ holds $\mathrm{FinS}(F, X) = \operatorname{card} \operatorname{dom}(F \restriction X) \longmapsto r$.

(79) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all sets X, Y such that $\operatorname{dom}(F \restriction (X \cup Y))$ is finite and $X \cap Y = \emptyset$ holds $\mathrm{FinS}(F, X \cup Y)$ and $(\mathrm{FinS}(F, X)) \frown \mathrm{FinS}(F, Y)$ are fiwerwise equipotent.

Let D be a non-empty set, and let F be a partial function from D to $\mathbb{R}$, and let X be a set. The functor $\sum_{\kappa=0}^{X} F(\kappa)$ yields a real number and is defined as follows:

(Def.15) $\sum_{\kappa=0}^{X} F(\kappa) = \sum \mathrm{FinS}(F, X)$.

One can prove the following propositions:

(80) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X and for every real number r such that $\operatorname{dom}(F \restriction X)$ is finite holds $\sum_{\kappa=0}^{X} (r\, F)(\kappa) = r \cdot \sum_{\kappa=0}^{X} F(\kappa)$.

(81) For every non-empty set D and for all partial functions F, G from D to

$\mathbb{R}$ and for every set X such that $\mathrm{dom}(F \restriction X)$ is finite and $\mathrm{dom}(F \restriction X) = \mathrm{dom}(G \restriction X)$ holds $\sum_{\kappa=0}^{X}(F+G)(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) + \sum_{\kappa=0}^{X} G(\kappa)$.

(82) For every non-empty set D and for all partial functions F, G from D to $\mathbb{R}$ and for every set X such that $\mathrm{dom}(F \restriction X)$ is finite and $\mathrm{dom}(F \restriction X) = \mathrm{dom}(G \restriction X)$ holds $\sum_{\kappa=0}^{X}(F-G)(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) - \sum_{\kappa=0}^{X} G(\kappa)$.

(83) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every set X and for every real number r such that $\mathrm{dom}(F \restriction X)$ is finite holds $\sum_{\kappa=0}^{X}(F-r)(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) - r \cdot \mathrm{card}\,\mathrm{dom}(F \restriction X)$.

(84) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ holds $\sum_{\kappa=0}^{\emptyset} F(\kappa) = 0$.

(85) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for every element d of D such that $d \in \mathrm{dom}\, F$ holds $\sum_{\kappa=0}^{\{d\}} F(\kappa) = F(d)$.

(86) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all sets X, Y such that $\mathrm{dom}(F \restriction (X \cup Y))$ is finite and $X \cap Y = \emptyset$ holds $\sum_{\kappa=0}^{X \cup Y} F(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) + \sum_{\kappa=0}^{Y} F(\kappa)$.

(87) For every non-empty set D and for every partial function F from D to $\mathbb{R}$ and for all sets X, Y such that $\mathrm{dom}(F \restriction (X \cup Y))$ is finite and $\mathrm{dom}(F \restriction X) \cap \mathrm{dom}(F \restriction Y) = \emptyset$ holds $\sum_{\kappa=0}^{X \cup Y} F(\kappa) = \sum_{\kappa=0}^{X} F(\kappa) + \sum_{\kappa=0}^{Y} F(\kappa)$.

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Domains of Submodules, Join and Meet of Finite Sequences of Submodules and Quotient Modules

Michał Muzalewski
Warsaw University
Białystok

Summary. Notions of domains of submodules, join and meet of finite sequences of submodules and quotient modules. A few basic theorems and schemes related to these notions are proved.

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The papers [17], [28], [3], [4], [2], [1], [16], [5], [29], [15], [24], [20], [25], [27], [21], [18], [7], [6], [8], [26], [23], [22], [19], [14], [13], [11], [12], [9], and [10] provide the terminology and notation for this paper.

1. Auxiliary theorems on free-modules

For simplicity we follow a convention: x is arbitrary, K is an associative ring, r is a scalar of K, V, M, N are left modules over K, a, b, a_1, a_2 are vectors of V, A, A_1, A_2 are subsets of V, l is a linear combination of A, W is a submodule of V, and L_1 is a finite sequence of elements of $\mathrm{Sub}(V)$. One can prove the following propositions:

(1) If K is non-trivial and A is linearly independent, then $0_V \notin A$.

(2) If $a \notin A$, then $l(a) = 0_K$.

(3) If K is trivial, then for every l holds support $l = \emptyset$ and $\mathrm{Lin}(A)$ is trivial.

(4) If V is non-trivial, then for every A such that A is base holds $A \neq \emptyset$.

(5) If $A_1 \cup A_2$ is linearly independent and $A_1 \cap A_2 = \emptyset$, then $\mathrm{Lin}(A_1) \cap \mathrm{Lin}(A_2) = \mathbf{0}_V$.

(6) If A is base and $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$, then V is the direct sum of $\mathrm{Lin}(A_1)$ and $\mathrm{Lin}(A_2)$.

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2. Domains of submodules

Let us consider K, V. A non-empty set is called a non empty set of submodules of V if:

(Def.1) if $x \in$ it, then x is a strict submodule of V.

Let us consider K, V. Then $\mathrm{Sub}(V)$ is a non empty set of submodules of V. Let us consider K, V, and let D be a non empty set of submodules of V. We see that the element of D is a strict submodule of V. Let us consider K, V, and let D be a non empty set of submodules of V. One can verify that there exists a strict element of D.

We now state two propositions:

(7) If x is an element of $\mathrm{Sub}(V)$ **qua** a non-empty set, then x is an element of $\mathrm{Sub}(V)$.

(8) If $x \in \mathrm{Sub}(V)$, then x is an element of $\mathrm{Sub}(V)$.

We now define two new modes. Let us consider K, V. Let us assume that V is non-trivial. A strict submodule of V is called a line of V if:

(Def.2) there exists a such that $a \neq 0_V$ and it $= \prod^* a$.

Let us consider K, V. A non-empty set is said to be a non empty set of lines of V if:

(Def.3) if $x \in$ it, then x is a line of V.

We now state two propositions:

(9) If W is strict and the group structure of W is strict, then W is an element of $\mathrm{Sub}(V)$ **qua** a non-empty set.

(10) If V is non-trivial, then every line of V is an element of $\mathrm{Sub}(V)$.

We now define three new constructions. Let us consider K, V. Let us assume that V is non-trivial. The functor $\mathrm{lines}(V)$ yields a non empty set of lines of V and is defined as follows:

(Def.4) $\mathrm{lines}(V)$ is the set of all lines of V.

Let us consider K, V, and let D be a non empty set of lines of V. We see that the element of D is a line of V. Let us consider K, V. Let us assume that V is non-trivial and V is free. A strict submodule of V is said to be a hiperplane of V if:

(Def.5) the group structure of it is strict and there exists a such that $a \neq 0_V$ and V is the direct sum of $\prod^* a$ and it.

Let us consider K, V. A non-empty set is called a non empty set of hiperplanes of V if:

(Def.6) if $x \in$ it, then x is a hiperplane of V.

One can prove the following proposition

(11) If V is non-trivial and V is free, then every hiperplane of V is an element of $\mathrm{Sub}(V)$.

Let us consider K, V. Let us assume that V is non-trivial and V is free. The functor hiperplanes(V) yielding a non empty set of hiperplanes of V is defined by:

(Def.7) hiperplanes(V) is the set of all hiperplanes of V.

Let us consider K, V, and let D be a non empty set of hiperplanes of V. We see that the element of D is a hiperplane of V.

3. Join and meet of finite sequences of submodules

We now define two new functors. Let us consider K, V, L_1. The functor $\sum L_1$ yielding an element of $\mathrm{Sub}(V)$ is defined as follows:

(Def.8) $\sum L_1 = \mathrm{SubJoin}\, V \circledast L_1$.

The functor $\bigcap L_1$ yields an element of $\mathrm{Sub}(V)$ and is defined as follows:

(Def.9) $\bigcap L_1 = \mathrm{SubMeet}\, V \circledast L_1$.

The following propositions are true:

(12) For every lattice G holds the join operation of G is commutative and the join operation of G is associative and the meet operation of G is commutative and the meet operation of G is associative.

(13) For every element a of $\mathrm{Sub}(V)$ holds the group structure of a is strict.

(14) $\mathrm{SubJoin}\, V$ is commutative and $\mathrm{SubJoin}\, V$ is associative and $\mathrm{SubJoin}\, V$ has a unity and $\mathbf{0}_V = \mathbf{1}_{\mathrm{SubJoin}\, V}$.

(15) If the group structure of V is strict, then $\mathrm{SubMeet}\, V$ is commutative and $\mathrm{SubMeet}\, V$ is associative and $\mathrm{SubMeet}\, V$ has a unity and $\Omega_V = \mathbf{1}_{\mathrm{SubMeet}\, V}$.

4. Sum of subsets of module

Let us consider K, V, A_1, A_2. The functor $A_1 + A_2$ yields a subset of V and is defined by:

(Def.10) $x \in A_1 + A_2$ if and only if there exist a_1, a_2 such that $a_1 \in A_1$ and $a_2 \in A_2$ and $x = a_1 + a_2$.

5. Vector of subset

Let us consider K, V, A. Let us assume that $A \neq \emptyset$. A vector of V is said to be a vector of A if:

(Def.11) it is an element of A.

One can prove the following propositions:

(16) If $A_1 \neq \emptyset$ and $A_1 \subseteq A_2$, then for every x such that x is a vector of A_1 holds x is a vector of A_2.

(17) $a_2 \in a_1 + W$ if and only if $a_1 - a_2 \in W$.

(18) $a_1 + W = a_2 + W$ if and only if $a_1 - a_2 \in W$.

We now define two new functors. Let us consider K, V, W. The functor $V \looparrowleft W$ yields a non-empty set and is defined by:

(Def.12) $x \in V \looparrowleft W$ if and only if there exists a such that $x = a + W$.

Let us consider K, V, W, a. The functor $a \looparrowleft W$ yields an element of $V \looparrowleft W$ and is defined as follows:

(Def.13) $a \looparrowleft W = a + W$.

We now state two propositions:

(19) For every element x of $V \looparrowleft W$ there exists a such that $x = a \looparrowleft W$.

(20) $a_1 \looparrowleft W = a_2 \looparrowleft W$ if and only if $a_1 - a_2 \in W$.

In the sequel S_1, S_2 will denote elements of $V \looparrowleft W$. We now define five new functors. Let us consider K, V, W, S_1. The functor $-S_1$ yields an element of $V \looparrowleft W$ and is defined by:

(Def.14) if $S_1 = a \looparrowleft W$, then $-S_1 = (-a) \looparrowleft W$.

Let us consider S_2. The functor $S_1 + S_2$ yields an element of $V \looparrowleft W$ and is defined by:

(Def.15) if $S_1 = a_1 \looparrowleft W$ and $S_2 = a_2 \looparrowleft W$, then $S_1 + S_2 = (a_1 + a_2) \looparrowleft W$.

Let us consider K, V, W. The functor $\mathrm{COMPL}(W)$ yields a unary operation on $V \looparrowleft W$ and is defined as follows:

(Def.16) $(\mathrm{COMPL}(W))(S_1) = -S_1$.

The functor $\mathrm{ADD}(W)$ yields a binary operation on $V \looparrowleft W$ and is defined by:

(Def.17) $(\mathrm{ADD}(W))(S_1, S_2) = S_1 + S_2$.

Let us consider K, V, W. The functor $V(W)$ yields a strict group structure and is defined by:

(Def.18) $V(W) = \langle V \looparrowleft W, \mathrm{ADD}(W), \mathrm{COMPL}(W), 0_V \looparrowleft W \rangle$.

One can prove the following proposition

(21) $a \looparrowleft W$ is an element of $V(W)$.

Let us consider K, V, W, a. The functor $a(W)$ yielding an element of $V(W)$ is defined by:

(Def.19) $a(W) = a \looparrowleft W$.

We now state four propositions:

(22) For every element x of $V(W)$ there exists a such that $x = a(W)$.

(23) $a_1(W) = a_2(W)$ if and only if $a_1 - a_2 \in W$.

(24) $a(W) + b(W) = (a + b)(W)$ and $-a(W) = (-a)(W)$ and $0_{V(W)} = 0_V(W)$.

(25) $V(W)$ is a strict Abelian group.

Let us consider K, V, W. Then $V(W)$ is a strict Abelian group.

In the sequel S is an element of $V(W)$. We now define three new functors. Let us consider K, V, W, r, S. The functor $r \cdot S$ yielding an element of $V(W)$ is defined by:

(Def.20) if $S = a(W)$, then $r \cdot S = (r \cdot a)(W)$.

Let us consider K, V, W. The functor LMULT(W) yielding a function from $[\!:$ the carrier of K, the carrier of $V(W)\, :\!]$ into the carrier of $V(W)$ is defined by:

(Def.21) $(\mathrm{LMULT}(W))(r, S) = r \cdot S$.

Let us consider K, V, W. The functor $\frac{V}{W}$ yielding a strict vector space structure over K is defined as follows:

(Def.22) $\frac{V}{W} = \langle$the carrier of $V(W)$, the addition of $V(W)$, the reverse-map of $V(W)$, the zero of $V(W)$, $\mathrm{LMULT}(W)\rangle$.

We now state two propositions:

(26) $a(W)$ is a vector of $\frac{V}{W}$.

(27) Every vector of $\frac{V}{W}$ is an element of $V(W)$.

Let us consider K, V, W, a. The functor $\frac{a}{W}$ yields a vector of $\frac{V}{W}$ and is defined as follows:

(Def.23) $\frac{a}{W} = a(W)$.

One can prove the following four propositions:

(28) For every vector x of $\frac{V}{W}$ there exists a such that $x = \frac{a}{W}$.

(29) $\frac{a_1}{W} = \frac{a_2}{W}$ if and only if $a_1 - a_2 \in W$.

(30) $\frac{a}{W} + \frac{b}{W} = \frac{a+b}{W}$ and $r \cdot \frac{a}{W} = \frac{r \cdot a}{W}$.

(31) $\frac{V}{W}$ is a strict left module over K.

Let us consider K, V, W. Then $\frac{V}{W}$ is a strict left module over K.

6. Quotient modules

In this article we present several logical schemes. The scheme *SetEq* deals with a unary predicate $\mathcal{P}$, and states that:

for all sets X_1, X_2 such that for an arbitrary x holds $x \in X_1$ if and only if $\mathcal{P}[x]$ and for an arbitrary x holds $x \in X_2$ if and only if $\mathcal{P}[x]$ holds $X_1 = X_2$
for all values of the parameter.

The scheme *DomainEq* deals with a unary predicate $\mathcal{P}$, and states that:

for all non-empty sets X_1, X_2 such that for an arbitrary x holds $x \in X_1$ if and only if $\mathcal{P}[x]$ and for an arbitrary x holds $x \in X_2$ if and only if $\mathcal{P}[x]$ holds
$X_1 = X_2$
for all values of the parameter.

The scheme *ElementEq* concerns a set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for all elements X_1, X_2 of $\mathcal{A}$ such that for an arbitrary x holds $x \in X_1$ if and only if $\mathcal{P}[x]$ and for an arbitrary x holds $x \in X_2$ if and only if $\mathcal{P}[x]$ holds $X_1 = X_2$

for all values of the parameters.

The scheme *TypeEq* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{A} = \mathcal{B}$

provided the parameters meet the following conditions:

- for an arbitrary x holds $x \in \mathcal{A}$ if and only if $\mathcal{P}[x]$,
- for an arbitrary x holds $x \in \mathcal{B}$ if and only if $\mathcal{P}[x]$.

The scheme *FuncEq* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

for all functions f_1, f_2 from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f_1(x) = \mathcal{F}(x)$ and for every element x of $\mathcal{A}$ holds $f_2(x) = \mathcal{F}(x)$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *UnOpEq* deals with a non-empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ and states that:

for all unary operations f_1, f_2 on $\mathcal{A}$ such that for every element a of $\mathcal{A}$ holds $f_1(a) = \mathcal{F}(a)$ and for every element a of $\mathcal{A}$ holds $f_2(a) = \mathcal{F}(a)$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *BinOpEq* concerns a non-empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ and states that:

for all binary operations f_1, f_2 on $\mathcal{A}$ such that for all elements a, b of $\mathcal{A}$ holds $f_1(a, b) = \mathcal{F}(a,b)$ and for all elements a, b of $\mathcal{A}$ holds $f_2(a, b) = \mathcal{F}(a,b)$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *TriOpEq* deals with a non-empty set $\mathcal{A}$ and a ternary functor $\mathcal{F}$ and states that:

for all ternary operations f_1, f_2 on $\mathcal{A}$ such that for all elements a, b, c of $\mathcal{A}$ holds $f_1(a, b, c) = \mathcal{F}(a,b,c)$ and for all elements a, b, c of $\mathcal{A}$ holds $f_2(a, b, c) = \mathcal{F}(a,b,c)$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *QuaOpEq* deals with a non-empty set $\mathcal{A}$ and a 4-ary functor $\mathcal{F}$ and states that:

for all quadrary operations f_1, f_2 on $\mathcal{A}$ such that for all elements a, b, c, d of $\mathcal{A}$ holds $f_1(a, b, c, d) = \mathcal{F}(a,b,c,d)$ and for all elements a, b, c, d of $\mathcal{A}$ holds $f_2(a, b, c, d) = \mathcal{F}(a,b,c,d)$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *Fraenkel1_Ex* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a subset S of $\mathcal{B}$ such that $S = \{\mathcal{F}(x) : \mathcal{P}[x]\}$, where x ranges over elements of $\mathcal{A}$

for all values of the parameters.

The scheme *Fr_0* concerns a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters meet the following requirement:

- $\mathcal{B} \in \{a : \mathcal{P}[a]\}$, where a ranges over elements of $\mathcal{A}$.

The scheme *Fr_1* deals with a set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{C} \in \mathcal{A}$ if and only if $\mathcal{P}[\mathcal{C}]$

provided the following condition is satisfied:

- $\mathcal{A} = \{a : \mathcal{P}[a]\}$, where a ranges over elements of $\mathcal{B}$.

The scheme *Fr_2* concerns a set $\mathcal{A}$, a non-empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{C}]$

provided the following conditions are met:

- $\mathcal{C} \in \mathcal{A}$,
- $\mathcal{A} = \{a : \mathcal{P}[a]\}$, where a ranges over elements of $\mathcal{B}$.

The scheme *Fr_3* concerns a constant $\mathcal{A}$, a set $\mathcal{B}$, a non-empty set $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{A} \in \mathcal{B}$ if and only if there exists an element a of $\mathcal{C}$ such that $\mathcal{A} = a$ and $\mathcal{P}[a]$

provided the parameters meet the following condition:

- $\mathcal{B} = \{a : \mathcal{P}[a]\}$, where a ranges over elements of $\mathcal{C}$.

The scheme *Fr_4* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a set $\mathcal{C}$, an element $\mathcal{D}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$, and two binary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\mathcal{D} \in \mathcal{F}(\mathcal{C})$ if and only if for every element b of $\mathcal{B}$ such that $b \in \mathcal{C}$ holds $\mathcal{P}[\mathcal{D}, b]$

provided the parameters meet the following conditions:

- $\mathcal{F}(\mathcal{C}) = \{a : \mathcal{Q}[a, \mathcal{C}]\}$, where a ranges over elements of $\mathcal{A}$,
- $\mathcal{Q}[\mathcal{D}, \mathcal{C}]$ if and only if for every element b of $\mathcal{B}$ such that $b \in \mathcal{C}$ holds $\mathcal{P}[\mathcal{D}, b]$.

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Remarks on Special Subsets of Topological Spaces

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X be a topological space and let A be a subset of X. Recall that A is *nowhere dense* in X if its closure is a boundary subset of X, i.e., if $\operatorname{Int}\overline{A} = \emptyset$ (see [2]). We introduce here the concept of everywhere dense subsets in X, which is dual to the above one. Namely, A is said to be *everywhere dense* in X if its interior is a dense subset of X, i.e., if $\overline{\operatorname{Int} A} =$ the carrier of X.

Our purpose is to list a number of properties of such sets (comp. [7]). As a sample we formulate their two dual characterizations. The first one characterizes thin sets in X : *A is nowhere dense iff for every open nonempty subset G of X there is an open nonempty subset of X contained in G and disjoint from A.* The corresponding second one characterizes thick sets in X : *A is everywhere dense iff for every closed subset F of X distinct from the carrier of X there is a closed subset of X distinct from the carrier of X, which contains F and together with A covers the carrier of X.* We also give some connections between both these concepts. Of course, *A is everywhere (nowhere) dense in X iff its complement is nowhere (everywhere) dense*. Moreover, *A is nowhere dense iff there are two subsets of X, C boundary closed and B everywhere dense, such that $A = C \cap B$ and $C \cup B$ covers the carrier of X*. Dually, *A is everywhere dense iff there are two disjoint subsets of X, C open dense and B nowhere dense, such that $A = C \cup B$.*

Note that some relationships between everywhere (nowhere) dense sets in X and everywhere (nowhere) dense sets in subspaces of X are also indicated.

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The notation and terminology used here are introduced in the following papers: [5], [6], [3], [7], [4], and [1].

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1. Selected Properties of Subsets of a Topological Space

In the sequel X will denote a topological space and A, B will denote subsets of X. We now state several propositions:

(1) $A = \emptyset_X$ if and only if $A^{c} = \Omega_X$ and also $A = \emptyset$ if and only if $A^{c} =$ the carrier of X.

(2) $A = \Omega_X$ if and only if $A^{c} = \emptyset_X$ and also $A =$ the carrier of X if and only if $A^{c} = \emptyset$.

(3) $\operatorname{Int} A \cap \overline{B} \subseteq \overline{A \cap B}$.

(4) $\operatorname{Int}(A \cup B) \subseteq \overline{A} \cup \operatorname{Int} B$.

(5) If A is closed, then $\operatorname{Int}(A \cup B) \subseteq A \cup \operatorname{Int} B$.

(6) If A is closed, then $\operatorname{Int}(A \cup B) = \operatorname{Int}(A \cup \operatorname{Int} B)$.

(7) If $A \cap \operatorname{Int} \overline{A} = \emptyset$, then $\operatorname{Int} \overline{A} = \emptyset$.

(8) If $A \cup \overline{\operatorname{Int} A} =$ the carrier of X, then $\overline{\operatorname{Int} A} =$ the carrier of X.

2. Special Subsets of a Topological Space

Let X be a topological space. Let us observe that a subset of X is boundary if:

(Def.1) $\operatorname{Int} \mathrm{it} = \emptyset$.

We now state several propositions:

(9) $\emptyset_X$ is boundary.

(10) If A is boundary, then $A \neq$ the carrier of X.

(11) If B is boundary and $A \subseteq B$, then A is boundary.

(12) A is boundary if and only if for every subset C of X such that $A^{c} \subseteq C$ and C is closed holds $C =$ the carrier of X.

(13) A is boundary if and only if for every subset G of X such that $G \neq \emptyset$ and G is open holds $A^{c} \cap G \neq \emptyset$.

(14) A is boundary if and only if for every subset F of X such that F is closed holds $\operatorname{Int} F = \operatorname{Int}(F \cup A)$.

(15) If A is boundary or B is boundary, then $A \cap B$ is boundary.

Let X be a topological space. Let us observe that a subset of X is dense if:

(Def.2) $\overline{\mathrm{it}} =$ the carrier of X.

Next we state several propositions:

(16) Ω_X is dense.

(17) If A is dense, then $A \neq \emptyset$.

(18) A is dense if and only if A^{c} is boundary.

(19) A is dense if and only if for every subset C of X such that $A \subseteq C$ and C is closed holds $C =$ the carrier of X.

(20) A is dense if and only if for every subset G of X such that G is open holds $\overline{G} = \overline{G \cap A}$.

(21) If A is dense or B is dense, then $A \cup B$ is dense.

Let X be a topological space. Let us observe that a subset of X is nowhere dense if:

(Def.3) $\operatorname{Int} \overline{\mathrm{it}} = \emptyset$.

The following propositions are true:

(22) $\emptyset_X$ is nowhere dense.

(23) If A is nowhere dense, then $A \neq$ the carrier of X.

(24) If A is nowhere dense, then $\overline{A}$ is nowhere dense.

(25) If A is nowhere dense, then A is not dense.

(26) If B is nowhere dense and $A \subseteq B$, then A is nowhere dense.

(27) A is nowhere dense if and only if there exists a subset C of X such that $A \subseteq C$ and C is closed and C is boundary.

(28) A is nowhere dense if and only if for every subset G of X such that $G \neq \emptyset$ and G is open there exists a subset H of X such that $H \subseteq G$ and $H \neq \emptyset$ and H is open and $A \cap H = \emptyset$.

(29) If A is nowhere dense or B is nowhere dense, then $A \cap B$ is nowhere dense.

(30) If A is nowhere dense and B is boundary, then $A \cup B$ is boundary.

Let X be a topological space. A subset of X is everywhere dense if:

(Def.4) $\overline{\operatorname{Int} \mathrm{it}} = \Omega_X$.

Let X be a topological space. Let us observe that a subset of X is everywhere dense if:

(Def.5) $\overline{\operatorname{Int} \mathrm{it}} =$ the carrier of X.

One can prove the following propositions:

(31) Ω_X is everywhere dense.

(32) If A is everywhere dense, then $\operatorname{Int} A$ is everywhere dense.

(33) If A is everywhere dense, then A is dense.

(34) If A is everywhere dense, then $A \neq \emptyset$.

(35) A is everywhere dense if and only if $\operatorname{Int} A$ is dense.

(36) If A is open and A is dense, then A is everywhere dense.

(37) If A is everywhere dense, then A is not boundary.

(38) If A is everywhere dense and $A \subseteq B$, then B is everywhere dense.

(39) A is everywhere dense if and only if A^{c} is nowhere dense.

(40) A is nowhere dense if and only if A^{c} is everywhere dense.

(41) A is everywhere dense if and only if there exists a subset C of X such that $C \subseteq A$ and C is open and C is dense.

(42) A is everywhere dense if and only if for every subset F of X such that $F \neq$ the carrier of X and F is closed there exists a subset H of X such that $F \subseteq H$ and $H \neq$ the carrier of X and H is closed and $A \cup H =$ the carrier of X.

(43) If A is everywhere dense or B is everywhere dense, then $A \cup B$ is everywhere dense.

(44) If A is everywhere dense and B is everywhere dense, then $A \cap B$ is everywhere dense.

(45) If A is everywhere dense and B is dense, then $A \cap B$ is dense.

(46) If A is dense and B is nowhere dense, then $A \setminus B$ is dense.

(47) If A is everywhere dense and B is boundary, then $A \setminus B$ is dense.

(48) If A is everywhere dense and B is nowhere dense, then $A \setminus B$ is everywhere dense.

In the sequel D denotes a subset of X. We now state four propositions:

(49) If D is everywhere dense, then there exist subsets C, B of X such that C is open and C is dense and B is nowhere dense and $C \cup B = D$ and $C \cap B = \emptyset$.

(50) If D is everywhere dense, then there exist subsets C, B of X such that C is open and C is dense and B is closed and B is boundary and $C \cup D \cap B = D$ and $C \cap B = \emptyset$ and $C \cup B =$ the carrier of X.

(51) If D is nowhere dense, then there exist subsets C, B of X such that C is closed and C is boundary and B is everywhere dense and $C \cap B = D$ and $C \cup B =$ the carrier of X.

(52) If D is nowhere dense, then there exist subsets C, B of X such that C is closed and C is boundary and B is open and B is dense and $C \cap (D \cup B) = D$ and $C \cap B = \emptyset$ and $C \cup B =$ the carrier of X.

3. Properties of Subsets in Subspaces

In the sequel Y_0 will denote a subspace of X. One can prove the following propositions:

(53) For every subset A of X and for every subset B of Y_0 such that $B \subseteq A$ holds $\overline{B} \subseteq \overline{A}$.

(54) For all subsets C, A of X and for every subset B of Y_0 such that C is closed and $C \subseteq$ the carrier of Y_0 and $A \subseteq C$ and $A = B$ holds $\overline{A} = \overline{B}$.

(55) For every closed subspace Y_0 of X and for every subset A of X and for every subset B of Y_0 such that $A = B$ holds $\overline{A} = \overline{B}$.

(56) For every subset A of X and for every subset B of Y_0 such that $A \subseteq B$ holds $\operatorname{Int} A \subseteq \operatorname{Int} B$.

(57) For all subsets C, A of X and for every subset B of Y_0 such that C is open and $C \subseteq$ the carrier of Y_0 and $A \subseteq C$ and $A = B$ holds $\operatorname{Int} A = \operatorname{Int} B$.

(58) For every open subspace Y_0 of X and for every subset A of X and for every subset B of Y_0 such that $A = B$ holds $\operatorname{Int} A = \operatorname{Int} B$.

In the sequel X_0 denotes a subspace of X. The following propositions are true:

(59) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if A is dense, then B is dense.

(60) For all subsets C, A of X and for every subset B of X_0 such that $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds C is dense and B is dense if and only if A is dense.

(61) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if A is everywhere dense, then B is everywhere dense.

(62) For all subsets C, A of X and for every subset B of X_0 such that C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds C is dense and B is everywhere dense if and only if A is everywhere dense.

(63) For every open subspace X_0 of X and for all subsets A, C of X and for every subset B of X_0 such that $C =$ the carrier of X_0 and $A = B$ holds C is dense and B is everywhere dense if and only if A is everywhere dense.

(64) For all subsets C, A of X and for every subset B of X_0 such that $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds C is everywhere dense and B is everywhere dense if and only if A is everywhere dense.

(65) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if B is boundary, then A is boundary.

(66) For all subsets C, A of X and for every subset B of X_0 such that C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds if A is boundary, then B is boundary.

(67) For every open subspace X_0 of X and for every subset A of X and for every subset B of X_0 such that $A = B$ holds A is boundary if and only if B is boundary.

(68) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if B is nowhere dense, then A is nowhere dense.

(69) For all subsets C, A of X and for every subset B of X_0 such that C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds if A is nowhere dense, then B is nowhere dense.

(70) For every open subspace X_0 of X and for every subset A of X and for every subset B of X_0 such that $A = B$ holds A is nowhere dense if and only if B is nowhere dense.

4. Subsets in Topological Spaces with the same Topological Structures

In the sequel X_1, X_2 will be topological spaces. Next we state several propositions:

(71) If the carrier of X_1 = the carrier of X_2, then for every subset C_1 of X_1 and for every subset C_2 of X_2 holds $C_1 = C_2$ if and only if ${C_1}^{\rm c} = {C_2}^{\rm c}$.

(72) If the carrier of X_1 = the carrier of X_2 and for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds C_1 is open if and only if C_2 is open, then the topological structure of X_1 = the topological structure of X_2.

(73) If the carrier of X_1 = the carrier of X_2 and for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds C_1 is closed if and only if C_2 is closed, then the topological structure of X_1 = the topological structure of X_2.

(74) If the carrier of X_1 = the carrier of X_2 and for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds $\operatorname{Int} C_1 = \operatorname{Int} C_2$, then the topological structure of X_1 = the topological structure of X_2.

(75) If the carrier of X_1 = the carrier of X_2 and for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds $\overline{C_1} = \overline{C_2}$, then the topological structure of X_1 = the topological structure of X_2.

In the sequel D_1 is a subset of X_1 and D_2 is a subset of X_2. One can prove the following propositions:

(76) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is open, then D_2 is open.

(77) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\operatorname{Int} D_1 = \operatorname{Int} D_2$.

(78) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\operatorname{Int} D_1 \subseteq \operatorname{Int} D_2$.

(79) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is closed, then D_2 is closed.

(80) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\overline{D_1} = \overline{D_2}$.

(81) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\overline{D_1} \subseteq \overline{D_2}$.

(82) If $D_2 \subseteq D_1$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is boundary, then D_2 is boundary.

(83) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is dense, then D_2 is dense.

(84) If $D_2 \subseteq D_1$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is nowhere dense, then D_2 is nowhere dense.

(85) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then if D_1 is everywhere dense, then D_2 is everywhere dense.

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On Discrete and Almost Discrete Topological Spaces

Zbigniew Karno
Warsaw University
Białystok

Summary. A topological space X is called *almost discrete* if every open subset of X is closed; equivalently, if every closed subset of X is open (comp. [6],[5]). Almost discrete spaces were investigated in Mizar formalism in [2]. We present here a few properties of such spaces supplementary to those given in [2].

Most interesting is the following characterization : *A topological space X is almost discrete iff every nonempty subset of X is not nowhere dense.* Hence, *X is non almost discrete iff there is an everywhere dense subset of X different from the carrier of X.* We have an analogous characterization of discrete spaces : *A topological space X is discrete iff every nonempty subset of X is not boundary.* Hence, *X is non discrete iff there is a dense subset of X different from the carrier of X.* It is well known that the class of all almost discrete spaces contains both the class of discrete spaces and the class of anti-discrete spaces (see e.g., [2]). Observations presented here show that the class of all almost discrete non discrete spaces is not contained in the class of anti-discrete spaces and the class of all almost discrete non anti-discrete spaces is not contained in the class of discrete spaces. Moreover, the class of almost discrete non discrete non anti-discrete spaces is nonempty. To analyse these interdependencies we use various examples of topological spaces constructed here in Mizar formalism.

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The papers [12], [14], [9], [11], [7], [13], [8], [15], [10], [4], [1], [2], and [3] provide the notation and terminology for this paper.

1. Properties of Subsets of a Topological Space with Modified Topology

In the sequel X will be a topological space and D will be a subset of X. One can prove the following propositions:

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(1) For every subset B of X and for every subset C of the X modified w.r.t. D such that $B = C$ holds if B is open, then C is open.

(2) For every subset B of X and for every subset C of the X modified w.r.t. D such that $B = C$ holds if B is closed, then C is closed.

(3) For every subset C of the X modified w.r.t. $D^{\rm c}$ such that $C = D$ holds C is closed.

(4) For every subset C of the X modified w.r.t. D such that $C = D$ holds if D is dense, then C is dense and C is open.

(5) For every subset C of the X modified w.r.t. D such that $D \subseteq C$ holds if D is dense, then C is everywhere dense.

(6) For every subset C of the X modified w.r.t. $D^{\rm c}$ such that $C = D$ holds if D is boundary, then C is boundary and C is closed.

(7) For every subset C of the X modified w.r.t. $D^{\rm c}$ such that $C \subseteq D$ holds if D is boundary, then C is nowhere dense.

2. Trivial Topological Spaces

Let us observe that a 1-sorted structure is trivial if:

(Def.1) there exists an element d of the carrier of it such that the carrier of it $= \{d\}$.

One can verify the following observations:

* there exists a 1-sorted structure which is trivial and strict,
* there exists a 1-sorted structure which is non trivial and strict,
* there exists a topological structure which is trivial and strict, and
* there exists a non trivial strict topological structure.

One can prove the following proposition

(8) For every Y being a trivial topological structure such that the topology of Y is non-empty holds if Y is almost discrete, then Y is topological space-like.

One can verify the following observations:

* there exists a trivial strict topological space,
* every topological space which is trivial is also anti-discrete and discrete,
* every discrete anti-discrete topological space is trivial,
* there exists a topological space which is non trivial and strict,
* every non discrete topological space is non trivial, and
* every non anti-discrete topological space is non trivial.

3. Examples of Discrete and Anti-discrete Topological Spaces

We now define two new functors. Let D be a set. The functor 2_*^D yielding a non-empty family of subsets of D is defined by:

(Def.2) $2_*^D = \{\emptyset, D\}$.

Let D be a non-empty set. The functor $\mathrm{ADTS}(D)$ yields an anti-discrete strict topological space and is defined as follows:

(Def.3) $\mathrm{ADTS}(D) = \langle D, 2_*^D \rangle$.

We now state several propositions:

(9) For every anti-discrete topological space X holds the topological structure of $X = \mathrm{ADTS}(\text{the carrier of } X)$.

(10) For every topological space X such that the topological structure of $X =$ the topological structure of ADTS(the carrier of X) holds X is anti-discrete.

(11) For every anti-discrete topological space X and for every subset A of X holds if A is empty, then $\overline{A} = \emptyset$ and also if A is non-empty, then $\overline{A} =$ the carrier of X.

(12) For every anti-discrete topological space X and for every subset A of X holds if $A \neq$ the carrier of X, then $\mathrm{Int}\, A = \emptyset$ and also if $A =$ the carrier of X, then $\mathrm{Int}\, A =$ the carrier of X.

(13) For every topological space X if for every subset A of X such that A is non-empty holds $\overline{A} =$ the carrier of X, then X is anti-discrete.

(14) For every topological space X if for every subset A of X such that $A \neq$ the carrier of X holds $\mathrm{Int}\, A = \emptyset$, then X is anti-discrete.

(15) For every anti-discrete topological space X and for every subset A of X holds if $A \neq \emptyset$, then A is dense and also if $A \neq$ the carrier of X, then A is boundary.

(16) For every topological space X if for every subset A of X such that $A \neq \emptyset$ holds A is dense, then X is anti-discrete.

(17) For every topological space X if for every subset A of X such that $A \neq$ the carrier of X holds A is boundary, then X is anti-discrete.

Let D be a set. Then 2^D is a non-empty family of subsets of D. Let D be a non-empty set. The functor $\mathrm{DTS}(D)$ yielding a discrete strict topological space is defined by:

(Def.4) $\mathrm{DTS}(D) = \langle D, 2^D \rangle$.

One can prove the following propositions:

(18) For every discrete topological space X holds the topological structure of $X = \mathrm{DTS}(\text{the carrier of } X)$.

(19) For every topological space X such that the topological structure of $X =$ the topological structure of DTS(the carrier of X) holds X is discrete.

(20) For every discrete topological space X and for every subset A of X holds $\overline{A} = A$ and $\operatorname{Int} A = A$.

(21) For every topological space X if for every subset A of X holds $\overline{A} = A$, then X is discrete.

(22) For every topological space X if for every subset A of X holds $\operatorname{Int} A = A$, then X is discrete.

(23) For every non-empty set D holds $\mathrm{ADTS}(D) = \mathrm{DTS}(D)$ if and only if there exists an element d_0 of D such that $D = \{d_0\}$.

Let us note that there exists a discrete non anti-discrete strict topological space and there exists an anti-discrete non discrete strict topological space.

4. An Example of a Topological Space

Let D be a set, and let F be a family of subsets of D, and let S be a set. Then $F \setminus S$ is a family of subsets of D. Let D be a non-empty set, and let d_0 be an element of D. The functor $\mathrm{STS}(D, d_0)$ yields a strict topological space and is defined as follows:

(Def.5) $\mathrm{STS}(D, d_0) = \langle D, 2^D \setminus \{A : d_0 \in A \wedge A \neq D\}\rangle$, where A ranges over subsets of D.

In the sequel D denotes a non-empty set and d_0 denotes an element of D. One can prove the following propositions:

(24) For every subset A of $\mathrm{STS}(D, d_0)$ holds if $\{d_0\} \subseteq A$, then A is closed and also if A is non-empty and A is closed, then $\{d_0\} \subseteq A$.

(25) If $D \setminus \{d_0\}$ is non-empty, then for every subset A of $\mathrm{STS}(D, d_0)$ holds if $A = \{d_0\}$, then A is closed and A is boundary and also if A is non-empty and A is closed and A is boundary, then $A = \{d_0\}$.

(26) For every subset A of $\mathrm{STS}(D, d_0)$ holds if $A \subseteq D \setminus \{d_0\}$, then A is open and also if $A \neq D$ and A is open, then $A \subseteq D \setminus \{d_0\}$.

(27) If $D \setminus \{d_0\}$ is non-empty, then for every subset A of $\mathrm{STS}(D, d_0)$ holds if $A = D \setminus \{d_0\}$, then A is open and A is dense and also if $A \neq D$ and A is open and A is dense, then $A = D \setminus \{d_0\}$.

Let us observe that there exists a non anti-discrete non discrete strict topological space.

The following propositions are true:

(28) For every topological space Y holds the topological structure of $Y =$ the topological structure of $\mathrm{STS}(D, d_0)$ if and only if the carrier of $Y = D$ and for every subset A of Y holds if $\{d_0\} \subseteq A$, then A is closed and also if A is non-empty and A is closed, then $\{d_0\} \subseteq A$.

(29) For every topological space Y holds the topological structure of $Y =$ the topological structure of $\mathrm{STS}(D, d_0)$ if and only if the carrier of $Y = D$ and for every subset A of Y holds if $A \subseteq D \setminus \{d_0\}$, then A is open and also if $A \neq D$ and A is open, then $A \subseteq D \setminus \{d_0\}$.

(30) For every topological space Y holds the topological structure of $Y =$ the topological structure of $\mathrm{STS}(D, d_0)$ if and only if the carrier of $Y = D$ and for every non-empty subset A of Y holds $\overline{A} = A \cup \{d_0\}$.

(31) For every topological space Y holds the topological structure of $Y =$ the topological structure of $\mathrm{STS}(D, d_0)$ if and only if the carrier of $Y = D$ and for every subset A of Y such that $A \neq D$ holds $\mathrm{Int}\, A = A \setminus \{d_0\}$.

(32) $\mathrm{STS}(D, d_0) = \mathrm{ADTS}(D)$ if and only if $D = \{d_0\}$.

(33) $\mathrm{STS}(D, d_0) = \mathrm{DTS}(D)$ if and only if $D = \{d_0\}$.

(34) For every non-empty set D and for every element d_0 of D and for every subset A of $\mathrm{STS}(D, d_0)$ such that $A = \{d_0\}$ holds $\mathrm{DTS}(D) =$ the $\mathrm{STS}(D, d_0)$ modified w.r.t. A.

5. Discrete and Almost Discrete Spaces

Let us observe that a topological space is discrete if:

(Def.6) for every non-empty subset A of it holds A is not boundary.

We now state the proposition

(35) X is discrete if and only if for every subset A of X such that $A \neq$ the carrier of X holds A is not dense.

One can verify that every non almost discrete topological space is non discrete and non anti-discrete.

Let us observe that a topological space is almost discrete if:

(Def.7) for every non-empty subset A of it holds A is not nowhere dense.

Next we state three propositions:

(36) X is almost discrete if and only if for every subset A of X such that $A \neq$ the carrier of X holds A is everywhere dense.

(37) X is non almost discrete if and only if there exists a non-empty subset A of X such that A is boundary and A is closed.

(38) X is non almost discrete if and only if there exists a subset A of X such that $A \neq$ the carrier of X and A is dense and A is open.

One can verify that there exists an almost discrete non discrete non anti-discrete strict topological space.

Next we state the proposition

(39) For every non-empty set C and for every element c_0 of C holds $C \setminus \{c_0\}$ is non-empty if and only if $\mathrm{STS}(C, c_0)$ is non almost discrete.

Let us observe that there exists a non almost discrete strict topological space.

We now state two propositions:

(40) For every non-empty subset A of X such that A is boundary holds the X modified w.r.t. A^{c} is non almost discrete.

(41) For every subset A of X such that $A \neq$ the carrier of X and A is dense holds the X modified w.r.t. A is non almost discrete.

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