# Simple Continued Fractions and Their Convergents 

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#### Abstract

Summary. The article introduces simple continued fractions. They are defined as an infinite sequence of integers. The characterization of rational numbers in terms of simple continued fractions is shown. We also give definitions of convergents of continued fractions, and several important properties of simple continued fractions and their convergents.


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The articles [15], [6], [18], [4], [2], [13], [3], [7], [8], [16], [17], [1], [19], [20], [14], [10], [5], [9], [11], and [12] provide the notation and terminology for this paper.

## 1. Preliminaries

For simplicity, we adopt the following convention: $a, b, k, n, m$ are natural numbers, $i$ is an integer, $r$ is a real number, $p$ is a rational number, $c$ is a complex number, $x$ is a set, and $f$ is a function.

Let us consider $n$. One can check the following observations:

* $n \div 0$ is zero,
* $n \bmod 0$ is zero,
* $0 \div n$ is zero, and
* $0 \bmod n$ is zero.

Let us consider $c$. One can verify that $c-c$ is zero and $\frac{c}{0}$ is zero.
Let us note that $\lfloor 0\rfloor$ is zero.
The following propositions are true:
(1) If $0<r$ and $r<1$, then $1<\frac{1}{r}$.
(2) If $i \leq r$ and $r<i+1$, then $\lfloor r\rfloor=i$.
(3) $\left\lfloor\frac{m}{n}\right\rfloor=m \div n$.
(4) If $m \bmod n=0$, then $\frac{m}{n}=m \div n$.
(5) If $\frac{m}{n}=m \div n$, then $m \bmod n=0$.
(6) $\operatorname{frac}\left(\frac{m}{n}\right)=\frac{m \bmod n}{n}$.
(7) If $p \geq 0$, then there exist natural numbers $m, n$ such that $n \neq 0$ and $p=\frac{m}{n}$.
Let $R$ be a binary relation. We say that $R$ is integer-yielding if and only if:
(Def. 1) $\quad \operatorname{rng} R \subseteq \mathbb{Z}$.
One can verify that every binary relation which is natural-yielding is also integer-yielding.

One can check the following observations:

* there exists a function which is natural-yielding,
* every binary relation which is empty is also integer-yielding, and
* every binary relation which is integer-yielding is also real-yielding.

Let $D$ be a set. One can verify that every partial function from $D$ to $\mathbb{Z}$ is integer-yielding.

Let $f$ be an integer-yielding function and let $n$ be a set. One can verify that $f(n)$ is integer.

Let us note that there exists a sequence of real numbers which is integeryielding.

An integer sequence is an integer-yielding sequence of real numbers.
One can prove the following proposition
(8) $\quad f$ is an integer sequence iff $\operatorname{dom} f=\mathbb{N}$ and for every $x$ such that $x \in \mathbb{N}$ holds $f(x)$ is integer.
Let $f$ be a natural-yielding function and let $n$ be a set. Note that $f(n)$ is natural.

We now state three propositions:
(9) $\quad f$ is a function from $\mathbb{N}$ into $\mathbb{Z}$ iff $f$ is an integer sequence.
(10) $\quad f$ is a sequence of naturals iff $\operatorname{dom} f=\mathbb{N}$ and for every $x$ such that $x \in \mathbb{N}$ holds $f(x)$ is natural.
(11) $\quad f$ is a function from $\mathbb{N}$ into $\mathbb{N}$ iff $f$ is a sequence of naturals.

## 2. On the Euclidean Algorithm

Let $m, n$ be natural numbers. The functor $\operatorname{modSeq}(m, n)$ yielding a sequence of naturals is defined by:
(Def. 2) $\quad(\operatorname{modSeq}(m, n))(0)=m \bmod n$ and $(\operatorname{modSeq}(m, n))(1)=n \bmod (m \bmod$ $n$ ) and for every natural number $k$ holds $(\operatorname{modSeq}(m, n))(k+2)=$ $(\operatorname{modSeq}(m, n))(k) \bmod (\operatorname{modSeq}(m, n))(k+1)$.
Let $m, n$ be natural numbers. The functor $\operatorname{divSeq}(m, n)$ yielding a sequence of naturals is defined as follows:
(Def. 3) $\quad(\operatorname{divSeq}(m, n))(0)=m \div n$ and $(\operatorname{divSeq}(m, n))(1)=n \div(m \bmod$ $n$ ) and for every natural number $k$ holds $(\operatorname{divSeq}(m, n))(k+2)=$ $(\operatorname{modSeq}(m, n))(k) \div(\operatorname{modSeq}(m, n))(k+1)$.
We now state several propositions:
(12) $\quad(\operatorname{divSeq}(m, n))(1)=n \div(\operatorname{modSeq}(m, n))(0)$.
(13) $\quad(\operatorname{modSeq}(m, n))(1)=n \bmod (\operatorname{modSeq}(m, n))(0)$.
(14) If $a \leq b$ and $(\operatorname{modSeq}(m, n))(a)=0$, then $(\operatorname{modSeq}(m, n))(b)=0$.
(15) If $a<b$, then $(\operatorname{modSeq}(m, n))(a)>(\operatorname{modSeq}(m, n))(b)$ or $(\operatorname{modSeq}(m, n))(a)=0$.
(16) If $(\operatorname{divSeq}(m, n))(a+1)=0$, then $(\operatorname{modSeq}(m, n))(a)=0$.
(17) If $a \neq 0$ and $a \leq b$ and $(\operatorname{divSeq}(m, n))(a)=0$, then $(\operatorname{divSeq}(m, n))(b)=$ 0.
(18) If $a<b$ and $(\operatorname{modSeq}(m, n))(a)=0$, then $(\operatorname{divSeq}(m, n))(b)=0$.
(19) If $n \neq 0$, then $m=(\operatorname{divSeq}(m, n))(0) \cdot n+(\operatorname{modSeq}(m, n))(0)$.
(20) If $n \neq 0$, then $\frac{m}{n}=(\operatorname{divSeq}(m, n))(0)+\frac{1}{(\operatorname{modSeq}(m, n))(0)}$.

One can prove the following propositions:
(21) $\operatorname{divSeq}(m, 0)=\mathbb{N} \longmapsto 0$.
(22) $\operatorname{modSeq}(m, 0)=\mathbb{N} \longmapsto 0$.
(23) $\operatorname{divSeq}(0, n)=\mathbb{N} \longmapsto 0$.
(24) $\operatorname{modSeq}(0, n)=\mathbb{N} \longmapsto 0$.
(25) There exists a natural number $k$ such that $(\operatorname{divSeq}(m, n))(k)=0$ and $(\operatorname{modSeq}(m, n))(k)=0$.

## 3. Simple Continued Fractions

Let $r$ be a real number. The remainders for s.c.f. of $r$ yields a sequence of real numbers and is defined by the conditions (Def. 4).
(Def. 4)(i) (The remainders for s.c.f. of $r)(0)=r$, and
(ii) for every natural number $n$ holds (the remainders for s.c.f. of $r)(n+1)=$ $\frac{1}{\text { frac (the remainders for s.c.f. of } r)(n)}$.

Let $r$ be a real number. We introduce $\operatorname{rfs} r$ as a synonym of the remainders for s.c.f. of $r$.

Let $r$ be a real number. The simple continued fraction of $r$ yielding an integer sequence is defined by:
(Def. 5) For every natural number $n$ holds (the simple continued fraction of $r)(n)=\lfloor(\operatorname{rfs} r)(n)\rfloor$.
Let $r$ be a real number. We introduce scf $r$ as a synonym of the simple continued fraction of $r$.

The following propositions are true:
(26) $\quad(\operatorname{rfs} r)(n+1)=\frac{1}{(\mathrm{rfs} r)(n)-(\operatorname{scf} r)(n)}$.
(27) If $(\operatorname{rfs} r)(n)=0$ and $n \leq m$, then $(\operatorname{rfs} r)(m)=0$.
(28) If $(\operatorname{rfs} r)(n)=0$ and $n \leq m$, then $(\operatorname{scf} r)(m)=0$.
(29) $\quad(\mathrm{rfs} i)(n+1)=0$.
(30) $\quad(\operatorname{scf} i)(0)=i$ and $(\operatorname{scf} i)(n+1)=0$.
(31) If $i>1$, then $\left(\operatorname{rfs}\left(\frac{1}{i}\right)\right)(1)=i$ and $\left(\operatorname{rfs}\left(\frac{1}{i}\right)\right)(n+2)=0$.
(32) If $i>1$, then $\left(\operatorname{scf}\left(\frac{1}{i}\right)\right)(0)=0$ and $\left(\operatorname{scf}\left(\frac{1}{i}\right)\right)(1)=i$ and $\left(\operatorname{scf}\left(\frac{1}{i}\right)\right)(n+2)=0$.
(33) If for every $n$ holds $(\operatorname{scf} r)(n)=0$, then $(\operatorname{rfs} r)(n)=0$.
(34) If for every $n$ holds $(\operatorname{scf} r)(n)=0$, then $r=0$.
(35) $\quad$ frac $r=r-(\operatorname{scf} r)(0)$.
(36) $\quad(\operatorname{rfs} r)(n+1)=\left(\operatorname{rfs}\left(\frac{1}{\operatorname{frac} r}\right)\right)(n)$.
(37) $\quad(\operatorname{scf} r)(n+1)=\left(\operatorname{scf}\left(\frac{1}{\operatorname{frac} r}\right)\right)(n)$.
(38) If $n \geq 1$, then $(\operatorname{scf} r)(n) \geq 0$.
(39) If $n \geq 1$, then $(\operatorname{scf} r)(n) \in \mathbb{N}$.
(40) If $n \geq 1$ and $(\operatorname{scf} r)(n) \neq 0$, then $(\operatorname{scf} r)(n) \geq 1$.
(41) $\quad\left(\operatorname{scf}\left(\frac{m}{n}\right)\right)(k)=(\operatorname{divSeq}(m, n))(k)$ and $\left(\operatorname{rfs}\left(\frac{m}{n}\right)\right)(1)=\frac{n}{(\operatorname{modSeq}(m, n))(0)}$ and $\left(\operatorname{rfs}\left(\frac{m}{n}\right)\right)(k+2)=\frac{(\operatorname{modSeq}(m, n))(k)}{(\operatorname{modSeq}(m, n))(k+1)}$.
(42) $\quad r$ is rational iff there exists $n$ such that for every $m$ such that $m \geq n$ holds $(\operatorname{scf} r)(m)=0$.
(43) If for every $n$ holds $(\operatorname{scf} r)(n) \neq 0$, then $r$ is irrational.

## 4. Convergents of Simple Continued Fractions

In the sequel $n_{1}, n_{2}$ are natural numbers.
Let $r$ be a real number. The convergent numerators of $r$ yielding a sequence of real numbers is defined by the conditions (Def. 6).
(Def. 6)(i) (The convergent numerators of $r)(0)=(\operatorname{scf} r)(0)$,
(ii) (the convergent numerators of $r)(1)=(\operatorname{scf} r)(1) \cdot(\operatorname{scf} r)(0)+1$, and
(iii) for every natural number $n$ holds (the convergent numerators of $r)(n+$ $2)=(\operatorname{scf} r)(n+2) \cdot($ the convergent numerators of $r)(n+1)+($ the convergent numerators of $r)(n)$.
Let $r$ be a real number. The convergent denominators of $r$ yields a sequence of real numbers and is defined by the conditions (Def. 7).
(Def. 7)(i) (The convergent denominators of $r)(0)=1$,
(ii) (the convergent denominators of $r)(1)=(\operatorname{scf} r)(1)$, and
(iii) for every natural number $n$ holds (the convergent denominators of $r)(n+2)=(\operatorname{scf} r)(n+2) \cdot($ the convergent denominators of $r)(n+1)+($ the convergent denominators of $r)(n)$.
Let $r$ be a real number. We introduce $c n r$ as a synonym of the convergent numerators of $r$. We introduce $c d r$ as a synonym of the convergent denominators of $r$.

One can prove the following propositions:
(44) If $(\operatorname{scf} r)(0)>0$, then for every $n$ holds $(c n r)(n) \in \mathbb{N}$.
(45) If $(\operatorname{scf} r)(0)>0$, then for every $n$ holds $(c n r)(n)>0$.
(46) If $(\operatorname{scf} r)(0)>0$, then for every $n$ holds $(c n r)(n+2)>(\operatorname{scf} r)(n+2)$. $(c n r)(n+1)$.
(47) If $(\operatorname{scf} r)(0)>0$, then for every $n$ such that $n_{1}=(c n r)(n+1)$ and $n_{2}=(c n r)(n)$ holds $\operatorname{gcd}\left(n_{1}, n_{2}\right)=1$.
(48) If $(\operatorname{scf} r)(0)>0$ and for every $n$ holds $(\operatorname{scf} r)(n) \neq 0$, then for every $n$ holds $(c n r)(n) \geq \tau^{n}$.
(49) If $(\operatorname{scf} r)(0)>0$ and for every $n$ holds $(\operatorname{scf} r)(n) \leq b$, then for every $n$ holds $(c n r)(n) \leq\left(\frac{b+\sqrt{b^{2}+4}}{2}\right)^{n+1}$.
(50) $\quad(c d r)(n) \in \mathbb{N}$.
(51) $\quad(c d r)(n) \geq 0$.
(52) If $(\operatorname{scf} r)(1)>0$, then for every $n$ holds $(c d r)(n)>0$.
(53) $(c d r)(n+2) \geq(\operatorname{scf} r)(n+2) \cdot(c d r)(n+1)$.
(54) If $(\operatorname{scf} r)(1)>0$, then for every $n$ holds $(c d r)(n+2)>(\operatorname{scf} r)(n+2)$. $(c d r)(n+1)$.
(55) If for every $n$ holds $(\operatorname{scf} r)(n)>0$, then for every $n$ such that $n \geq 1$ holds $\frac{1}{(c d r)(n) \cdot(c d r)(n+1)}<\frac{1}{(\operatorname{scf} r)(n+1) \cdot(c d r)(n)^{2}}$.
(56) If for every $n$ holds $(\operatorname{scf} r)(n) \leq b$, then for every $n$ holds $(c d r)(n+1) \leq$ $\left(\frac{b+\sqrt{b^{2}+4}}{2}\right)^{n+1}$.
(57) If $n_{1}=(c d r)(n+1)$ and $n_{2}=(c d r)(n)$, then $\operatorname{gcd}\left(n_{1}, n_{2}\right)=1$.
(58) If for every $n$ holds $(\operatorname{scf} r)(n)>0$, then for every $n$ holds $\frac{(c d r)(n+1)}{(c d r)(n)} \geq$ $\frac{1}{(\operatorname{scf} r)(n+2)}$.
(59) If for every $n$ holds $(\operatorname{scf} r)(n)>0$, then for every $n$ holds $(c d r)(n+2) \leq$ $2 \cdot(\operatorname{scf} r)(n+2) \cdot(c d r)(n+1)$.
(60) If for every $n$ holds $(\operatorname{scf} r)(n) \neq 0$, then for every $n$ holds $\frac{1}{(\operatorname{scf} r)(n+1) \cdot(c d r)(n)^{2}} \leq \frac{1}{(c d r)(n)^{2}}$.
(61) If for every $n$ holds (scf $r)(n) \neq 0$, then for every $n$ holds $(c d r)(n+1) \geq$ $\tau^{n}$.
(62) If $a>0$ and for every $n$ holds ( $\operatorname{scf} r)(n) \geq a$, then for every $n$ holds $(c d r)(n+1) \geq\left(\frac{a+\sqrt{a^{2}+4}}{2}\right)^{n}$.
(63) $\frac{(c n r)(n+2)}{(c d r)(n+2)}=\frac{(\operatorname{scf} r)(n+2) \cdot(c n r)(n+1)+(c n r)(n)}{(\operatorname{scf} r)(n+2) \cdot(c d r)(n+1)+(c d r)(n)}$.
(64) $(c n r)(n+1) \cdot(c d r)(n)-(c n r)(n) \cdot(c d r)(n+1)=(-1)^{n}$.
(65) If for every $n$ holds $(c d r)(n) \neq 0$, then $\frac{(c n r)(n+1)}{(c d r)(n+1)}-\frac{(c n r)(n)}{(c d r)(n)}=$ $\frac{(-1)^{n}}{(c d r)(n+1) \cdot(c d r)(n)}$.
(66) $(c n r)(n+2) \cdot(c d r)(n)-(c n r)(n) \cdot(c d r)(n+2)=(-1)^{n} \cdot(\operatorname{scf} r)(n+2)$.
(67) If for every $n$ holds $(c d r)(n) \neq 0$, then $\frac{(c n r)(n+2)}{(c d r)(n+2)}-\frac{(c n r)(n)}{(c d r)(n)}=$ $\frac{(-1)^{n} \cdot(\operatorname{scf} r)(n+2)}{(c d r)(n+2) \cdot(c d r)(n)}$.
(68) If for every $n$ holds (scf $r)(n) \neq 0$, then for every $n$ such that $n \geq 1$ holds $\frac{(c n r)(n)}{(c d r)(n)}=\frac{(c n r)(n+1)-(c n r)(n-1)}{(c d r)(n+1)-(c d r)(n-1)}$.
(69) If for every $n$ holds $(c d r)(n) \neq 0$, then for every $n$ holds $\left\lvert\, \frac{(c n r)(n+1)}{(c d r)(n+1)}-\right.$ $\frac{(c n r)(n)}{(c d r)(n) \mid} \left\lvert\,=\frac{1}{|(c d r)(n+1) \cdot(c d r)(n)|}\right.$.
(70) If $(\operatorname{scf} r)(1)>0$, then for every $n$ holds $\frac{(c n r)(2 \cdot n+1)}{(c d r)(2 \cdot n+1)}>\frac{(c n r)(2 \cdot n)}{(c d r)(2 \cdot n)}$.

Let $r$ be a real number. The convergents of continued fractions of $r$ yielding a sequence of real numbers is defined as follows:
(Def. 8) The convergents of continued fractions of $r=c n r / c d r$.
Let $r$ be a real number. We introduce $\operatorname{cocf} r$ as a synonym of the convergents of continued fractions of $r$.

One can prove the following propositions:
(71) $(\operatorname{cocf} r)(0)=(\operatorname{scf} r)(0)$.
(72) If $(\operatorname{scf} r)(1) \neq 0$, then $(\operatorname{cocf} r)(1)=(\operatorname{scf} r)(0)+\frac{1}{(\operatorname{scf} r)(1)}$.
(73) If for every $n$ holds $(\operatorname{scf} r)(n)>0$, then $(\operatorname{cocf} r)(2)=(\operatorname{scf} r)(0)+$ $\frac{1}{(\operatorname{scf} r)(1)+\frac{1}{(\operatorname{scf} r)(2)}}$.
(74) If for every $n$ holds $(\operatorname{scf} r)(n)>0$, then $(\operatorname{cocf} r)(3)=(\operatorname{scf} r)(0)+$ $\frac{1}{(\operatorname{scf} r)(1)+\frac{1}{(\operatorname{scf} r)(2)+\frac{1}{(\operatorname{scf} r)(3)}}}$.
(75) If for every $n$ holds (scf $r)(n)>0$, then for every $n$ such that $n \geq 1$ holds $\frac{(c n r)(2 \cdot n+1)}{(c d r)(2 \cdot n+1)}<\frac{(c n r)(2 \cdot n-1)}{(c d r)(2 \cdot n-1)}$.
(76) If for every $n$ holds $(\operatorname{scf} r)(n)>0$, then for every $n$ such that $n \geq 1$ holds $\frac{(c n r)(2 \cdot n)}{(c d r)(2 \cdot n)}>\frac{(c n r)(2 \cdot n-2)}{(c d r)(2 \cdot n-2)}$.
(77) If for every $n$ holds $(\operatorname{scf} r)(n)>0$, then for every $n$ such that $n \geq 1$ holds $\frac{(c n r)(2 \cdot n)}{(c d r)(2 \cdot n)}<\frac{(c n r)(2 \cdot n-1)}{(c d r)(2 \cdot n-1)}$.
Let $r$ be a real number. The back continued fraction of $r$ yields a sequence of real numbers and is defined by the conditions (Def. 9).
(Def. 9)(i) (The back continued fraction of $r)(0)=(\operatorname{scf} r)(0)$, and
(ii) for every natural number $n$ holds (the back continued fraction of $r)(n+$ $1)=\frac{1}{(\text { the back continued fraction of } r)(n)}+(\operatorname{scf} r)(n+1)$.
Let $r$ be a real number. We introduce bcf $r$ as a synonym of the back continued fraction of $r$.

One can prove the following proposition
(78) If $(\operatorname{scf} r)(0)>0$, then for every $n$ holds $(\operatorname{bcf} r)(n+1)=\frac{(c n r)(n+1)}{(c n r)(n)}$.

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# Chordal Graphs ${ }^{1}$ 

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#### Abstract

Summary. We are formalizing [9, pp. 81-84] where chordal graphs are defined and their basic characterization is given. This formalization is a part of the M.Sc. work of the first author under supervision of the second author.


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The terminology and notation used here are introduced in the following articles: [18], [21], [3], [16], [22], [5], [6], [4], [1], [8], [19], [2], [12], [11], [10], [7], [14], [17], [20], [15], and [13].

## 1. Preliminaries

One can prove the following propositions:
(1) For every non zero natural number $n$ holds $n-1$ is a natural number and $1 \leq n$.
(2) For every odd natural number $n$ holds $n-1$ is a natural number and $1 \leq n$.
(3) For all odd integers $n$, $m$ such that $n<m$ holds $n \leq m-2$.
(4) For all odd integers $n, m$ such that $m<n$ holds $m+2 \leq n$.
(5) For every odd natural number $n$ such that $1 \neq n$ there exists an odd natural number $m$ such that $m+2=n$.
(6) For every odd natural number $n$ such that $n \leq 2$ holds $n=1$.
(7) For every odd natural number $n$ such that $n \leq 4$ holds $n=1$ or $n=3$.
(8) For every odd natural number $n$ such that $n \leq 6$ holds $n=1$ or $n=3$ or $n=5$.

[^0](9) For every odd natural number $n$ such that $n \leq 8$ holds $n=1$ or $n=3$ or $n=5$ or $n=7$.
(10) For every even natural number $n$ such that $n \leq 1$ holds $n=0$.
(11) For every even natural number $n$ such that $n \leq 3$ holds $n=0$ or $n=2$.
(12) For every even natural number $n$ such that $n \leq 5$ holds $n=0$ or $n=2$ or $n=4$.
(13) For every even natural number $n$ such that $n \leq 7$ holds $n=0$ or $n=2$ or $n=4$ or $n=6$.
(14) For every finite sequence $p$ and for every non zero natural number $n$ such that $p$ is one-to-one and $n \leq \operatorname{len} p$ holds $p(n) \leftarrow p=n$.
(15) Let $p$ be a non empty finite sequence and $T$ be a non empty subset of $\operatorname{rng} p$. Then there exists a set $x$ such that $x \in T$ and for every set $y$ such that $y \in T$ holds $x \leftarrow p \leq y \leftarrow p$.
Let $p$ be a finite sequence and let $n$ be a natural number. The functor $p$.followSet $(n)$ yields a finite set and is defined as follows:
(Def. 1) $p$.followSet $(n)=\operatorname{rng}\langle p(n), \ldots, p(\operatorname{len} p)\rangle$.
The following three propositions are true:
(16) Let $p$ be a finite sequence, $x$ be a set, and $n$ be a natural number. Suppose $x \in \operatorname{rng} p$ and $n \in \operatorname{dom} p$ and $p$ is one-to-one. Then $x \in p$.followSet $(n)$ if and only if $x \leftrightarrow p \geq n$.
(17) Let $p, q$ be finite sequences and $x$ be a set. If $p=\langle x\rangle^{\wedge} q$, then for every non zero natural number $n$ holds $p$.followSet $(n+1)=q$.followSet $(n)$.
(18) Let $X$ be a set, $f$ be a finite sequence of elements of $X$, and $g$ be a FinSubsequence of $f$. If len $\operatorname{Seq} g=\operatorname{len} f$, then $\operatorname{Seq} g=f$.

## 2. Miscellany on Graphs

Next we state a number of propositions:
(19) Let $G$ be a graph, $S$ be a subset of the vertices of $G, H$ be a subgraph of $G$ induced by $S$, and $u, v$ be sets. Suppose $u \in S$ and $v \in S$. Let $e$ be a set. If $e$ joins $u$ and $v$ in $G$, then $e$ joins $u$ and $v$ in $H$.
(20) For every graph $G$ and for every walk $W$ of $G$ holds $W$ is trail-like iff len $W=2 \cdot \operatorname{card}(W \cdot \operatorname{edges}())+1$.
(21) Let $G$ be a graph, $S$ be a subset of the vertices of $G, H$ be a subgraph of $G$ with vertices $S$ removed, and $W$ be a walk of $G$. Suppose that for every odd natural number $n$ such that $n \leq$ len $W$ holds $W(n) \notin S$. Then $W$ is a walk of $H$.
(22) Let $G$ be a graph and $a, b$ be sets. Suppose $a \neq b$. Let $W$ be a walk of $G$. If $W$.vertices ()$=\{a, b\}$, then there exists a set $e$ such that $e$ joins $a$ and $b$ in $G$.
(23) Let $G$ be a graph, $S$ be a non empty subset of the vertices of $G, H$ be a subgraph of $G$ induced by $S$, and $W$ be a walk of $G$. If $W$.vertices () $\subseteq S$, then $W$ is a walk of $H$.
(24) Let $G_{1}, G_{2}$ be graphs. Suppose $G_{1}={ }_{G} G_{2}$. Let $W_{1}$ be a walk of $G_{1}$ and $W_{2}$ be a walk of $G_{2}$. If $W_{1}=W_{2}$, then if $W_{1}$ is cycle-like, then $W_{2}$ is cycle-like.
(25) Let $G$ be a graph, $P$ be a path of $G$, and $m, n$ be odd natural numbers. Suppose $m \leq \operatorname{len} P$ and $n \leq \operatorname{len} P$ and $P(m)=P(n)$. Then $m=n$ or $m=1$ and $n=\operatorname{len} P$ or $m=\operatorname{len} P$ and $n=1$.
(26) Let $G$ be a graph and $P$ be a path of $G$. Suppose $P$ is open. Let $a, e, b$ be sets. Suppose $a \notin P$.vertices() and $b=P$.first() and $e$ joins $a$ and $b$ in $G$. Then $(G$.walkOf $(a, e, b))$.append $(P)$ is path-like.
(27) Let $G$ be a graph and $P, H$ be paths of $G$. Suppose $P$.edges() misses $H$.edges () and $P$ is non trivial and open and $H$ is non trivial and open and $P$.vertices ()$\cap H$.vertices ()$=\{P$.first ()$, P$.last ()$\}$ and $H$.first ()$=P$.last () and $H$.last ()$=P$.first () . Then $P$.append $(H)$ is cycle-like.
(28) For every graph $G$ and for all walks $W_{1}, W_{2}$ of $G$ such that $W_{1} \cdot \operatorname{last}()=$ $W_{2} \cdot$ first () holds $\left(W_{1} \cdot \operatorname{append}\left(W_{2}\right)\right) \cdot \operatorname{length}()=W_{1} \cdot$ length ()$+W_{2} \cdot$ length () .
(29) Let $G$ be a graph and $A, B$ be non empty subsets of the vertices of $G$. Suppose $B \subseteq A$. Let $H_{1}$ be a subgraph of $G$ induced by $A$. Then every subgraph of $H_{1}$ induced by $B$ is a subgraph of $G$ induced by $B$.
(30) Let $G$ be a graph and $A, B$ be non empty subsets of the vertices of $G$. Suppose $B \subseteq A$. Let $H_{1}$ be a subgraph of $G$ induced by $A$. Then every subgraph of $G$ induced by $B$ is a subgraph of $H_{1}$ induced by $B$.
(31) Let $G$ be a graph and $S, T$ be non empty subsets of the vertices of $G$. If $T \subseteq S$, then for every subgraph $G_{2}$ of $G$ induced by $S$ holds $G_{2}$.edgesBetween $(T)=G$.edgesBetween $(T)$.
The scheme FinGraphOrderCompInd concerns a unary predicate $\mathcal{P}$, and states that:

For every finite graph $G$ holds $\mathcal{P}[G]$
provided the parameters meet the following condition:

- Let $k$ be a non zero natural number. Suppose that for every finite graph $G_{3}$ such that $G_{3}$.order ()$<k$ holds $\mathcal{P}\left[G_{3}\right]$. Let $G_{4}$ be a finite graph. If $G_{4}$.order ()$=k$, then $\mathcal{P}\left[G_{4}\right]$.
We now state two propositions:
(32) For every graph $G$ and for every walk $W$ of $G$ such that $W$ is open and path-like holds $W$ is vertex-distinct.
(33) Let $G$ be a graph and $P$ be a path of $G$. Suppose $P$ is open and len $P>3$. Let $e$ be a set. If $e$ joins $P$.last() and $P$.first() in $G$, then $P$.addEdge ( $e$ ) is cycle-like.


## 3. Shortest Topological Path

Let $G$ be a graph and let $W$ be a walk of $G$. We say that $W$ is minimum length if and only if:
(Def. 2) For every walk $W_{2}$ of $G$ such that $W_{2}$ is walk from $W$.first() to $W$.last() holds len $W_{2} \geq$ len $W$.
The following propositions are true:
(34) For every graph $G$ and for every walk $W$ of $G$ and for every subwalk $S$ of $W$ such that $S . \operatorname{first}()=W . \operatorname{first}()$ and $S . \operatorname{edgeSeq}()=W . \operatorname{edgeSeq}()$ holds $S=W$.
(35) For every graph $G$ and for every walk $W$ of $G$ and for every subwalk $S$ of $W$ such that len $S=$ len $W$ holds $S=W$.
(36) For every graph $G$ and for every walk $W$ of $G$ such that $W$ is minimum length holds $W$ is path-like.
(37) For every graph $G$ and for every walk $W$ of $G$ such that $W$ is minimum length holds $W$ is path-like.
(38) Let $G$ be a graph and $W$ be a walk of $G$. Suppose that for every path $P$ of $G$ such that $P$ is walk from $W$.first() to $W$.last() holds len $P \geq \operatorname{len} W$. Then $W$ is minimum length.
(39) For every graph $G$ and for every walk $W$ of $G$ holds there exists a path of $G$ which is walk from $W$.first() to $W$.last() and minimum length.
(40) Let $G$ be a graph and $W$ be a walk of $G$. Suppose $W$ is minimum length. Let $m, n$ be odd natural numbers. Suppose $m+2<n$ and $n \leq$ len $W$. Then it is not true that there exists a set $e$ such that $e$ joins $W(m)$ and $W(n)$ in $G$.
(41) Let $G$ be a graph, $S$ be a non empty subset of the vertices of $G, H$ be a subgraph of $G$ induced by $S$, and $W$ be a walk of $H$. Suppose $W$ is minimum length. Let $m, n$ be odd natural numbers. Suppose $m+2<n$ and $n \leq$ len $W$. Then it is not true that there exists a set $e$ such that $e$ joins $W(m)$ and $W(n)$ in $G$.
(42) Let $G$ be a graph and $W$ be a walk of $G$. Suppose $W$ is minimum length. Let $m, n$ be odd natural numbers. If $m \leq n$ and $n \leq$ len $W$, then $W \operatorname{ccut}(m, n)$ is minimum length.
(43) Let $G$ be a graph. Suppose $G$ is connected. Let $A, B$ be non empty subsets of the vertices of $G$. Suppose $A$ misses $B$. Then there exists a path $P$ of $G$ such that
(i) $\quad P$ is minimum length and non trivial,
(ii) $\quad P$.first ()$\in A$,
(iii) $P$.last ()$\in B$, and
(iv) for every odd natural number $n$ such that $1<n$ and $n<\operatorname{len} P$ holds $P(n) \notin A$ and $P(n) \notin B$.

## 4. Adjacency and Complete Graphs

Let $G$ be a graph and let $a, b$ be vertices of $G$. We say that $a$ and $b$ are adjacent if and only if:
(Def. 3) There exists a set $e$ such that $e$ joins $a$ and $b$ in $G$.
Let us note that the predicate $a$ and $b$ are adjacent is symmetric.
Next we state several propositions:
(44) Let $G_{1}, G_{2}$ be graphs. Suppose $G_{1}={ }_{G} G_{2}$. Let $u_{1}, v_{1}$ be vertices of $G_{1}$. Suppose $u_{1}$ and $v_{1}$ are adjacent. Let $u_{2}, v_{2}$ be vertices of $G_{2}$. If $u_{1}=u_{2}$ and $v_{1}=v_{2}$, then $u_{2}$ and $v_{2}$ are adjacent.
(45) Let $G$ be a graph, $S$ be a non empty subset of the vertices of $G, H$ be a subgraph of $G$ induced by $S, u, v$ be vertices of $G$, and $t, w$ be vertices of $H$. Suppose $u=t$ and $v=w$. Then $u$ and $v$ are adjacent if and only if $t$ and $w$ are adjacent.
(46) For every graph $G$ and for every walk $W$ of $G$ such that $W$.first ()$\neq$ $W . \operatorname{last}()$ and $W . \operatorname{first}()$ and $W \cdot \operatorname{last}()$ are not adjacent holds $W \cdot \operatorname{length}() \geq$ 2.
(47) Let $G$ be a graph and $v_{1}, v_{2}, v_{3}$ be vertices of $G$. Suppose $v_{1} \neq v_{2}$ and $v_{1} \neq v_{3}$ and $v_{2} \neq v_{3}$ and $v_{1}$ and $v_{2}$ are adjacent and $v_{2}$ and $v_{3}$ are adjacent. Then there exists a path $P$ of $G$ and there exist sets $e_{1}, e_{2}$ such that $P$ is open and len $P=5$ and $P$.length ()$=2$ and $e_{1}$ joins $v_{1}$ and $v_{2}$ in $G$ and $e_{2}$ joins $v_{2}$ and $v_{3}$ in $G$ and $P$.edges ()$=\left\{e_{1}, e_{2}\right\}$ and $P$.vertices ()$=$ $\left\{v_{1}, v_{2}, v_{3}\right\}$ and $P(1)=v_{1}$ and $P(3)=v_{2}$ and $P(5)=v_{3}$.
(48) Let $G$ be a graph and $v_{1}, v_{2}, v_{3}, v_{4}$ be vertices of $G$. Suppose that $v_{1} \neq v_{2}$ and $v_{1} \neq v_{3}$ and $v_{2} \neq v_{3}$ and $v_{2} \neq v_{4}$ and $v_{3} \neq v_{4}$ and $v_{1}$ and $v_{2}$ are adjacent and $v_{2}$ and $v_{3}$ are adjacent and $v_{3}$ and $v_{4}$ are adjacent. Then there exists a path $P$ of $G$ such that len $P=7$ and $P . l e n g t h()=3$ and $P$.vertices ()$=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and $P(1)=v_{1}$ and $P(3)=v_{2}$ and $P(5)=v_{3}$ and $P(7)=v_{4}$.
Let $G$ be a graph and let $S$ be a set. The functor $G$.adjacentSet $(S)$ yields a subset of the vertices of $G$ and is defined as follows:
(Def. 4) G.adjacentSet $(S)=\{u ; u$ ranges over vertices of $G$ : $u \notin S \wedge$ $\bigvee_{v: \text { vertex of } G}(v \in S \wedge u$ and $v$ are adjacent $\left.)\right\}$.
One can prove the following propositions:
(49) For every graph $G$ and for all sets $S, x$ such that $x \in G \cdot \operatorname{adjacentSet}(S)$ holds $x \notin S$.
(50) Let $G$ be a graph, $S$ be a set, and $u$ be a vertex of $G$. Then $u \in$ $G$.adjacentSet $(S)$ if and only if the following conditions are satisfied:
(i) $u \notin S$, and
(ii) there exists a vertex $v$ of $G$ such that $v \in S$ and $u$ and $v$ are adjacent.
(51) For all graphs $G_{1}, G_{2}$ such that $G_{1}={ }_{G} G_{2}$ and for every set $S$ holds $G_{1} \cdot \operatorname{adjacentSet}(S)=G_{2} \cdot \operatorname{adjacentSet}(S)$.
(52) For every graph $G$ and for all vertices $u, v$ of $G$ holds $u \in$ $G$.adjacentSet $(\{v\})$ iff $u \neq v$ and $v$ and $u$ are adjacent.
(53) For every graph $G$ and for all sets $x, y$ holds $x \in G$.adjacentSet( $\{y\}$ ) iff $y \in G \cdot \operatorname{adjacentSet}(\{x\})$.
(54) Let $G$ be a graph and $C$ be a path of $G$. Suppose $C$ is cycle-like and $C$.length ()$>3$. Let $x$ be a vertex of $G$. Suppose $x \in C$.vertices(). Then there exist odd natural numbers $m, n$ such that $m+2<n$ and $n \leq \operatorname{len} C$ and $m=1$ and $n=\operatorname{len} C$ and $m=1$ and $n=\operatorname{len} C-2$ and $m=3$ and $n=\operatorname{len} C$ and $C(m) \neq C(n)$ and $C(m) \in G$.adjacentSet $(\{x\})$ and $C(n) \in G$.adjacentSet $(\{x\})$.
(55) Let $G$ be a graph and $C$ be a path of $G$. Suppose $C$ is cycle-like and $C$.length ()$>3$. Let $x$ be a vertex of $G$. Suppose $x \in C$.vertices(). Then there exist odd natural numbers $m, n$ such that
(i) $m+2<n$,
(ii) $n \leq \operatorname{len} C$,
(iii) $\quad C(m) \neq C(n)$,
(iv) $C(m) \in G$.adjacentSet $(\{x\})$,
(v) $C(n) \in G \cdot a d j a c e n t S e t(\{x\})$, and
(vi) for every set $e$ such that $e \in C$.edges() holds $e$ does not join $C(m)$ and $C(n)$ in $G$.
(56) For every loopless graph $G$ and for every vertex $u$ of $G$ holds $G$.adjacentSet $(\{u\})=\emptyset$ iff $u$ is isolated.
(57) Let $G$ be a graph, $G_{0}$ be a subgraph of $G, S$ be a non empty subset of the vertices of $G, x$ be a vertex of $G, G_{1}$ be a subgraph of $G$ induced by $S$, and $G_{2}$ be a subgraph of $G$ induced by $S \cup\{x\}$. If $G_{1}$ is connected and $x \in G$.adjacentSet(the vertices of $G_{1}$ ), then $G_{2}$ is connected.
(58) Let $G$ be a graph, $S$ be a non empty subset of the vertices of $G, H$ be a subgraph of $G$ induced by $S$, and $u$ be a vertex of $G$. Suppose $u \in S$ and $G$.adjacentSet $(\{u\}) \subseteq S$. Let $v$ be a vertex of $H$. If $u=v$, then $G \cdot \operatorname{adjacentSet}(\{u\})=H \cdot \operatorname{adjacentSet}(\{v\})$.
Let $G$ be a graph and let $S$ be a set. A subgraph of $G$ is called an adjacency graph of $S$ in $G$ if:
(Def. 5) It is a subgraph of $G$ induced by $G$.adjacentSet $(S)$ if $S$ is a subset of the vertices of $G$.

Next we state two propositions:
(59) Let $G_{1}, G_{2}$ be graphs. Suppose $G_{1}={ }_{G} G_{2}$. Let $u_{1}$ be a vertex of $G_{1}$ and $u_{2}$ be a vertex of $G_{2}$. Suppose $u_{1}=u_{2}$. Let $H_{1}$ be an adjacency graph of $\left\{u_{1}\right\}$ in $G_{1}$ and $H_{2}$ be an adjacency graph of $\left\{u_{2}\right\}$ in $G_{2}$. Then $H_{1}={ }_{G} H_{2}$.
(60) Let $G$ be a graph, $S$ be a non empty subset of the vertices of $G, H$ be a subgraph of $G$ induced by $S$, and $u$ be a vertex of $G$. Suppose $u \in S$ and $G$.adjacentSet $(\{u\}) \subseteq S$ and $G$.adjacentSet $(\{u\}) \neq \emptyset$. Let $v$ be a vertex of $H$. Suppose $u=v$. Let $G_{5}$ be an adjacency graph of $\{u\}$ in $G$ and $H_{3}$ be an adjacency graph of $\{v\}$ in $H$. Then $G_{5}={ }_{G} H_{3}$.
Let $G$ be a graph. We say that $G$ is complete if and only if:
(Def. 6) For all vertices $u, v$ of $G$ such that $u \neq v$ holds $u$ and $v$ are adjacent.
We now state the proposition
(61) For every graph $G$ such that $G$ is trivial holds $G$ is complete.

One can check that every graph which is trivial is also complete.
Let us note that there exists a graph which is trivial, simple, and complete and there exists a graph which is non trivial, finite, simple, and complete.

The following propositions are true:
(62) For all graphs $G_{1}, G_{2}$ such that $G_{1}={ }_{G} G_{2}$ holds if $G_{1}$ is complete, then $G_{2}$ is complete.
(63) For every complete graph $G$ and for every subset $S$ of the vertices of $G$ holds every subgraph of $G$ induced by $S$ is complete.

## 5. Simplicial Vertex

Let $G$ be a graph and let $v$ be a vertex of $G$. We say that $v$ is simplicial if and only if:
(Def. 7) If $G$.adjacentSet $(\{v\}) \neq \emptyset$, then every adjacency graph of $\{v\}$ in $G$ is complete.
The following propositions are true:
(64) For every complete graph $G$ holds every vertex of $G$ is simplicial.
(65) For every trivial graph $G$ holds every vertex of $G$ is simplicial.
(66) Let $G_{1}, G_{2}$ be graphs. Suppose $G_{1}={ }_{G} G_{2}$. Let $u_{1}$ be a vertex of $G_{1}$ and $u_{2}$ be a vertex of $G_{2}$. If $u_{1}=u_{2}$ and $u_{1}$ is simplicial, then $u_{2}$ is simplicial.
(67) Let $G$ be a graph, $S$ be a non empty subset of the vertices of $G, H$ be a subgraph of $G$ induced by $S$, and $u$ be a vertex of $G$. Suppose $u \in S$ and $G$.adjacentSet $(\{u\}) \subseteq S$. Let $v$ be a vertex of $H$. If $u=v$, then $u$ is simplicial iff $v$ is simplicial.
(68) Let $G$ be a graph and $v$ be a vertex of $G$. Suppose $v$ is simplicial. Let $a, b$ be sets. Suppose $a \neq b$ and $a \in G$.adjacentSet $(\{v\})$ and $b \in$ $G$.adjacentSet $(\{v\})$. Then there exists a set $e$ such that $e$ joins $a$ and $b$ in $G$.
(69) Let $G$ be a graph and $v$ be a vertex of $G$. Suppose $v$ is not simplicial. Then there exist vertices $a, b$ of $G$ such that $a \neq b$ and $v \neq a$ and $v \neq b$ and $v$ and $a$ are adjacent and $v$ and $b$ are adjacent and $a$ and $b$ are not adjacent.

## 6. Vertex Separator

Let $G$ be a graph and let $a, b$ be vertices of $G$. Let us assume that $a \neq b$ and $a$ and $b$ are not adjacent. A subset of the vertices of $G$ is said to be a vertex separator of $a$ and $b$ if:
(Def. 8) $\quad a \notin$ it and $b \notin$ it and for every subgraph $G_{2}$ of $G$ with vertices it removed holds there exists no walk of $G_{2}$ which is walk from $a$ to $b$.
Next we state several propositions:
(70) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Then every vertex separator of $a$ and $b$ is a vertex separator of $b$ and $a$.
(71) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a subset of the vertices of $G$. Then $S$ is a vertex separator of $a$ and $b$ if and only if $a \notin S$ and $b \notin S$ and for every walk $W$ of $G$ such that $W$ is walk from $a$ to $b$ there exists a vertex $x$ of $G$ such that $x \in S$ and $x \in W$.vertices().
(72) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$ and $W$ be a walk of $G$. Suppose $W$ is walk from $a$ to $b$. Then there exists an odd natural number $k$ such that $1<k$ and $k<$ len $W$ and $W(k) \in S$.
(73) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$. If $S=\emptyset$, then there exists no walk of $G$ which is walk from $a$ to $b$.
(74) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent and there exists no walk of $G$ which is walk from $a$ to $b$. Then $\emptyset$ is a vertex separator of $a$ and $b$.
(75) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b, G_{2}$ be a subgraph of $G$ with vertices $S$ removed, and $a_{2}$ be a vertex of $G_{2}$. If $a_{2}=a$, then $\left(G_{2}\right.$.reachableFrom $\left.\left(a_{2}\right)\right) \cap S=\emptyset$.
(76) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b, G_{2}$ be a subgraph of $G$ with vertices $S$ removed, and $a_{2}, b_{2}$ be vertices of $G_{2}$. If $a_{2}=a$ and $b_{2}=b$, then $\left(G_{2}\right.$.reachableFrom $\left.\left(a_{2}\right)\right) \cap\left(G_{2}\right.$.reachableFrom $\left.\left(b_{2}\right)\right)=\emptyset$.
(77) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$ and $G_{2}$ be a subgraph of $G$ with vertices $S$ removed. Then $a$ is a vertex of $G_{2}$ and $b$ is a vertex of $G_{2}$.
Let $G$ be a graph, let $a, b$ be vertices of $G$, and let $S$ be a vertex separator of $a$ and $b$. We say that $S$ is minimal if and only if:
(Def. 9) For every subset $T$ of $S$ such that $T \neq S$ holds $T$ is not a vertex separator of $a$ and $b$.
Next we state several propositions:
(78) Let $G$ be a graph, $a, b$ be vertices of $G$, and $S$ be a vertex separator of $a$ and $b$. If $S=\emptyset$, then $S$ is minimal.
(79) For every finite graph $G$ and for all vertices $a, b$ of $G$ holds there exists a vertex separator of $a$ and $b$ which is minimal.
(80) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$. Suppose $S$ is minimal. Let $T$ be a vertex separator of $b$ and $a$. If $S=T$, then $T$ is minimal.
(81) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$. Suppose $S$ is minimal. Let $x$ be a vertex of $G$. If $x \in S$, then there exists a walk $W$ of $G$ such that $W$ is walk from $a$ to $b$ and $x \in W$.vertices().
(82) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$. Suppose $S$ is minimal. Let $H$ be a subgraph of $G$ with vertices $S$ removed and $a_{1}$ be a vertex of $H$. Suppose $a_{1}=a$. Let $x$ be a vertex of $G$. Suppose $x \in S$. Then there exists a vertex $y$ of $G$ such that $y \in H$.reachableFrom $\left(a_{1}\right)$ and $x$ and $y$ are adjacent.
(83) Let $G$ be a graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$. Suppose $S$ is minimal. Let $H$ be a subgraph of $G$ with vertices $S$ removed and $a_{1}$ be a vertex of $H$. Suppose $a_{1}=b$. Let $x$ be a vertex of $G$. Suppose $x \in S$. Then there exists a vertex $y$ of $G$ such that $y \in H$.reachableFrom $\left(a_{1}\right)$ and $x$ and $y$ are adjacent.

## 7. Chordal Graphs

Let $G$ be a graph and let $W$ be a walk of $G$. We say that $W$ is chordal if and only if the condition (Def. 10) is satisfied.
(Def. 10) There exist odd natural numbers $m, n$ such that
(i) $m+2<n$,
(ii) $n \leq \operatorname{len} W$,
(iii) $\quad W(m) \neq W(n)$,
(iv) there exists a set $e$ such that $e$ joins $W(m)$ and $W(n)$ in $G$, and
(v) for every set $f$ such that $f \in W$.edges() holds $f$ does not join $W(m)$ and $W(n)$ in $G$.
Let $G$ be a graph and let $W$ be a walk of $G$. We introduce $W$ is chordless as an antonym of $W$ is chordal.

Next we state a number of propositions:
(84) Let $G$ be a graph and $W$ be a walk of $G$. Suppose $W$ is chordal. Then there exist odd natural numbers $m, n$ such that
(i) $m+2<n$,
(ii) $n \leq \operatorname{len} W$,
(iii) $W(m) \neq W(n)$,
(iv) there exists a set $e$ such that $e$ joins $W(m)$ and $W(n)$ in $G$, and
(v) if $W$ is cycle-like, then $m=1$ and $n=\operatorname{len} W$ and $m=1$ and $n=$ len $W-2$ and $m=3$ and $n=\operatorname{len} W$.
(85) Let $G$ be a graph and $P$ be a path of $G$. Given odd natural numbers $m$, $n$ such that
(i) $m+2<n$,
(ii) $n \leq \operatorname{len} P$,
(iii) there exists a set $e$ such that $e$ joins $P(m)$ and $P(n)$ in $G$, and
(iv) if $P$ is cycle-like, then $m=1$ and $n=\operatorname{len} P$ and $m=1$ and $n=\operatorname{len} P-2$ and $m=3$ and $n=$ len $P$. Then $P$ is chordal.
(86) Let $G_{1}, G_{2}$ be graphs. Suppose $G_{1}={ }_{G} G_{2}$. Let $W_{1}$ be a walk of $G_{1}$ and $W_{2}$ be a walk of $G_{2}$. If $W_{1}=W_{2}$, then if $W_{1}$ is chordal, then $W_{2}$ is chordal.
(87) Let $G$ be a graph, $S$ be a non empty subset of the vertices of $G, H$ be a subgraph of $G$ induced by $S, W_{1}$ be a walk of $G$, and $W_{2}$ be a walk of $H$. If $W_{1}=W_{2}$, then $W_{2}$ is chordal iff $W_{1}$ is chordal.
(88) Let $G$ be a graph and $W$ be a walk of $G$. Suppose $W$ is cycle-like and chordal and $W$.length ()$=4$. Then there exists a set $e$ such that $e$ joins $W(1)$ and $W(5)$ in $G$ or $e$ joins $W(3)$ and $W(7)$ in $G$.
(89) For every graph $G$ and for every walk $W$ of $G$ such that $W$ is minimum length holds $W$ is chordless.
(90) Let $G$ be a graph and $W$ be a walk of $G$. Suppose $W$ is open and len $W=$ 5 and $W$.first() and $W$.last() are not adjacent. Then $W$ is chordless.
(91) For every graph $G$ and for every walk $W$ of $G$ holds $W$ is chordal iff $W$.reverse() is chordal.
(92) Let $G$ be a graph and $P$ be a path of $G$. Suppose $P$ is open and chordless. Let $m, n$ be odd natural numbers. Suppose $m<n$ and $n \leq \operatorname{len} P$. Then there exists a set $e$ such that $e$ joins $P(m)$ and $P(n)$ in $G$ if and only if $m+2=n$.
(93) Let $G$ be a graph and $P$ be a path of $G$. Suppose $P$ is open and chordless. Let $m, n$ be odd natural numbers. If $m<n$ and $n \leq \operatorname{len} P$, then $P$.cut $(m, n)$ is chordless and $P . \operatorname{cut}(m, n)$ is open.
(94) Let $G$ be a graph, $S$ be a non empty subset of the vertices of $G, H$ be a subgraph of $G$ induced by $S, W$ be a walk of $G$, and $V$ be a walk of $H$. If $W=V$, then $W$ is chordless iff $V$ is chordless.

Let $G$ be a graph. We say that $G$ is chordal if and only if:
(Def. 11) For every walk $P$ of $G$ such that $P$.length ()$>3$ and $P$ is cycle-like holds $P$ is chordal.
Next we state two propositions:
(95) For all graphs $G_{1}, G_{2}$ such that $G_{1}={ }_{G} G_{2}$ holds if $G_{1}$ is chordal, then $G_{2}$ is chordal.
(96) For every finite graph $G$ such that card (the vertices of $G$ ) $\leq 3$ holds $G$ is chordal.

One can verify the following observations:

* there exists a graph which is trivial, finite, and chordal,
* there exists a graph which is non trivial, finite, simple, and chordal, and
* every graph which is complete is also chordal.

Let $G$ be a chordal graph and let $V$ be a set. One can check that every subgraph of $G$ induced by $V$ is chordal.

Next we state several propositions:
(97) Let $G$ be a chordal graph and $P$ be a path of $G$. Suppose $P$ is open and chordless. Let $x, e$ be sets. Suppose $x \notin P$.vertices() and $e$ joins $P$.last() and $x$ in $G$ and it is not true that there exists a set $f$ such that $f$ joins $P(\operatorname{len} P-2)$ and $x$ in $G$. Then $P$.addEdge $(e)$ is path-like and $P$.addEdge $(e)$ is open and $P$.addEdge $(e)$ is chordless.
(98) Let $G$ be a chordal graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$. If $S$ is minimal and non empty, then every subgraph of $G$ induced by $S$ is complete.
(99) Let $G$ be a finite graph. Suppose that for all vertices $a, b$ of $G$ such that
$a \neq b$ and $a$ and $b$ are not adjacent and for every vertex separator $S$ of $a$ and $b$ such that $S$ is minimal and non empty holds every subgraph of $G$ induced by $S$ is complete. Then $G$ is chordal.
(100) Let $G$ be a finite chordal graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$. Suppose $S$ is minimal. Let $H$ be a subgraph of $G$ with vertices $S$ removed and $a_{3}$ be a vertex of $H$. Suppose $a=a_{3}$. Then there exists a vertex $c$ of $G$ such that $c \in H$.reachableFrom $\left(a_{3}\right)$ and for every vertex $x$ of $G$ such that $x \in S$ holds $c$ and $x$ are adjacent.
(101) Let $G$ be a finite chordal graph and $a, b$ be vertices of $G$. Suppose $a \neq b$ and $a$ and $b$ are not adjacent. Let $S$ be a vertex separator of $a$ and $b$. Suppose $S$ is minimal. Let $H$ be a subgraph of $G$ with vertices $S$ removed and $a_{3}$ be a vertex of $H$. Suppose $a=a_{3}$. Let $x, y$ be vertices of $G$. Suppose $x \in S$ and $y \in S$. Then there exists a vertex $c$ of $G$ such that $c \in H$.reachableFrom $\left(a_{3}\right)$ and $c$ and $x$ are adjacent and $c$ and $y$ are adjacent.
(102) Let $G$ be a non trivial finite chordal graph. Suppose $G$ is not complete. Then there exist vertices $a, b$ of $G$ such that $a \neq b$ and $a$ and $b$ are not adjacent and $a$ is simplicial and $b$ is simplicial.
(103) For every finite chordal graph $G$ holds there exists a vertex of $G$ which is simplicial.

## 8. Vertex Elimination Scheme

Let $G$ be a finite graph. A finite sequence of elements of the vertices of $G$ is said to be a vertex scheme of $G$ if:
(Def. 12) It is one-to-one and rng it $=$ the vertices of $G$.
Let $G$ be a finite graph. Note that every vertex scheme of $G$ is non empty. The following three propositions are true:
(104) For every finite graph $G$ and for every vertex scheme $S$ of $G$ holds len $S=$ card (the vertices of $G$ ).
(105) For every finite graph $G$ and for every vertex scheme $S$ of $G$ holds $1 \leq$ len $S$.
(106) For all finite graphs $G, H$ and for every vertex scheme $g$ of $G$ such that $G={ }_{G} H$ holds $g$ is a vertex scheme of $H$.
Let $G$ be a finite graph, let $S$ be a vertex scheme of $G$, and let $x$ be a vertex of $G$. Then $x \leftrightarrow S$ is a non zero element of $\mathbb{N}$.

Let $G$ be a finite graph, let $S$ be a vertex scheme of $G$, and let $n$ be a natural number. Then $S$.followSet $(n)$ is a subset of the vertices of $G$.

Next we state the proposition
(107) Let $G$ be a finite graph, $S$ be a vertex scheme of $G$, and $n$ be a non zero natural number. If $n \leq \operatorname{len} S$, then $S$.followSet $(n)$ is non empty.

Let $G$ be a finite graph and let $S$ be a vertex scheme of $G$. We say that $S$ is perfect if and only if the condition (Def. 13) is satisfied.
(Def. 13) Let $n$ be a non zero natural number. Suppose $n \leq \operatorname{len} S$. Let $G_{6}$ be a subgraph of $G$ induced by $S$.followSet $(n)$ and $v$ be a vertex of $G_{6}$. If $v=S(n)$, then $v$ is simplicial.
One can prove the following propositions:
(108) Let $G$ be a finite trivial graph and $v$ be a vertex of $G$. Then there exists a vertex scheme $S$ of $G$ such that $S=\langle v\rangle$ and $S$ is perfect.
(109) Let $G$ be a finite graph and $V$ be a vertex scheme of $G$. Then $V$ is perfect if and only if for all vertices $a, b, c$ of $G$ such that $b \neq c$ and $a$ and $b$ are adjacent and $a$ and $c$ are adjacent and for all natural numbers $v_{5}, v_{6}$, $v_{7}$ such that $v_{5} \in \operatorname{dom} V$ and $v_{6} \in \operatorname{dom} V$ and $v_{7} \in \operatorname{dom} V$ and $V\left(v_{5}\right)=a$ and $V\left(v_{6}\right)=b$ and $V\left(v_{7}\right)=c$ and $v_{5}<v_{6}$ and $v_{5}<v_{7}$ holds $b$ and $c$ are adjacent.
Let $G$ be a finite chordal graph. One can check that there exists a vertex scheme of $G$ which is perfect.

The following propositions are true:
(110) Let $G, H$ be finite chordal graphs and $g$ be a perfect vertex scheme of $G$. If $G={ }_{G} H$, then $g$ is a perfect vertex scheme of $H$.
(111) For every finite graph $G$ such that there exists a vertex scheme of $G$ which is perfect holds $G$ is chordal.

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# Connectedness and Continuous Sequences in Finite Topological Spaces 

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#### Abstract

Summary. First, equivalence conditions for connectedness are examined for a finite topological space (originated in [9]). Secondly, definitions of subspace, and components of the subspace of a finite topological space are given. Lastly, concepts of continuous finite sequence and minimum path of finite topological space are proposed.


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The articles [16], [5], [18], [13], [1], [19], [14], [3], [4], [2], [6], [12], [10], [15], [7], [11], [8], and [17] provide the terminology and notation for this paper.

## 1. Connectedness and Subspaces

In this paper $F_{1}$ denotes a non empty finite topology space and $A, B, C$ denote subsets of $F_{1}$.

Let us consider $F_{1}$. One can check that $\emptyset_{\left(F_{1}\right)}$ is connected.
We now state two propositions:
(1) For all subsets $A, B$ of $F_{1}$ holds $(A \cup B)^{b}=A^{b} \cup B^{b}$.
(2) $\left(\emptyset_{\left(F_{1}\right)}\right)^{b}=\emptyset$.

Let us consider $F_{1}$. Observe that $\left(\emptyset_{\left(F_{1}\right)}\right)^{b}$ is empty.
Next we state the proposition
(3) Let $A$ be a subset of $F_{1}$. Suppose that for all subsets $B, C$ of $F_{1}$ such that $A=B \cup C$ and $B \neq \emptyset$ and $C \neq \emptyset$ and $B$ misses $C$ holds $B^{b}$ meets $C$ and $B$ meets $C^{b}$. Then $A$ is connected.

Let $F_{1}$ be a non empty finite topology space. We say that $F_{1}$ is connected if and only if:
(Def. 1) $\Omega_{\left(F_{1}\right)}$ is connected.
We now state four propositions:
(4) Let $A$ be a subset of $F_{1}$. Suppose $A$ is connected. Let $A_{2}, B_{2}$ be subsets of $F_{1}$. Suppose $A=A_{2} \cup B_{2}$ and $A_{2}$ misses $B_{2}$ and $A_{2}$ and $B_{2}$ are separated. Then $A_{2}=\emptyset_{\left(F_{1}\right)}$ or $B_{2}=\emptyset_{\left(F_{1}\right)}$.
(5) Suppose $F_{1}$ is connected. Let $A, B$ be subsets of $F_{1}$. Suppose $\Omega_{\left(F_{1}\right)}=$ $A \cup B$ and $A$ misses $B$ and $A$ and $B$ are separated. Then $A=\emptyset_{\left(F_{1}\right)}$ or $B=\emptyset_{\left(F_{1}\right)}$.
(6) For all subsets $A, B$ of $F_{1}$ such that $F_{1}$ is symmetric and $A^{b}$ misses $B$ holds $A$ misses $B^{b}$.
(7) Let $A$ be a subset of $F_{1}$. Suppose that
(i) $\quad F_{1}$ is symmetric, and
(ii) for all subsets $A_{2}, B_{2}$ of $F_{1}$ such that $A=A_{2} \cup B_{2}$ and $A_{2}$ misses $B_{2}$ and $A_{2}$ and $B_{2}$ are separated holds $A_{2}=\emptyset_{\left(F_{1}\right)}$ or $B_{2}=\emptyset_{\left(F_{1}\right)}$.
Then $A$ is connected.
Let $T$ be a finite topology space. A finite topology space is said to be a subspace of $T$ if it satisfies the conditions (Def. 2).
(Def. 2)(i) The carrier of it $\subseteq$ the carrier of $T$,
(ii) $\operatorname{dom}($ the neighbour-map of it) $=$ the carrier of it, and
(iii) for every element $x$ of it such that $x \in$ the carrier of it holds (the neighbour-map of it) $(x)=$ (the neighbour-map of $T)(x) \cap$ the carrier of it.
Let $T$ be a finite topology space. Note that there exists a subspace of $T$ which is strict.

Let $T$ be a non empty finite topology space. Note that there exists a subspace of $T$ which is strict and non empty.

Let $T$ be a non empty finite topology space and let $P$ be a non empty subset of $T$. The functor $T \upharpoonright P$ yields a strict non empty subspace of $T$ and is defined as follows:
(Def. 3) $\quad \Omega_{T \upharpoonright P}=P$.
We now state the proposition
(8) For every non empty subspace $X$ of $F_{1}$ such that $F_{1}$ is filled holds $X$ is filled.
Let $F_{1}$ be a filled non empty finite topology space. Note that every non empty subspace of $F_{1}$ is filled.

Next we state a number of propositions:
(9) For every non empty subspace $X$ of $F_{1}$ such that $F_{1}$ is symmetric holds $X$ is symmetric.
(10) For every subspace $X^{\prime}$ of $F_{1}$ holds every subset of $X^{\prime}$ is a subset of $F_{1}$.
(11) For every subset $P$ of $F_{1}$ holds $P$ is closed iff $P^{\mathrm{c}}$ is open.
(12) Let $A$ be a subset of $F_{1}$. Then $A$ is open if and only if the following conditions are satisfied:
(i) for every element $z$ of $F_{1}$ such that $U(z) \subseteq A$ holds $z \in A$, and
(ii) for every element $x$ of $F_{1}$ such that $x \in A$ holds $U(x) \subseteq A$.
(13) Let $X^{\prime}$ be a non empty subspace of $F_{1}, A$ be a subset of $F_{1}$, and $A_{1}$ be a subset of $X^{\prime}$. If $A=A_{1}$, then $A_{1}^{b}=A^{b} \cap \Omega_{X^{\prime}}$.
(14) Let $X^{\prime}$ be a non empty subspace of $F_{1}, P_{1}, Q_{1}$ be subsets of $F_{1}$, and $P, Q$ be subsets of $X^{\prime}$. Suppose $P=P_{1}$ and $Q=Q_{1}$. If $P$ and $Q$ are separated, then $P_{1}$ and $Q_{1}$ are separated.
(15) Let $X^{\prime}$ be a non empty subspace of $F_{1}, P, Q$ be subsets of $F_{1}$, and $P_{1}$, $Q_{1}$ be subsets of $X^{\prime}$. Suppose $P=P_{1}$ and $Q=Q_{1}$ and $P \cup Q \subseteq \Omega_{X^{\prime}}$. If $P$ and $Q$ are separated, then $P_{1}$ and $Q_{1}$ are separated.
(16) For every non empty subset $A$ of $F_{1}$ holds $A$ is connected iff $F_{1} \upharpoonright A$ is connected.
(17) Let $F_{1}$ be a filled non empty finite topology space and $A$ be a non empty subset of $F_{1}$. Suppose $F_{1}$ is symmetric. Then $A$ is connected if and only if for all subsets $P, Q$ of $F_{1}$ such that $A=P \cup Q$ and $P$ misses $Q$ and $P$ and $Q$ are separated holds $P=\emptyset_{\left(F_{1}\right)}$ or $Q=\emptyset_{\left(F_{1}\right)}$.
(18) For every subset $A$ of $F_{1}$ such that $F_{1}$ is filled and connected and $A \neq \emptyset$ and $A^{\mathrm{c}} \neq \emptyset$ holds $A^{\delta} \neq \emptyset$.
(19) For every subset $A$ of $F_{1}$ such that $F_{1}$ is filled, symmetric, and connected and $A \neq \emptyset$ and $A^{\mathrm{c}} \neq \emptyset$ holds $A^{\delta_{i}} \neq \emptyset$.
(20) For every subset $A$ of $F_{1}$ such that $F_{1}$ is filled, symmetric, and connected and $A \neq \emptyset$ and $A^{\mathrm{c}} \neq \emptyset$ holds $A^{\delta_{o}} \neq \emptyset$.
(21) For every subset $A$ of $F_{1}$ holds $A^{\delta_{i}}$ misses $A^{\delta_{o}}$.
(22) For every filled non empty finite topology space $F_{1}$ and for every subset $A$ of $F_{1}$ holds $A^{\delta_{o}}=A^{b} \backslash A$.
(23) For all subsets $A, B$ of $F_{1}$ such that $A$ and $B$ are separated holds $A^{\delta_{o}}$ misses $B$.
(24) Let $A, B$ be subsets of $F_{1}$. Suppose $F_{1}$ is filled and $A$ misses $B$ and $A^{\delta_{o}}$ misses $B$ and $B^{\delta_{o}}$ misses $A$. Then $A$ and $B$ are separated.
(25) For every point $x$ of $F_{1}$ holds $\{x\}$ is connected.

Let us consider $F_{1}$ and let $x$ be a point of $F_{1}$. Note that $\{x\}$ is connected.
Let $F_{1}$ be a non empty finite topology space and let $A$ be a subset of $F_{1}$.
We say that $A$ is a component of $F_{1}$ if and only if:
(Def. 4) $\quad A$ is connected and for every subset $B$ of $F_{1}$ such that $B$ is connected holds if $A \subseteq B$, then $A=B$.
One can prove the following propositions:
(26) For every subset $A$ of $F_{1}$ such that $A$ is a component of $F_{1}$ holds $A \neq$ $\emptyset_{\left(F_{1}\right)}$.
(27) If $A$ is closed and $B$ is closed and $A$ misses $B$, then $A$ and $B$ are separated.
(28) If $F_{1}$ is filled and $\Omega_{\left(F_{1}\right)}=A \cup B$ and $A$ and $B$ are separated, then $A$ is open and closed.
(29) For all subsets $A, B, A_{1}, B_{1}$ of $F_{1}$ such that $A$ and $B$ are separated and $A_{1} \subseteq A$ and $B_{1} \subseteq B$ holds $A_{1}$ and $B_{1}$ are separated.
(30) If $A$ and $B$ are separated and $A$ and $C$ are separated, then $A$ and $B \cup C$ are separated.
(31) Suppose that
(i) $\quad F_{1}$ is filled and symmetric, and
(ii) for all subsets $A, B$ of $F_{1}$ such that $\Omega_{\left(F_{1}\right)}=A \cup B$ and $A \neq \emptyset_{\left(F_{1}\right)}$ and $B \neq \emptyset_{\left(F_{1}\right)}$ and $A$ is closed and $B$ is closed holds $A$ meets $B$. Then $F_{1}$ is connected.
(32) Suppose $F_{1}$ is connected. Let $A, B$ be subsets of $F_{1}$. Suppose $\Omega_{\left(F_{1}\right)}=$ $A \cup B$ and $A \neq \emptyset_{\left(F_{1}\right)}$ and $B \neq \emptyset_{\left(F_{1}\right)}$ and $A$ is closed and $B$ is closed. Then $A$ meets $B$.
(33) If $F_{1}$ is filled and $A$ is connected and $A \subseteq B \cup C$ and $B$ and $C$ are separated, then $A \subseteq B$ or $A \subseteq C$.
(34) Let $A, B$ be subsets of $F_{1}$. Suppose $F_{1}$ is symmetric and $A$ is connected and $B$ is connected and $A$ and $B$ are not separated. Then $A \cup B$ is connected.
(35) For all subsets $A, C$ of $F_{1}$ such that $F_{1}$ is symmetric and $C$ is connected and $C \subseteq A$ and $A \subseteq C^{b}$ holds $A$ is connected.
(36) For every subset $C$ of $F_{1}$ such that $F_{1}$ is filled and symmetric and $C$ is connected holds $C^{b}$ is connected.
(37) Suppose $F_{1}$ is filled, symmetric, and connected and $A$ is connected and $\Omega_{\left(F_{1}\right)} \backslash A=B \cup C$ and $B$ and $C$ are separated. Then $A \cup B$ is connected.
(38) Let $X^{\prime}$ be a non empty subspace of $F_{1}, A$ be a subset of $F_{1}$, and $B$ be a subset of $X^{\prime}$. Suppose $F_{1}$ is symmetric and $A=B$. Then $A$ is connected if and only if $B$ is connected.
(39) For every subset $A$ of $F_{1}$ such that $F_{1}$ is filled and symmetric and $A$ is a component of $F_{1}$ holds $A$ is closed.
(40) Let $A, B$ be subsets of $F_{1}$. Suppose $F_{1}$ is symmetric and $A$ is a component of $F_{1}$ and $B$ is a component of $F_{1}$. Then $A=B$ or $A$ and $B$ are separated.
(41) Let $A, B$ be subsets of $F_{1}$. Suppose $F_{1}$ is filled and symmetric and $A$ is a component of $F_{1}$ and $B$ is a component of $F_{1}$. Then $A=B$ or $A$ misses $B$.
(42) Let $C$ be a subset of $F_{1}$. Suppose $F_{1}$ is filled and symmetric and $C$ is connected. Let $S$ be a subset of $F_{1}$. If $S$ is a component of $F_{1}$, then $C$ misses $S$ or $C \subseteq S$.
Let $F_{1}$ be a non empty finite topology space, let $A$ be a non empty subset of $F_{1}$, and let $B$ be a subset of $F_{1}$. We say that $B$ is a component of $A$ if and only if:
(Def. 5) There exists a subset $B_{1}$ of $F_{1} \upharpoonright A$ such that $B_{1}=B$ and $B_{1}$ is a component of $F_{1} \upharpoonright A$.
We now state the proposition
(43) Let $D$ be a non empty subset of $F_{1}$. Suppose $F_{1}$ is filled and symmetric and $D=\Omega_{\left(F_{1}\right)} \backslash A$. Suppose $F_{1}$ is connected and $A$ is connected and $C$ is a component of $D$. Then $\Omega_{\left(F_{1}\right)} \backslash C$ is connected.

## 2. Continuous Finite Sequences and Minimum Path

Let us consider $F_{1}$ and let $f$ be a finite sequence of elements of $F_{1}$. We say that $f$ is continuous if and only if the conditions (Def. 6) are satisfied.
(Def. 6)(i) $1 \leq \operatorname{len} f$, and
(ii) for every natural number $i$ and for every element $x_{1}$ of $F_{1}$ such that $1 \leq i$ and $i<\operatorname{len} f$ and $x_{1}=f(i)$ holds $f(i+1) \in U\left(x_{1}\right)$.
Let us consider $F_{1}$ and let $x$ be an element of $F_{1}$. Observe that $\langle x\rangle$ is continuous.

One can prove the following two propositions:
(44) Let $f$ be a finite sequence of elements of $F_{1}$ and $x, y$ be elements of $F_{1}$. If $f$ is continuous and $y=f(\operatorname{len} f)$ and $x \in U(y)$, then $f \frown\langle x\rangle$ is continuous.
(45) Let $f, g$ be finite sequences of elements of $F_{1}$. Suppose $f$ is continuous and $g$ is continuous and $g(1) \in U\left(f_{\operatorname{len} f}\right)$. Then $f \frown g$ is continuous.
Let us consider $F_{1}$ and let $A$ be a subset of $F_{1}$. We say that $A$ is arcwise connected if and only if the condition (Def. 7) is satisfied.
(Def. 7) Let $x_{1}, x_{2}$ be elements of $F_{1}$. Suppose $x_{1} \in A$ and $x_{2} \in A$. Then there exists a finite sequence $f$ of elements of $F_{1}$ such that $f$ is continuous and $\operatorname{rng} f \subseteq A$ and $f(1)=x_{1}$ and $f(\operatorname{len} f)=x_{2}$.
Let us consider $F_{1}$. Observe that $\emptyset_{\left(F_{1}\right)}$ is arcwise connected.
Let us consider $F_{1}$ and let $x$ be an element of $F_{1}$. One can verify that $\{x\}$ is arcwise connected.

The following three propositions are true:
(46) For every subset $A$ of $F_{1}$ such that $F_{1}$ is symmetric holds $A$ is connected iff $A$ is arcwise connected.
(47) Let $g$ be a finite sequence of elements of $F_{1}$ and $k$ be a natural number. If $g$ is continuous and $1 \leq k$, then $g \upharpoonright k$ is continuous.
(48) Let $g$ be a finite sequence of elements of $F_{1}$ and $k$ be an element of $\mathbb{N}$. If $g$ is continuous and $k<\operatorname{len} g$, then $g_{\lfloor k}$ is continuous.

Let us consider $F_{1}$, let $g$ be a finite sequence of elements of $F_{1}$, let $A$ be a subset of $F_{1}$, and let $x_{1}, x_{2}$ be elements of $F_{1}$. We say that $g$ is minimum path in $A$ between $x_{1}$ and $x_{2}$ if and only if the conditions (Def. 8) are satisfied.
(Def. 8)(i) $g$ is continuous,
(ii) $\operatorname{rng} g \subseteq A$,
(iii) $g(1)=x_{1}$,
(iv) $g(\operatorname{len} g)=x_{2}$, and
(v) for every finite sequence $h$ of elements of $F_{1}$ such that $h$ is continuous and $\operatorname{rng} h \subseteq A$ and $h(1)=x_{1}$ and $h($ len $h)=x_{2}$ holds len $g \leq$ len $h$.
One can prove the following propositions:
(49) For every subset $A$ of $F_{1}$ and for every element $x$ of $F_{1}$ such that $x \in A$ holds $\langle x\rangle$ is minimum path in $A$ between $x$ and $x$.
(50) Let $A$ be a subset of $F_{1}$. Then $A$ is arcwise connected if and only if for all elements $x_{1}, x_{2}$ of $F_{1}$ such that $x_{1} \in A$ and $x_{2} \in A$ holds there exists a finite sequence of elements of $F_{1}$ which is minimum path in $A$ between $x_{1}$ and $x_{2}$.
(51) Let $A$ be a subset of $F_{1}$ and $x_{1}, x_{2}$ be elements of $F_{1}$. Given a finite sequence $f$ of elements of $F_{1}$ such that $f$ is continuous and $\operatorname{rng} f \subseteq A$ and $f(1)=x_{1}$ and $f(\operatorname{len} f)=x_{2}$. Then there exists a finite sequence of elements of $F_{1}$ which is minimum path in $A$ between $x_{1}$ and $x_{2}$.
(52) Let $g$ be a finite sequence of elements of $F_{1}, A$ be a subset of $F_{1}, x_{1}, x_{2}$ be elements of $F_{1}$, and $k$ be an element of $\mathbb{N}$. Suppose $g$ is minimum path in $A$ between $x_{1}$ and $x_{2}$ and $1 \leq k$ and $k \leq \operatorname{len} g$. Then $g \upharpoonright k$ is continuous and $\operatorname{rng}(g \upharpoonright k) \subseteq A$ and $(g \upharpoonright k)(1)=x_{1}$ and $(g \upharpoonright k)(\operatorname{len}(g \upharpoonright k))=g_{k}$.
(53) Let $g$ be a finite sequence of elements of $F_{1}, A$ be a subset of $F_{1}, x_{1}$, $x_{2}$ be elements of $F_{1}$, and $k$ be an element of $\mathbb{N}$. Suppose $g$ is minimum path in $A$ between $x_{1}$ and $x_{2}$ and $k<\operatorname{len} g$. Then $g_{\mid k}$ is continuous and $\operatorname{rng}\left(g_{\downarrow k}\right) \subseteq A$ and $g_{\downarrow k}(1)=g_{1+k}$ and $g_{\downarrow k}\left(\operatorname{len}\left(g_{\backslash k}\right)\right)=x_{2}$.
(54) Let $g$ be a finite sequence of elements of $F_{1}, A$ be a subset of $F_{1}$, and $x_{1}, x_{2}$ be elements of $F_{1}$. Suppose $g$ is minimum path in $A$ between $x_{1}$ and $x_{2}$. Let $k$ be a natural number. If $1 \leq k$ and $k \leq \operatorname{len} g$, then $g \upharpoonright k$ is minimum path in $A$ between $x_{1}$ and $g_{k}$.
(55) Let $g$ be a finite sequence of elements of $F_{1}, A$ be a subset of $F_{1}$, and $x_{1}, x_{2}$ be elements of $F_{1}$. If $g$ is minimum path in $A$ between $x_{1}$ and $x_{2}$, then $g$ is one-to-one.

Let us consider $F_{1}$ and let $f$ be a finite sequence of elements of $F_{1}$. We say that $f$ is inversely continuous if and only if the conditions (Def. 9) are satisfied.
(Def. 9)(i) $1 \leq \operatorname{len} f$, and
(ii) for all natural numbers $i, j$ and for every element $y$ of $F_{1}$ such that $1 \leq i$ and $i \leq \operatorname{len} f$ and $1 \leq j$ and $j \leq \operatorname{len} f$ and $y=f(i)$ and $i \neq j$ and $f(j) \in U(y)$ holds $i=j+1$ or $j=i+1$.
We now state three propositions:
(56) Let $g$ be a finite sequence of elements of $F_{1}, A$ be a subset of $F_{1}$, and $x_{1}$, $x_{2}$ be elements of $F_{1}$. Suppose $g$ is minimum path in $A$ between $x_{1}$ and $x_{2}$ and $F_{1}$ is symmetric. Then $g$ is inversely continuous.
(57) Let $g$ be a finite sequence of elements of $F_{1}, A$ be a subset of $F_{1}$, and $x_{1}$, $x_{2}$ be elements of $F_{1}$. Suppose $g$ is minimum path in $A$ between $x_{1}$ and $x_{2}$ and $F_{1}$ is filled and symmetric and $x_{1} \neq x_{2}$. Then
(i) for every natural number $i$ such that $1<i$ and $i<\operatorname{len} g$ holds $\operatorname{rng} g \cap$ $U\left(g_{i}\right)=\left\{g\left(i-^{\prime} 1\right), g(i), g(i+1)\right\}$,
(ii) $\quad \operatorname{rng} g \cap U\left(g_{1}\right)=\{g(1), g(2)\}$, and
(iii) $\quad \operatorname{rng} g \cap U\left(g_{\operatorname{len} g}\right)=\left\{g\left(\operatorname{len} g-^{\prime} 1\right), g(\operatorname{len} g)\right\}$.
(58) Let $g$ be a finite sequence of elements of $F_{1}, A$ be a non empty subset of $F_{1}, x_{1}, x_{2}$ be elements of $F_{1}$, and $B_{0}$ be a subset of $F_{1} \upharpoonright A$. Suppose $g$ is minimum path in $A$ between $x_{1}$ and $x_{2}$ and $F_{1}$ is filled and symmetric and $x_{1} \neq x_{2}$ and $B_{0}=\left\{x_{1}\right\}$. Let $i$ be an element of $\mathbb{N}$. If $i<\operatorname{len} g$, then $g(i+1) \in \operatorname{Finf}\left(B_{0}, i\right)$ and if $i \geq 1$, then $g(i+1) \notin \operatorname{Finf}\left(B_{0}, i-^{\prime} 1\right)$.

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# The Definition of Finite Sequences and Matrices of Probability, and Addition of Matrices of Real Elements 

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#### Abstract

Summary. In this article, we first define finite sequences of probability distribution and matrices of joint probability and conditional probability. We discuss also the concept of marginal probability. Further, we describe some theorems of matrices of real elements including quadratic form.


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The papers [20], [23], [2], [19], [24], [6], [12], [8], [21], [4], [1], [22], [18], [14], [7], [9], [25], [3], [5], [15], [16], [17], [11], [13], and [10] provide the terminology and notation for this paper.

For simplicity, we use the following convention: $D$ denotes a non empty set, $i, j, k$ denote elements of $\mathbb{N}, n, m$ denote natural numbers, and $e$ denotes a finite sequence of elements of $\mathbb{R}$.

Let $d$ be a set, let $g$ be a finite sequence of elements of $d^{*}$, and let $n$ be a natural number. Then $g(n)$ is a finite sequence of elements of $d$.

Let $x$ be a real number. Then $\langle x\rangle$ is a finite sequence of elements of $\mathbb{R}$.
Next we state a number of propositions:
(1) Let $a$ be an element of $D, m$ be a non empty natural number, and $g$ be a finite sequence of elements of $D$. Then len $g=m$ and for every natural number $i$ such that $i \in \operatorname{dom} g$ holds $g(i)=a$ if and only if $g=m \mapsto a$.
(2) Let $a, b$ be elements of $D$. Then there exists a finite sequence $g$ of elements of $D$ such that len $g=n$ and for every natural number $i$ such that $i \in \operatorname{Seg} n$ holds if $i \in \operatorname{Seg} k$, then $g(i)=a$ and if $i \notin \operatorname{Seg} k$, then $g(i)=b$.
(3) Suppose that for every natural number $i$ such that $i \in$ dome holds $0 \leq e(i)$. Let $f$ be a sequence of real numbers. Suppose $f(1)=e(1)$ and for every natural number $n$ such that $0 \neq n$ and $n<$ len $e$ holds $f(n+1)=f(n)+e(n+1)$. Let $n, m$ be natural numbers. If $n \in \operatorname{dom} e$ and $m \in \operatorname{dom} e$ and $n \leq m$, then $f(n) \leq f(m)$.
(4) Suppose len $e \geq 1$ and for every natural number $i$ such that $i \in \operatorname{dome}$ holds $0 \leq e(i)$. Let $f$ be a sequence of real numbers. Suppose $f(1)=e(1)$ and for every natural number $n$ such that $0 \neq n$ and $n<$ len $e$ holds $f(n+1)=f(n)+e(n+1)$. Let $n$ be a natural number. If $n \in \operatorname{dom} e$, then $e(n) \leq f(n)$.
(5) Suppose that for every natural number $i$ such that $i \in$ dome holds $0 \leq e(i)$. Let $k$ be a natural number. If $k \in \operatorname{dom} e$, then $e(k) \leq \sum e$.
(6) Let $r_{1}, r_{2}$ be real numbers, $k$ be a natural number, and $s_{2}$ be a sequence of real numbers. Then there exists a sequence $s_{1}$ of real numbers such that $s_{1}(0)=r_{1}$ and for every $n$ holds if $n \neq 0$ and $n \leq k$, then $s_{1}(n)=s_{2}(n)$ and if $n>k$, then $s_{1}(n)=r_{2}$.
(7) Let $F$ be a finite sequence of elements of $\mathbb{R}$. Then there exists a sequence $f$ of real numbers such that $f(0)=0$ and for every natural number $i$ such that $i<\operatorname{len} F$ holds $f(i+1)=f(i)+F(i+1)$ and $\sum F=f(\operatorname{len} F)$.
(8) Let $D$ be a set and $e_{1}$ be a finite sequence of elements of $D$. Then $n \mapsto e_{1}$ is a finite sequence of elements of $D^{*}$.
(9) Let $D$ be a set and $e_{1}, e_{2}$ be finite sequences of elements of $D$. Then there exists a finite sequence $e$ of elements of $D^{*}$ such that len $e=n$ and for every natural number $i$ such that $i \in \operatorname{Seg} n$ holds if $i \in \operatorname{Seg} k$, then $e(i)=e_{1}$ and if $i \notin \operatorname{Seg} k$, then $e(i)=e_{2}$.
(10) Let $D$ be a set and $s$ be a finite sequence. Then $s$ is a matrix over $D$ if and only if there exists $n$ such that for every $i$ such that $i \in \operatorname{dom} s$ there exists a finite sequence $p$ of elements of $D$ such that $s(i)=p$ and len $p=n$.
(11) Let $D$ be a set and $e$ be a finite sequence of elements of $D^{*}$. Then there exists $n$ such that for every $i$ such that $i \in$ dom $e$ holds len $e(i)=n$ if and only if $e$ is a matrix over $D$.
(12) For every tabular finite sequence $M$ holds $\langle i, j\rangle \in$ the indices of $M$ iff $i \in \operatorname{Seg} \operatorname{len} M$ and $j \in \operatorname{Seg}$ width $M$.
(13) Let $D$ be a non empty set and $M$ be a matrix over $D$. Then $\langle i, j\rangle \in$ the indices of $M$ if and only if $i \in \operatorname{dom} M$ and $j \in \operatorname{dom} M(i)$.
(14) For every non empty set $D$ and for every matrix $M$ over $D$ such that $\langle i$, $j\rangle \in$ the indices of $M$ holds $M_{i, j}=M(i)(j)$.
(15) Let $D$ be a non empty set and $M$ be a matrix over $D$. Then $\langle i, j\rangle \in$ the indices of $M$ if and only if $i \in \operatorname{dom}\left(M_{\square, j}\right)$ and $j \in \operatorname{dom} \operatorname{Line}(M, i)$.
(16) Let $D_{1}, D_{2}$ be non empty sets, $M_{1}$ be a matrix over $D_{1}$, and $M_{2}$ be a
matrix over $D_{2}$. If $M_{1}=M_{2}$, then for every $i$ such that $i \in \operatorname{dom} M_{1}$ holds $\operatorname{Line}\left(M_{1}, i\right)=\operatorname{Line}\left(M_{2}, i\right)$.
(17) Let $D_{1}, D_{2}$ be non empty sets, $M_{1}$ be a matrix over $D_{1}$, and $M_{2}$ be a matrix over $D_{2}$. If $M_{1}=M_{2}$, then for every $j$ such that $j \in \operatorname{Seg}$ width $M_{1}$ holds $\left(M_{1}\right)_{\square, j}=\left(M_{2}\right)_{\square, j}$.
(18) Let $e_{1}$ be a finite sequence of elements of $D$. If len $e_{1}=m$, then $n \mapsto e_{1}$ is a matrix over $D$ of dimension $n \times m$.
(19) Let $e_{1}, e_{2}$ be finite sequences of elements of $D$. Suppose len $e_{1}=m$ and len $e_{2}=m$. Then there exists a matrix $M$ over $D$ of dimension $n \times m$ such that for every natural number $i$ holds
(i) if $i \in \operatorname{Seg} k$, then $M(i)=e_{1}$, and
(ii) if $i \notin \operatorname{Seg} k$, then $M(i)=e_{2}$.

Let $e$ be a finite sequence of elements of $\mathbb{R}^{*}$. The functor $\sum e$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:
(Def. 1) len $\sum e=$ len $e$ and for every $k$ such that $k \in \operatorname{dom} \sum e \operatorname{holds}\left(\sum e\right)(k)=$ $\sum e(k)$.
Let $m$ be a matrix over $\mathbb{R}$. We introduce LineSum $m$ as a synonym of $\sum m$. We now state the proposition
(20) For every matrix $m$ over $\mathbb{R}$ holds len $\sum m=\operatorname{len} m$ and for every $i$ such that $i \in \operatorname{Seg}$ len $m$ holds $\left(\sum m\right)(i)=\sum \operatorname{Line}(m, i)$.
Let $m$ be a matrix over $\mathbb{R}$. The functor $\operatorname{ColSum} m$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:
(Def. 2) len ColSum $m=$ width $m$ and for every $j$ such that $j \in \operatorname{Seg}$ width $m$ holds $(\operatorname{ColSum} m)(j)=\sum\left(m_{\square, j}\right)$.
We now state two propositions:
(21) For every matrix $M$ over $\mathbb{R}$ such that width $M>0$ holds LineSum $M=$ $\operatorname{ColSum}\left(M^{\mathrm{T}}\right)$.
(22) For every matrix $M$ over $\mathbb{R}$ holds $\operatorname{ColSum} M=\operatorname{LineSum}\left(M^{\mathrm{T}}\right)$.

Let $M$ be a matrix over $\mathbb{R}$. The functor SumAll $M$ yields an element of $\mathbb{R}$ and is defined as follows:
(Def. 3) SumAll $M=\sum \sum M$.
The following propositions are true:
(23) For every matrix $M$ over $\mathbb{R}$ such that len $M=0$ holds SumAll $M=0$.
(24) For every matrix $M$ over $\mathbb{R}$ of dimension $m \times 0$ holds SumAll $M=0$.
(25) Let $M_{1}$ be a matrix over $\mathbb{R}$ of dimension $n \times k$ and $M_{2}$ be a matrix over $\mathbb{R}$ of dimension $m \times k$. Then $\sum\left(M_{1} \frown M_{2}\right)=\left(\sum M_{1}\right)^{\wedge} \sum M_{2}$.
(26) For all matrices $M_{1}, M_{2}$ over $\mathbb{R}$ holds $\sum M_{1}+\sum M_{2}=\sum\left(M_{1} \frown M_{2}\right)$.
(27) For all matrices $M_{1}, M_{2}$ over $\mathbb{R}$ such that len $M_{1}=\operatorname{len} M_{2}$ holds SumAll $M_{1}+\operatorname{SumAll} M_{2}=\operatorname{SumAll}\left(M_{1} \frown M_{2}\right)$.
(28) For every matrix $M$ over $\mathbb{R}$ holds SumAll $M=\operatorname{SumAll}\left(M^{\mathrm{T}}\right)$.
(29) For every matrix $M$ over $\mathbb{R}$ holds SumAll $M=\sum$ ColSum $M$.
(30) For all finite sequences $x, y$ of elements of $\mathbb{R}$ such that len $x=\operatorname{len} y$ holds $\operatorname{len}(x \bullet y)=\operatorname{len} x$.
(31) For every $i$ and for every element $R$ of $\mathbb{R}^{i}$ holds $i \mapsto 1 \bullet R=R$.
(32) For every finite sequence $x$ of elements of $\mathbb{R}$ holds len $x \mapsto 1 \bullet x=x$.
(33) Let $x, y$ be finite sequences of elements of $\mathbb{R}$. Suppose for every $i$ such that $i \in \operatorname{dom} x$ holds $x(i) \geq 0$ and for every $i$ such that $i \in \operatorname{dom} y$ holds $y(i) \geq 0$. Let given $k$. If $k \in \operatorname{dom}(x \bullet y)$, then $(x \bullet y)(k) \geq 0$.
(34) Let given $i, e_{1}, e_{2}$ be elements of $\mathbb{R}^{i}$, and $f_{1}, f_{2}$ be elements of (the carrier of $\left.\mathbb{R}_{\mathrm{F}}\right)^{i}$. If $e_{1}=f_{1}$ and $e_{2}=f_{2}$, then $e_{1} \bullet e_{2}=f_{1} \bullet f_{2}$.
(35) Let $e_{1}, e_{2}$ be finite sequences of elements of $\mathbb{R}$ and $f_{1}, f_{2}$ be finite sequences of elements of $\mathbb{R}_{\mathrm{F}}$. If len $e_{1}=\operatorname{len} e_{2}$ and $e_{1}=f_{1}$ and $e_{2}=f_{2}$, then $e_{1} \bullet e_{2}=f_{1} \bullet f_{2}$.
(36) Let $e$ be a finite sequence of elements of $\mathbb{R}$ and $f$ be a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$. If $e=f$, then $\sum e=\sum f$.
Let $e_{1}, e_{2}$ be finite sequences of elements of $\mathbb{R}$. We introduce $e_{1} \cdot e_{2}$ as a synonym of $\left|\left(e_{1}, e_{2}\right)\right|$.

We now state several propositions:
(37) Let given $i, e_{1}, e_{2}$ be elements of $\mathbb{R}^{i}$, and $f_{1}, f_{2}$ be elements of (the carrier of $\left.\mathbb{R}_{\mathrm{F}}\right)^{i}$. If $e_{1}=f_{1}$ and $e_{2}=f_{2}$, then $e_{1} \cdot e_{2}=f_{1} \cdot f_{2}$.
(38) Let $e_{1}, e_{2}$ be finite sequences of elements of $\mathbb{R}$ and $f_{1}, f_{2}$ be finite sequences of elements of $\mathbb{R}_{\mathrm{F}}$. If len $e_{1}=\operatorname{len} e_{2}$ and $e_{1}=f_{1}$ and $e_{2}=f_{2}$, then $e_{1} \cdot e_{2}=f_{1} \cdot f_{2}$.
(39) Let $M, M_{1}, M_{2}$ be matrices over $\mathbb{R}$. Suppose width $M_{1}=$ len $M_{2}$. Then $M=M_{1} \cdot M_{2}$ if and only if the following conditions are satisfied:
(i) $\operatorname{len} M=\operatorname{len} M_{1}$,
(ii) width $M=$ width $M_{2}$, and
(iii) for all $i, j$ such that $\langle i, j\rangle \in$ the indices of $M$ holds $M_{i, j}=\operatorname{Line}\left(M_{1}, i\right)$. $\left(M_{2}\right)_{\square, j}$.
(40) Let $M$ be a matrix over $\mathbb{R}$ and $p$ be a finite sequence of elements of $\mathbb{R}$. If len $M=\operatorname{len} p$, then for every $i$ such that $i \in \operatorname{Seg} \operatorname{len}(p \cdot M)$ holds $(p \cdot M)(i)=p \cdot M_{\square, i}$.
(41) Let $M$ be a matrix over $\mathbb{R}$ and $p$ be a finite sequence of elements of $\mathbb{R}$. If width $M=\operatorname{len} p$ and width $M>0$, then for every $i$ such that $i \in \operatorname{Seg} \operatorname{len}(M \cdot p)$ holds $(M \cdot p)(i)=\operatorname{Line}(M, i) \cdot p$.
(42) Let $M, M_{1}, M_{2}$ be matrices over $\mathbb{R}$. Suppose width $M_{1}=\operatorname{len} M_{2}$ and width $M_{1}>0$ and width $M_{2}>0$. Then $M=M_{1} \cdot M_{2}$ if and only if the following conditions are satisfied:
(i) $\operatorname{len} M=\operatorname{len} M_{1}$,
(ii) width $M=$ width $M_{2}$, and
(iii) for every $i$ such that $i \in \operatorname{Seg}$ len $M$ holds $\operatorname{Line}(M, i)=\operatorname{Line}\left(M_{1}, i\right) \cdot M_{2}$.

Let $n, m, k$ be non empty natural numbers, let $M_{1}$ be a matrix over $\mathbb{R}$ of dimension $n \times k$, and let $M_{2}$ be a matrix over $\mathbb{R}$ of dimension $k \times m$. Note that $M_{1} \cdot M_{2}$
let $x, y$ be finite sequences of elements of $\mathbb{R}$ and let $M$ be a matrix over $\mathbb{R}$. Let us assume that $\operatorname{len} x=\operatorname{len} M$ and len $y=$ width $M$. The functor QuadraticForm $(x, M, y)$ yields a matrix over $\mathbb{R}$ and is defined by the conditions (Def. 4).
(Def. 4)(i) len QuadraticForm $(x, M, y)=\operatorname{len} x$,
(ii) width QuadraticForm $(x, M, y)=\operatorname{len} y$, and
(iii) for all natural numbers $i, j$ such that $\langle i, j\rangle \in$ the indices of $M$ holds (QuadraticForm $(x, M, y))_{i, j}=x(i) \cdot M_{i, j} \cdot y(j)$.
The following propositions are true:
(43) Let $x, y$ be finite sequences of elements of $\mathbb{R}$ and $M$ be a matrix over $\mathbb{R}$. If len $x=\operatorname{len} M$ and len $y=$ width $M$ and len $x>0$ and len $y>0$, then (QuadraticForm $(x, M, y))^{\mathrm{T}}=$ QuadraticForm $\left(y, M^{\mathrm{T}}, x\right)$.
(44) Let $x, y$ be finite sequences of elements of $\mathbb{R}$ and $M$ be a matrix over $\mathbb{R}$. If len $x=\operatorname{len} M$ and len $y=$ width $M$ and len $x>0$ and len $y>0$, then $|(x, M \cdot y)|=\operatorname{SumAll}$ QuadraticForm $(x, M, y)$.
(45) For every finite sequence $x$ of elements of $\mathbb{R}$ holds $\mid(x$, len $x \mapsto 1) \mid=\sum x$.
(46) Let $x, y$ be finite sequences of elements of $\mathbb{R}$ and $M$ be a matrix over $\mathbb{R}$. If len $x=\operatorname{len} M$ and len $y=$ width $M$ and len $x>0$ and len $y>0$, then $|(x \cdot M, y)|=\operatorname{SumAll}$ QuadraticForm $(x, M, y)$.
(47) Let $x, y$ be finite sequences of elements of $\mathbb{R}$ and $M$ be a matrix over $\mathbb{R}$. If len $x=\operatorname{len} M$ and len $y=$ width $M$ and len $x>0$ and len $y>0$, then $|(x \cdot M, y)|=|(x, M \cdot y)|$.
(48) Let $x, y$ be finite sequences of elements of $\mathbb{R}$ and $M$ be a matrix over $\mathbb{R}$. If len $y=\operatorname{len} M$ and len $x=$ width $M$ and len $x>0$ and len $y>0$, then $|(M \cdot x, y)|=\left|\left(x, M^{\mathrm{T}} \cdot y\right)\right|$.
(49) Let $x, y$ be finite sequences of elements of $\mathbb{R}$ and $M$ be a matrix over $\mathbb{R}$. If len $y=\operatorname{len} M$ and len $x=$ width $M$ and len $x>0$ and len $y>0$, then $|(x, y \cdot M)|=\left|\left(x \cdot M^{\mathrm{T}}, y\right)\right|$.
(50) Let $x$ be a finite sequence of elements of $\mathbb{R}$ and $M$ be a matrix over $\mathbb{R}$. If len $x=\operatorname{len} M$ and $x=\operatorname{len} x \mapsto 1$, then for every $k$ such that $k \in \operatorname{Seg} \operatorname{len}(x \cdot M)$ holds $(x \cdot M)(k)=\sum\left(M_{\square, k}\right)$.
(51) Let $x$ be a finite sequence of elements of $\mathbb{R}$ and $M$ be a matrix over $\mathbb{R}$. Suppose len $x=$ width $M$ and width $M>0$ and $x=\operatorname{len} x \mapsto 1$. Let given $k$. If $k \in \operatorname{Seg} \operatorname{len}(M \cdot x)$, then $(M \cdot x)(k)=\sum \operatorname{Line}(M, k)$.
(52) Let $n$ be a non empty natural number. Then there exists a finite sequence $P$ of elements of $\mathbb{R}$ such that len $P=n$ and for every $i$ such that $i \in \operatorname{dom} P$ holds $P(i) \geq 0$ and $\sum P=1$.
Let $p$ be a finite sequence of elements of $\mathbb{R}$. We say that $p$ is finite probability distribution if and only if:
(Def. 5) For every $i$ such that $i \in \operatorname{dom} p$ holds $p(i) \geq 0$ and $\sum p=1$.
One can check that there exists a finite sequence of elements of $\mathbb{R}$ which is non empty and finite probability distribution.

One can prove the following propositions:
(53) Let $p$ be a non empty finite probability distribution finite sequence of elements of $\mathbb{R}$ and given $k$. If $k \in \operatorname{dom} p$, then $p(k) \leq 1$.
(54) For every non empty yielding matrix $M$ over $D$ holds $1 \leq$ len $M$ and $1 \leq$ width $M$.
Let $M$ be a matrix over $\mathbb{R}$. We say that $M$ is nonnegative if and only if:
(Def. 6) For all $i, j$ such that $\langle i, j\rangle \in$ the indices of $M$ holds $M_{i, j} \geq 0$.
Let $M$ be a matrix over $\mathbb{R}$. We say that $M$ is summable-to- 1 if and only if:
(Def. 7) SumAll $M=1$.
Let $M$ be a matrix over $\mathbb{R}$. We say that $M$ is joint probability if and only if:
(Def. 8) $\quad M$ is nonnegative and summable-to-1.
Let us mention that every matrix over $\mathbb{R}$ which is joint probability is also nonnegative and summable-to-1 and every matrix over $\mathbb{R}$ which is nonnegative and summable-to- 1 is also joint probability.

We now state the proposition
(55) Let $n, m$ be non empty natural numbers. Then there exists a matrix $M$ over $\mathbb{R}$ of dimension $n \times m$ such that $M$ is nonnegative and SumAll $M=1$.
One can check that there exists a matrix over $\mathbb{R}$ which is non empty yielding and joint probability.

Let $n, m$ be non empty natural numbers, let $D$ be a non empty set, and let $M$ be a matrix over $D$ of dimension $n \times m$. Observe that $M^{\mathrm{T}}$

Next we state two propositions:
(56) Let $M$ be a non empty yielding joint probability matrix over $\mathbb{R}$. Then $M^{\mathrm{T}}$ is a non empty yielding joint probability matrix over $\mathbb{R}$.
(57) Let $M$ be a non empty yielding joint probability matrix over $\mathbb{R}$ and given $i, j$. If $\langle i, j\rangle \in$ the indices of $M$, then $M_{i, j} \leq 1$.
Let $M$ be a matrix over $\mathbb{R}$. We say that $M$ has lines summable-to- 1 if and only if:
(Def. 9) For every $k$ such that $k \in \operatorname{dom} M$ holds $\sum M(k)=1$.
The following proposition is true
(58) For all non empty natural numbers $n, m$ holds there exists a matrix over $\mathbb{R}$ of dimension $n \times m$ which is nonnegative and has lines summable-to- 1 .
Let $M$ be a matrix over $\mathbb{R}$. We say that $M$ is conditional probability if and only if:
(Def. 10) $\quad M$ is nonnegative and has lines summable-to-1.
Let us observe that every matrix over $\mathbb{R}$ which is conditional probability is also nonnegative and has lines summable-to- 1 and every matrix over $\mathbb{R}$ which is nonnegative and has lines summable-to- 1 is also conditional probability.

Let us mention that there exists a matrix over $\mathbb{R}$ which is non empty yielding and conditional probability.

Next we state three propositions:
(59) Let $M$ be a non empty yielding conditional probability matrix over $\mathbb{R}$ and given $i, j$. If $\langle i, j\rangle \in$ the indices of $M$, then $M_{i, j} \leq 1$.
(60) Let $M$ be a non empty yielding matrix over $\mathbb{R}$. Then the following statements are equivalent
(i) $\quad M$ is a non empty yielding conditional probability matrix over $\mathbb{R}$,
(ii) for every $i$ such that $i \in \operatorname{dom} M$ holds Line $(M, i)$ is a non empty finite probability distribution finite sequence of elements of $\mathbb{R}$.
(61) For every non empty yielding matrix $M$ over $\mathbb{R}$ with lines summable-to-1 holds SumAll $M=\operatorname{len} M$.
Let $M$ be a matrix over $\mathbb{R}$. We introduce the row marginal $M$ as a synonym of LineSum $M$. We introduce the column marginal $M$ as a synonym of ColSum M.

Let $M$ be a non empty yielding joint probability matrix over $\mathbb{R}$. Note that the row marginal $M$ is non empty and finite probability distribution and the column marginal $M$ is non empty and finite probability distribution.

Let $M$ be a non empty yielding matrix over $\mathbb{R}$. Observe that $M^{\mathrm{T}}$ is non empty yielding.

Let $M$ be a non empty yielding joint probability matrix over $\mathbb{R}$. Note that $M^{\mathrm{T}}$ is joint probability.

The following propositions are true:
(62) Let $p$ be a non empty finite probability distribution finite sequence of elements of $\mathbb{R}$ and $P$ be a non empty yielding conditional probability matrix over $\mathbb{R}$. Suppose len $p=\operatorname{len} P$. Then $p \cdot P$ is a non empty finite probability distribution finite sequence of elements of $\mathbb{R}$ and $\operatorname{len}(p \cdot P)=$ width $P$.
(63) Let $P_{1}, P_{2}$ be non empty yielding conditional probability matrices over $\mathbb{R}$. Suppose width $P_{1}=\operatorname{len} P_{2}$. Then $P_{1} \cdot P_{2}$ is a non empty yielding conditional probability matrix over $\mathbb{R}$ and $\operatorname{len}\left(P_{1} \cdot P_{2}\right)=\operatorname{len} P_{1}$ and $\operatorname{width}\left(P_{1} \cdot P_{2}\right)=$ width $P_{2}$.

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# Several Differentiation Formulas of Special Functions. Part IV 

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#### Abstract

Summary. In this article, we give several differentiation formulas of special and composite functions including trigonometric function, polynomial function and logarithmic function.


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The notation and terminology used here are introduced in the following papers: [13], [15], [1], [16], [2], [4], [10], [11], [17], [5], [14], [12], [3], [7], [6], [9], and [8].

For simplicity, we adopt the following convention: $x, a, b, c$ denote real numbers, $n$ denotes a natural number, $Z$ denotes an open subset of $\mathbb{R}$, and $f$, $f_{1}, f_{2}$ denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:
(1) If $x \in \operatorname{dom}($ the function $\tan )$, then (the function $\cos )(x) \neq 0$.
(2) If $x \in \operatorname{dom}($ the function $\cot )$, then (the function $\sin )(x) \neq 0$.
(3) If $Z \subseteq \operatorname{dom}\left(\frac{f_{1}}{f_{2}}\right)$, then for every $x$ such that $x \in Z$ holds $\left(\frac{f_{1}}{f_{2}}\right)(x)_{\mathbb{Z}}^{n}=\frac{f_{1}(x)_{\mathbb{Z}}^{n}}{f_{2}(x)_{\mathbb{Z}}^{n}}$.
(4) Suppose $Z \subseteq \operatorname{dom}\left(\frac{f_{1}}{f_{2}}\right)$ and for every $x$ such that $x \in Z$ holds $f_{1}(x)=$ $x+a$ and $f_{2}(x)=x-b$. Then $\frac{f_{1}}{f_{2}}$ is differentiable on $Z$ and for every $x$ such that $x \in Z$ holds $\left(\frac{f_{1}}{f_{2}}\right)^{\prime}{ }_{Y}(x)=\frac{-a-b}{(x-b)^{2}}$.
(5) Suppose $Z \subseteq \operatorname{dom}\left((\right.$ the function $\left.\ln ) \cdot \frac{1}{f}\right)$ and for every $x$ such that $x \in Z$ holds $f(x)=x$. Then (the function $\ln$ ) $\cdot \frac{1}{f}$ is differentiable on $Z$ and for every $x$ such that $x \in Z$ holds $\left((\text { the function } \ln ) \cdot \frac{1}{f}\right)_{Y Z}^{\prime}(x)=-\frac{1}{x}$.
(6) Suppose $Z \subseteq \operatorname{dom}(($ the function $\tan ) \cdot f)$ and for every $x$ such that $x \in Z$ holds $f(x)=a \cdot x+b$. Then
(i) (the function $\tan ) \cdot f$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function tan) $\cdot f)^{\prime}{ }_{Y}(x)=$ $\frac{a}{\text { (the function } \cos )(a \cdot x+b)^{2}}$.
(7) Suppose $Z \subseteq \operatorname{dom}(($ the function cot) $\cdot f)$ and for every $x$ such that $x \in Z$ holds $f(x)=a \cdot x+b$. Then
(i) (the function cot) $\cdot f$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\cot ) \cdot f)^{\prime}{ }_{Z}(x)=$ $-\frac{a}{\text { (the function } \sin )(a \cdot x+b)^{2}}$.
(8) Suppose $Z \subseteq \operatorname{dom}\left((\right.$ the function $\left.\tan ) \cdot \frac{1}{f}\right)$ and for every $x$ such that $x \in Z$ holds $f(x)=x$. Then
(i) (the function $\tan ) \cdot \frac{1}{f}$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\left.\tan ) \cdot \frac{1}{f}\right)^{\prime}{ }_{Y}(x)=$ $-\frac{1}{\left.x^{2} \text {.(the function } \cos \right)\left(\frac{1}{x}\right)^{2}}$.
(9) Suppose $Z \subseteq \operatorname{dom}\left((\right.$ the function cot $\left.) \cdot \frac{1}{f}\right)$ and for every $x$ such that $x \in Z$ holds $f(x)=x$. Then
(i) (the function cot) $\cdot \frac{1}{f}$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function cot) $\left.\cdot \frac{1}{f}\right)^{\prime}{ }_{Y}(x)=$ $\frac{1}{\left.x^{2} \text {.(the function } \sin \right)\left(\frac{1}{x}\right)^{2}}$.
(10) Suppose $Z \subseteq \operatorname{dom}\left(\left(\right.\right.$ the function tan) $\left.\cdot\left(f_{1}+c f_{2}\right)\right)$ and $f_{2}={ }_{\mathbb{Z}}^{2}$ and for every $x$ such that $x \in Z$ holds $f_{1}(x)=a+b \cdot x$. Then
(i) (the function tan) $\cdot\left(f_{1}+c f_{2}\right)$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds $\left((\text { the function } \tan ) \cdot\left(f_{1}+c f_{2}\right)\right)_{Z}^{\prime}(x)=$ $\frac{b+2 \cdot c \cdot x}{\left(\text { the function cos) }\left(a+b \cdot x+c \cdot x^{2}\right)^{2}\right.}$.
(11) Suppose $Z \subseteq \operatorname{dom}\left(\left(\right.\right.$ the function cot) $\left.\cdot\left(f_{1}+c f_{2}\right)\right)$ and $f_{2}=\frac{2}{\mathbb{Z}}$ and for every $x$ such that $x \in Z$ holds $f_{1}(x)=a+b \cdot x$. Then
(i) (the function cot) $\cdot\left(f_{1}+c f_{2}\right)$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function cot) $\left.\cdot\left(f_{1}+c f_{2}\right)\right)^{\prime}{ }_{Z}^{\prime}(x)=$ $-\frac{b+2 \cdot c \cdot x}{(\text { the function } \sin )\left(a+b \cdot x+c \cdot x^{2}\right)^{2}}$.
(12) Suppose $Z \subseteq \operatorname{dom}(($ the function $\tan ) \cdot($ the function $\exp ))$. Then
(i) (the function $\tan ) \cdot($ the function $\exp$ ) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\tan$ ) (the function $\exp ))^{\prime}(x)=\frac{(\text { the function } \exp )(x)}{(\text { the function cos) (the function exp)(x) })^{2}}$.
(13) Suppose $Z \subseteq \operatorname{dom}(($ the function cot) $\cdot($ the function $\exp ))$. Then
(i) (the function cot) •(the function $\exp$ ) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function cot) •(the function $\exp ))^{\prime}{ }_{Z}(x)=-\frac{(\text { the function } \exp )(x)}{(\text { the function sin) })(\text { (the function } \exp )(x))^{2}}$.
(14) Suppose $Z \subseteq \operatorname{dom}(($ the function $\tan ) \cdot($ the function $\ln ))$. Then
(i) (the function $\tan ) \cdot($ the function $\ln$ ) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function tan) •(the function $\ln ))^{\prime}{ }^{\prime}(x)=\frac{1}{x \cdot(\text { the function } \cos )((\text { the function } \ln )(x))^{2}}$.
(15) Suppose $Z \subseteq \operatorname{dom}(($ the function cot) $\cdot($ the function $\ln ))$. Then
(i) (the function cot) • (the function $\ln$ ) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function cot) $\cdot$ (the function $\ln ))^{{ }_{\gamma}}(x)=-\frac{1}{x \cdot(\text { the function } \sin )((\text { the function } \ln )(x))^{2}}$.
(16) Suppose $Z \subseteq \operatorname{dom}(($ the function $\exp ) \cdot($ the function tan $))$. Then
(i) (the function $\exp ) \cdot($ the function $\tan )$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\exp )$ •(the function $\tan ))^{\prime}{ }_{Z}(x)=\frac{(\text { the function } \exp )((\text { the function } \tan )(x))}{\text { (the function } \cos )(x)^{2}}$.
(17) Suppose $Z \subseteq \operatorname{dom}(($ the function $\exp ) \cdot($ the function cot $))$. Then
(i) (the function $\exp ) \cdot($ the function cot) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\exp ) \cdot$ (the function $\cot ))^{\dagger}(x)=-\frac{\text { (the function exp) }(\text { (the function } \cot )(x))}{\text { (the function sin) }(x)^{2}}$.
(18) Suppose $Z \subseteq \operatorname{dom}(($ the function $\ln ) \cdot($ the function $\tan ))$. Then
(i) (the function $\ln$ ) •(the function $\tan )$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\ln ) \cdot($ the function $\tan ))^{\prime}{ }^{\prime}(x)=\frac{1}{(\text { the function } \cos )(x) \cdot(\text { the function } \sin )(x)}$.
(19) Suppose $Z \subseteq \operatorname{dom}(($ the function $\ln ) \cdot($ the function cot $))$. Then
(i) (the function $\ln ) \cdot($ the function cot) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\ln ) \cdot($ the function $\cot ))^{\prime}{ }_{Z}(x)=-\frac{1}{(\text { the function sin) }(x) \cdot(\text { the function } \cos )(x)}$.
(20) Suppose $Z \subseteq \operatorname{dom}\left(\left(_{\mathbb{Z}}^{n}\right) \cdot(\right.$ the function $\left.\tan )\right)$ and $1 \leq n$. Then
(i) $\binom{n}{\mathbb{Z}} \cdot($ the function $\tan )$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds $\left(\binom{n}{\mathbb{Z}} \cdot(\text { the function } \tan )\right)^{\prime}{ }_{Z}(x)=$ $\frac{n \cdot(\text { the function } \sin )(x)_{\mathbb{Z}}^{n-1}}{\text { (the function } \cos )(x)_{\mathbb{Z}}^{n+1}}$.
(21) Suppose $Z \subseteq \operatorname{dom}((\underset{\mathbb{Z}}{n}) \cdot($ the function cot) $)$ and $1 \leq n$. Then
(i) $\binom{n}{\mathbb{Z}} \cdot($ the function cot) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds $\left(\left(\begin{array}{l}\mathbb{Z}\end{array}\right) \cdot(\text { the function } \cot )\right)^{\prime}{ }_{Z}(x)=$ $-\frac{n \cdot(\text { the function } \cos )(x)_{\mathbb{Z}}^{n-1}}{(\text { the function } \sin )(x)_{\mathbb{Z}}^{n+1}}$.
(22) Suppose that
(i) $Z \subseteq \operatorname{dom}\left((\right.$ the function $\left.\tan )+\frac{1}{\text { the function cos }}\right)$, and
(ii) for every $x$ such that $x \in Z$ holds $1+($ the function $\sin )(x) \neq 0$ and $1-($ the function $\sin )(x) \neq 0$.
Then
(iii) (the function $\tan )+\frac{1}{\text { the function cos }}$ is differentiable on $Z$, and
(iv) for every $x$ such that $x \in Z$ holds $\left((\text { the function } \tan )+\frac{1}{\text { the function } \cos }\right)^{\prime}{ }_{Y}(x)=$ $\frac{1}{1-(\text { the function } \sin )(x)}$.
(23) Suppose that
(i) $Z \subseteq \operatorname{dom}\left((\right.$ the function $\left.\tan )-\frac{1}{\text { the function cos }}\right)$, and
(ii) for every $x$ such that $x \in Z$ holds $1-$ (the function $\sin )(x) \neq 0$ and $1+($ the function $\sin )(x) \neq 0$.
Then
(iii) (the function $\tan$ ) $-\frac{1}{\text { the function cos }}$ is differentiable on $Z$, and
(iv) for every $x$ such that $x \in Z$ holds $\left((\text { the function } \tan )-\frac{1}{\text { the function cos }}\right)^{\prime}{ }_{Z}(x)=$ $\frac{1}{1+(\text { the } \text { function } \sin )(x)}$.
(24) Suppose $Z \subseteq \operatorname{dom}\left((\right.$ the function $\left.\tan )-\mathrm{id}_{Z}\right)$. Then
(i) (the function $\tan )-\mathrm{id}_{Z}$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\left.\tan )-\mathrm{id}_{Z}\right)_{\mid Z}^{\prime}(x)=$ $\frac{(\text { the function } \sin )(x)^{2}}{(\text { the }}$.
(25) Suppose $Z \subseteq \operatorname{dom}\left(-\right.$ the function $\left.\cot -\mathrm{id}_{Z}\right)$. Then
(i) -the function $\cot -\mathrm{id}_{Z}$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds (-the function $\left.\cot -\operatorname{id}_{Z}\right)_{\mid Z}^{\prime}(x)=$ $\frac{(\text { the function } \cos )(x)^{2}}{(\text { the function } \sin )(x)^{2}}$.
(26) Suppose $Z \subseteq \operatorname{dom}\left(\frac{1}{a}((\right.$ the function $\left.\tan ) \cdot f)-\operatorname{id}_{Z}\right)$ and for every $x$ such that $x \in Z$ holds $f(x)=a \cdot x$ and $a \neq 0$. Then
(i) $\frac{1}{a}(($ the function $\tan ) \cdot f)-\mathrm{id}_{Z}$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds $\left(\frac{1}{a}((\right.$ the function tan $) \cdot f)-$ $\left.\mathrm{id}_{Z}\right)^{\prime}{ }_{Y}(x)=\frac{(\text { the function } \sin )(a \cdot x)^{2}}{(\text { (the function } \cos )(a \cdot x)^{2}}$.
(27) Suppose $Z \subseteq \operatorname{dom}\left(\left(-\frac{1}{a}\right)((\right.$ the function $\left.\cot ) \cdot f)-\operatorname{id}_{Z}\right)$ and for every $x$ such that $x \in Z$ holds $f(x)=a \cdot x$ and $a \neq 0$. Then
(i) $\quad\left(-\frac{1}{a}\right)(($ the function cot $) \cdot f)-\mathrm{id}_{Z}$ is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds $\left(\left(-\frac{1}{a}\right)((\right.$ the function $\cot ) \cdot f)-$ $\left.\operatorname{id}_{Z}\right)^{\prime}{ }_{Y}(x)=\frac{(\text { the function } \cos )(a \cdot x)^{2}}{\text { (the function sin) }(a \cdot x)^{2}}$.
(28) Suppose $Z \subseteq \operatorname{dom}(f$ (the function $\tan ))$ and for every $x$ such that $x \in Z$ holds $f(x)=a \cdot x+b$. Then
(i) $\quad f$ (the function $\tan$ ) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds $(f(\text { the function } \tan ))^{\prime}{ }_{Y}(x)=$ $\frac{a \cdot(\text { the function } \sin )(x)}{\text { (the function } \cos )(x)}+\frac{a \cdot x+b}{(\text { the function } \cos )(x)^{2}}$.
(29) Suppose $Z \subseteq \operatorname{dom}(f$ (the function cot)) and for every $x$ such that $x \in Z$ holds $f(x)=a \cdot x+b$. Then
(i) $f$ (the function cot) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds $(f$ (the function $\cot ))^{\prime}{ }_{Z}(x)=$ $\frac{a \cdot(\text { the } \text { function } \cos )(x)}{(\text { the function } \sin )(x)}-\frac{a \cdot x+b}{(\text { the function } \sin )(x)^{2}}$.
(30) Suppose $Z \subseteq \operatorname{dom}(($ the function $\exp )$ (the function tan)). Then
(i) (the function $\exp$ ) (the function $\tan$ ) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function exp) (the function $\tan ))_{\mid Z}^{\prime}(x)=\frac{(\text { the function exp })(x) \cdot(\text { the function } \sin )(x)}{(\text { the function } \cos )(x)}+\frac{\text { (the function } \exp )(x)}{\left(\text { the function cos) }(x)^{2}\right.}$.
(31) Suppose $Z \subseteq \operatorname{dom}(($ the function $\exp )$ (the function cot)). Then
(i) (the function $\exp$ ) (the function cot) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function exp) (the function $\cot ))^{\prime}(x)=\frac{(\text { the function } \exp )(x) \cdot(\text { the function } \cos )(x)}{(\text { the function } \sin )(x)}-\frac{(\text { the function } \exp )(x)}{(\text { the function } \sin )(x)^{2}}$.
(32) Suppose $Z \subseteq \operatorname{dom}(($ the function $\ln )$ (the function $\tan )$ ). Then
(i) (the function $\ln$ ) (the function $\tan$ ) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\ln$ ) (the function $\tan ))^{\prime}(x)=\frac{\frac{(\text { the function } \sin )(x)}{(\text { (the function } \cos )(x)}}{x}+\frac{(\text { the function } \ln )(x)}{(\text { the function } \cos )(x)^{2}}$.
(33) Suppose $Z \subseteq \operatorname{dom}(($ the function $\ln )$ (the function cot)). Then
(i) (the function $\ln$ ) (the function cot) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds ((the function $\ln$ ) (the function $\cot ))^{\prime}(x)=\frac{\frac{(\text { the function } \cos )(x)}{(\text { the function sin) })(x)}}{x}-\frac{(\text { the function } \ln )(x)}{(\text { the function } \sin )(x)^{2}}$.
(34) Suppose $Z \subseteq \operatorname{dom}\left(\frac{1}{f}\right.$ (the function $\left.\left.\tan \right)\right)$ and for every $x$ such that $x \in Z$ holds $f(x)=x$. Then
(i) $\frac{1}{f}$ (the function $\tan$ ) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds $\left(\frac{1}{f}(\text { the function } \tan )\right)^{\prime}{ }_{Z}(x)=$ $-\frac{\frac{(\text { the function } \sin )(x)}{(\text { the function } \cos )(x)}}{x^{2}}+\frac{\frac{1}{x}}{(\text { the function } \cos )(x)^{2}}$.
(35) Suppose $Z \subseteq \operatorname{dom}\left(\frac{1}{f}\right.$ (the function cot)) and for every $x$ such that $x \in Z$ holds $f(x)=x$. Then
(i) $\frac{1}{f}$ (the function cot) is differentiable on $Z$, and
(ii) for every $x$ such that $x \in Z$ holds $\left(\frac{1}{f}(\text { the function } \cot )\right)^{\prime}{ }_{Y}(x)=$ $-\frac{\frac{(\text { the function } \cos )(x)}{(\text { the function } \sin )(x)}}{x^{2}}-\frac{\frac{1}{x}}{(\text { the function } \sin )(x)^{2}}$.

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# Difference and Difference Quotient 

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#### Abstract

Summary. In this article, we give the definitions of forward difference, backward difference, central difference and difference quotient, and some of their important properties.


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The articles [2], [6], [1], [13], [16], [17], [14], [4], [5], [9], [8], [12], [18], [7], [15], [11], [10], [3], and [19] provide the terminology and notation for this paper.

For simplicity, we follow the rules: $n, m, i$ are elements of $\mathbb{N}, h, r, r_{1}, r_{2}$, $x_{0}, x_{1}, x_{2}, x$ are real numbers, $f$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$, and $S$ is a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$.

Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. The functor $\operatorname{Shift}(f, h)$ yields a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:
(Def. 1) $\quad \operatorname{dom} \operatorname{Shift}(f, h)=-h+\operatorname{dom} f$ and for every $x$ such that $x \in-h+\operatorname{dom} f$ holds $(\operatorname{Shift}(f, h))(x)=f(x+h)$.
Let $f$ be a function from $\mathbb{R}$ into $\mathbb{R}$ and let $h$ be a real number. Then $\operatorname{Shift}(f, h)$ is a function from $\mathbb{R}$ into $\mathbb{R}$ and it can be characterized by the condition:
(Def. 2) For every $x$ holds $(\operatorname{Shift}(f, h))(x)=f(x+h)$.
Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. The functor $\mathrm{fD}(f, h)$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:
(Def. 3) $\quad \mathrm{fD}(f, h)=\operatorname{Shift}(f, h)-f$.
Let $f$ be a function from $\mathbb{R}$ into $\mathbb{R}$ and let $h$ be a real number. Then $\mathrm{fD}(f, h)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. The functor $\mathrm{bD}(f, h)$ yields a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:
(Def. 4) $\mathrm{bD}(f, h)=f-\operatorname{Shift}(f,-h)$.
Let $f$ be a function from $\mathbb{R}$ into $\mathbb{R}$ and let $h$ be a real number. Then $\mathrm{bD}(f, h)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

We now state the proposition
(1) $\mathrm{bD}(f, h)=-\mathrm{fD}(f,-h)$.

Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. The functor $\mathrm{c}(f, h)$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by:
(Def. 5) $\quad \mathrm{cD}(f, h)=\operatorname{Shift}\left(f, \frac{h}{2}\right)-\operatorname{Shift}\left(f,-\frac{h}{2}\right)$.
Let $f$ be a function from $\mathbb{R}$ into $\mathbb{R}$ and let $h$ be a real number. Then $\mathrm{cD}(f, h)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. The forward difference of $f$ and $h$ yields a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ and is defined by the conditions (Def. 6).
(Def. 6)(i) (The forward difference of $f$ and $h)(0)=f$, and
(ii) for every $n$ holds (the forward difference of $f$ and $h)(n+1)=\mathrm{fD}($ (the forward difference of $f$ and $h)(n), h)$.
Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. We introduce $\operatorname{fdif}(f, h)$ as a synonym of the forward difference of $f$ and $h$.

In the sequel $f, f_{1}, f_{2}$ denote functions from $\mathbb{R}$ into $\mathbb{R}$.
The following propositions are true:
(2) For every $n$ holds $(f d i f(f, h))(n)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.
(3) For every $x$ holds $(\mathrm{fD}(f, h))(x)=f(x+h)-f(x)$.
(4) For every $x$ holds $(\mathrm{bD}(f, h))(x)=f(x)-f(x-h)$.
(5) For every $x$ holds $(\mathrm{cD}(f, h))(x)=f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)$.
(6) If $f$ is constant, then for every $x$ holds $(\operatorname{fdif}(f, h))(n+1)(x)=0$.
(7) $\quad(\operatorname{fdif}(r f, h))(n+1)(x)=r \cdot(\operatorname{fdif}(f, h))(n+1)(x)$.
(8) $\quad\left(\operatorname{fdif}\left(f_{1}+f_{2}, h\right)\right)(n+1)(x)=\left(\operatorname{fdif}\left(f_{1}, h\right)\right)(n+1)(x)+\left(\operatorname{fdif}\left(f_{2}, h\right)\right)(n+1)(x)$.
(9) $\quad\left(\operatorname{fdif}\left(f_{1}-f_{2}, h\right)\right)(n+1)(x)=\left(\operatorname{fdif}\left(f_{1}, h\right)\right)(n+1)(x)-\left(f d i f\left(f_{2}, h\right)\right)(n+1)(x)$.
(10) If $f=r_{1} f_{1}+r_{2} f_{2}$, then for every $x$ holds $(\operatorname{fdif}(f, h))(n+1)(x)=$ $r_{1} \cdot\left(\operatorname{fdif}\left(f_{1}, h\right)\right)(n+1)(x)+r_{2} \cdot\left(\operatorname{fdif}\left(f_{2}, h\right)\right)(n+1)(x)$.
(11) For every $x$ holds $(\operatorname{fdif}(f, h))(1)(x)=(\operatorname{Shift}(f, h))(x)-f(x)$.

Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. The backward difference of $f$ and $h$ yielding a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ is defined by the conditions (Def. 7).
(Def. 7)(i) (The backward difference of $f$ and $h)(0)=f$, and
(ii) for every $n$ holds (the backward difference of $f$ and $h)(n+1)=\mathrm{bD}(($ the backward difference of $f$ and $h)(n), h)$.
Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. We introduce $\operatorname{bdif}(f, h)$ as a synonym of the backward difference of $f$ and $h$.

We now state several propositions:
(12) For every $n$ holds $(\operatorname{bdif}(f, h))(n)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.
(13) If $f$ is constant, then for every $x \operatorname{holds}(\operatorname{bdif}(f, h))(n+1)(x)=0$.
(14) $\quad(\operatorname{bdif}(r f, h))(n+1)(x)=r \cdot(\operatorname{bdif}(f, h))(n+1)(x)$.
(15) $\quad\left(\operatorname{bdif}\left(f_{1}+f_{2}, h\right)\right)(n+1)(x)=\left(\operatorname{bdif}\left(f_{1}, h\right)\right)(n+1)(x)+\left(\operatorname{bdif}\left(f_{2}, h\right)\right)(n+$ 1) $(x)$.
(16) $\quad\left(\operatorname{bdif}\left(f_{1}-f_{2}, h\right)\right)(n+1)(x)=\left(\operatorname{bdif}\left(f_{1}, h\right)\right)(n+1)(x)-\left(\operatorname{bdif}\left(f_{2}, h\right)\right)(n+$ 1) $(x)$.
(17) If $f=r_{1} f_{1}+r_{2} f_{2}$, then for every $x$ holds $(\operatorname{bdif}(f, h))(n+1)(x)=$ $r_{1} \cdot\left(\operatorname{bdif}\left(f_{1}, h\right)\right)(n+1)(x)+r_{2} \cdot\left(\operatorname{bdif}\left(f_{2}, h\right)\right)(n+1)(x)$.
(18) $\quad(\operatorname{bdif}(f, h))(1)(x)=f(x)-(\operatorname{Shift}(f,-h))(x)$.

Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. The central difference of $f$ and $h$ yielding a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ is defined by the conditions (Def. 8).
(Def. 8)(i) (The central difference of $f$ and $h)(0)=f$, and
(ii) for every $n$ holds (the central difference of $f$ and $h)(n+1)=\mathrm{c}(($ the central difference of $f$ and $h)(n), h)$.
Let $f$ be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let $h$ be a real number. We introduce cdif $(f, h)$ as a synonym of the central difference of $f$ and $h$.

One can prove the following propositions:
(19) For every $n$ holds $(\operatorname{cdif}(f, h))(n)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.
(20) If $f$ is constant, then for every $x$ holds $(\operatorname{cdif}(f, h))(n+1)(x)=0$.
(21) $\quad(\operatorname{cdif}(r f, h))(n+1)(x)=r \cdot(\operatorname{cdif}(f, h))(n+1)(x)$.
(22) $\quad\left(\operatorname{cdif}\left(f_{1}+f_{2}, h\right)\right)(n+1)(x)=\left(\operatorname{cdif}\left(f_{1}, h\right)\right)(n+1)(x)+\left(\operatorname{cdif}\left(f_{2}, h\right)\right)(n+$ 1) $(x)$.
(23) $\quad\left(\operatorname{cdif}\left(f_{1}-f_{2}, h\right)\right)(n+1)(x)=\left(\operatorname{cdif}\left(f_{1}, h\right)\right)(n+1)(x)-\left(\operatorname{cdif}\left(f_{2}, h\right)\right)(n+$ 1) $(x)$.
(24) If $f=r_{1} f_{1}+r_{2} f_{2}$, then for every $x \operatorname{holds}(\operatorname{cdif}(f, h))(n+1)(x)=$ $r_{1} \cdot\left(\operatorname{cdif}\left(f_{1}, h\right)\right)(n+1)(x)+r_{2} \cdot\left(\operatorname{cdif}\left(f_{2}, h\right)\right)(n+1)(x)$.
(25) $\quad(\operatorname{cdif}(f, h))(1)(x)=\left(\operatorname{Shift}\left(f, \frac{h}{2}\right)\right)(x)-\left(\operatorname{Shift}\left(f,-\frac{h}{2}\right)\right)(x)$.
(26) $\quad(\operatorname{fdif}(f, h))(n)(x)=(\operatorname{bdif}(f, h))(n)(x+n \cdot h)$.
(27) $\quad(\operatorname{fdif}(f, h))(2 \cdot n)(x)=(\operatorname{cdif}(f, h))(2 \cdot n)(x+n \cdot h)$.
(28) $\quad(\operatorname{fdif}(f, h))(2 \cdot n+1)(x)=(\operatorname{cdif}(f, h))(2 \cdot n+1)\left(x+n \cdot h+\frac{h}{2}\right)$.

Let $f$ be a function from $\mathbb{R}$ into $\mathbb{R}$ and let us consider $x_{0}, x_{1}$. The functor $\Delta\left(f, x_{0}, x_{1}\right)$ yielding a real number is defined as follows:
(Def. 9)(i) $\Delta\left(f, x_{0}, x_{1}\right)=\frac{f\left(x_{0}\right)-f\left(x_{1}\right)}{x_{0}-x_{1}}$ if $x_{0} \neq x_{1}$,
(ii) $x_{0} \neq x_{1}$, otherwise.

Let $x_{0}, x_{1}, x_{2}$ be real numbers and let $f$ be a function from $\mathbb{R}$ into $\mathbb{R}$. The functor $\left[!f, x_{0}, x_{1}, x_{2}!\right]$ yielding a real number is defined as follows:
(Def. 10)(i) $\quad\left[!f, x_{0}, x_{1}, x_{2}!\right]=\frac{\Delta\left(f, x_{0}, x_{1}\right)-\Delta\left(f, x_{1}, x_{2}\right)}{x_{0}-x_{2}}$ if $x_{0} \neq x_{2}$,
(ii) $x_{0} \neq x_{2}$, otherwise.

Let $x_{0}, x_{1}, x_{2}, x_{3}$ be real numbers and let $f$ be a function from $\mathbb{R}$ into $\mathbb{R}$. The functor $\left[!f, x_{0}, x_{1}, x_{2}, x_{3}!\right]$ yielding a real number is defined by:
(Def. 11)(i) $\quad\left[!f, x_{0}, x_{1}, x_{2}, x_{3}!\right]=\frac{\left[!f, x_{0}, x_{1}, x_{2}!!-\left[!f, x_{1}, x_{2}, x_{3}!\right]\right.}{x_{0}-x_{3}}$ if $x_{0} \neq x_{3}$,
(ii) $x_{0} \neq x_{3}$, otherwise.

We now state several propositions:
(29) If $x_{0} \neq x_{1}$, then $\Delta\left(f, x_{0}, x_{1}\right)=\Delta\left(f, x_{1}, x_{0}\right)$.
(30) If $f$ is constant and $x_{0} \neq x_{1}$, then $\Delta\left(f, x_{0}, x_{1}\right)=0$.
(31) If $x_{0} \neq x_{1}$, then $\Delta\left(r f, x_{0}, x_{1}\right)=r \cdot \Delta\left(f, x_{0}, x_{1}\right)$.
(32) If $x_{0} \neq x_{1}$, then $\Delta\left(f_{1}+f_{2}, x_{0}, x_{1}\right)=\Delta\left(f_{1}, x_{0}, x_{1}\right)+\Delta\left(f_{2}, x_{0}, x_{1}\right)$.
(33) If $x_{0} \neq x_{1}$, then $\Delta\left(r_{1} f_{1}+r_{2} f_{2}, x_{0}, x_{1}\right)=r_{1} \cdot \Delta\left(f_{1}, x_{0}, x_{1}\right)+r_{2}$. $\Delta\left(f_{2}, x_{0}, x_{1}\right)$.
(34) If $x_{0} \neq x_{1}$ and $x_{0} \neq x_{2}$ and $x_{1} \neq x_{2}$, then $\left[!f, x_{0}, x_{1}, x_{2}!\right]=\left[!f, x_{1}, x_{2}, x_{0}\right.$ !] and $\left[!f, x_{0}, x_{1}, x_{2}!\right]=\left[!f, x_{2}, x_{1}, x_{0}!\right]$.
(35) If $x_{0} \neq x_{1}$ and $x_{0} \neq x_{2}$ and $x_{1} \neq x_{2}$, then $\left[!f, x_{0}, x_{1}, x_{2}!\right]=\left[!f, x_{2}, x_{0}, x_{1}!\right]$ and $\left[!f, x_{0}, x_{1}, x_{2}!\right]=\left[!f, x_{1}, x_{0}, x_{2}!\right]$.
(36) $\quad(\operatorname{fdif}((f d i f(f, h))(m), h))(n)(x)=(f d i f(f, h))(m+n)(x)$.

Let us consider $S$. We say that $S$ is sequence-yielding if and only if:
(Def. 12) For every $n$ holds $S(n)$ is a sequence of real numbers.
Let us note that there exists a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ which is sequence-yielding.

A seq sequence is a sequence-yielding sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$.

Let $S$ be a seq sequence and let us consider $n$. Then $S(n)$ is a sequence of real numbers.

In the sequel $S$ denotes a seq sequence.
Next we state the proposition
(37) Suppose that for every $n$ and for every $i$ such that $i \leq n$ holds $S(n)(i)=\binom{n}{i} \cdot\left(\operatorname{fdif}\left(f_{1}, h\right)\right)(i)(x) \cdot\left(f d i f\left(f_{2}, h\right)\right)\left(n-^{\prime} i\right)(x+i \cdot h)$. Then $\left(\operatorname{fdif}\left(f_{1} f_{2}, h\right)\right)(1)(x)=\sum_{\kappa=0}^{1} S(1)(\kappa)$ and $\left(f d i f\left(f_{1} f_{2}, h\right)\right)(2)(x)=$ $\sum_{\kappa=0}^{2} S(2)(\kappa)$.

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## Contents

Simple Continued Fractions and Their Convergents
By Bo Li and Yan Zhang and Artur Kornilowicz ..... 71
Chordal Graphs
By Broderick Arneson and Piotr Rudnicki ..... 79
Connectedness and Continuous Sequences in Finite Topological SpacesBy Yatsuka Nakamura93
The Definition of Finite Sequences and Matrices of Probability, and Addition of Matrices of Real Elements
By Bo Zhang and Yatsuka Nakamura ..... 101
Several Differentiation Formulas of Special Functions. Part IV
By Bo Li and Peng Wang ..... 109
Difference and Difference Quotient
By Bo Li and Yan Zhang and Xiquan Liang ..... 115


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