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Partial Differentiation of Vector-Valued Functions on n-Dimensional Real Normed Linear Spaces

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Summary. In this article, we define and develop partial differentiation of vector-valued functions on n-dimensional real normed linear spaces (refer to [19] and [20]).

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The notation and terminology used in this paper have been introduced in the following papers: [7], [15], [2], [3], [24], [4], [5], [1], [11], [16], [6], [9], [12], [17], [18], [10], [8], [23], [14], [21], [13], and [22].

For simplicity, we use the following convention: n, m denote non empty elements of $\mathbb{N}$, i, j denote elements of $\mathbb{N}$, f denotes a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, g denotes a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, h denotes a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, x denotes a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, y denotes an element of $\mathcal{R}^m$, and X denotes a set.

We now state a number of propositions:

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(1) If $i \leq j$, then $\langle \underbrace{0, \ldots, 0}_{j} \rangle \restriction i = \langle \underbrace{0, \ldots, 0}_{i} \rangle$.

(2) If $i \leq j$, then $\langle \underbrace{0, \ldots, 0}_{j} \rangle \restriction (i -' 1) = \langle \underbrace{0, \ldots, 0}_{i -' 1} \rangle$.

(3) $\langle \underbrace{0, \ldots, 0}_{j} \rangle_{\mid i} = \langle \underbrace{0, \ldots, 0}_{j -' i} \rangle$.

(4) If $i \leq j$, then $\langle \underbrace{0, \ldots, 0}_{j} \rangle \restriction (i -' 1) = \langle \underbrace{0, \ldots, 0}_{i -' 1} \rangle$ and $\langle \underbrace{0, \ldots, 0}_{j} \rangle_{\mid i} = \langle \underbrace{0, \ldots, 0}_{j -' i} \rangle$.

(5) For every element x_1 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $1 \leq i \leq j$ holds $\|(\mathrm{reproj}(i, 0_{\langle \mathcal{E}^j, \| \cdot \| \rangle}))(x_1)\| = \|x_1\|$.

(6) Let m, i be elements of $\mathbb{N}$, x be an element of $\mathcal{R}^m$, and r be a real number. Then $(\mathrm{reproj}(i, x))(r) - x = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(r - (\mathrm{proj}(i, m))(x))$ and $x - (\mathrm{reproj}(i, x))(r) = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))((\mathrm{proj}(i, m))(x) - r)$.

(7) Let m, i be elements of $\mathbb{N}$, x be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and p be a point of $\langle \mathcal{E}^1, \| \cdot \| \rangle$. Then $(\mathrm{reproj}(i, x))(p) - x = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(p - (\mathrm{Proj}(i, m))(x))$ and $x - (\mathrm{reproj}(i, x))(p) = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))((\mathrm{Proj}(i, m))(x) - p)$.

(8) Let m, n be non empty elements of $\mathbb{N}$, i be an element of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and Z be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose Z is open and $1 \leq i \leq m$. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \mathrm{dom}\, f$ and for every point x of $\langle \mathcal{E}^m, \| \cdot \| \rangle$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i.

(9) For all elements x, y of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $\mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x + y) = \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x) + \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, y)$.

(10) For all elements x, a of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $\mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, a \cdot x) = a \cdot \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x)$.

(11) For every element x of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ and $x \neq 0$ holds $\mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x) \neq \langle \underbrace{0, \ldots, 0}_{m} \rangle$.

(12) Let x, y be elements of $\mathbb{R}$, z be an element of $\mathcal{R}^m$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq m$ and $y = (\mathrm{proj}(i, m))(z)$. Then $\mathrm{Replace}(z, i, x) - z = \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x - y)$ and $z - \mathrm{Replace}(z, i, x) = \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, y - x)$.

(13) For all elements x, y of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $(\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x + y) = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x) + (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(y)$.

(14) For all points x, y of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $(\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x + y) = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x) + (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(y)$.

(15) For all elements x, a of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $(\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(a \cdot x) = a \cdot (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x)$.

(16) Let x be a point of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a be an element of $\mathbb{R}$, and i be an element of $\mathbb{N}$. If $1 \leq i \leq m$, then $(\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(a \cdot x) = a \cdot (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x)$.

(17) For every element x of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ and $x \neq 0$ holds $(\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x) \neq \langle \underbrace{0, \ldots, 0}_{m} \rangle$.

(18) For every point x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ and $x \neq 0_{\langle \mathcal{E}^1, \| \cdot \| \rangle}$ holds $(\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x) \neq 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}$.

(19) Let x, y be elements of $\mathbb{R}$, z be an element of $\mathcal{R}^m$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq m$ and $y = (\mathrm{proj}(i, m))(z)$. Then $(\mathrm{reproj}(i, z))(x) - z = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x - y)$ and $z - (\mathrm{reproj}(i, z))(x) = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(y - x)$.

(20) Let x, y be points of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, i be an element of $\mathbb{N}$, and z be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose $1 \leq i \leq m$ and $y = (\mathrm{Proj}(i, m))(z)$. Then $(\mathrm{reproj}(i, z))(x) - z = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x - y)$ and $z - (\mathrm{reproj}(i, z))(x) = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(y - x)$.

(21) Suppose f is differentiable in x and $1 \leq i \leq m$. Then f is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f, x, i) = f'(x) \cdot \mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle})$.

(22) Suppose g is differentiable in y and $1 \leq i \leq m$. Then g is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(g, y, i) = (g'(y) \cdot \mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(\langle 1 \rangle)$.

Let n be a non empty element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. We say that f is differentiable in x if and only if:

(Def. 1) $\langle f \rangle$ is differentiable in x.

Let n be a non empty element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. The functor $f'(x)$ yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined as follows:

(Def. 2) $f'(x) = \mathrm{proj}(1, 1) \cdot \langle f \rangle'(x)$.

Next we state several propositions:

(23) Suppose h is differentiable in y and $1 \leq i \leq m$. Then h is partially differentiable in y w.r.t. i and
partdiff$(h, y, i) = (h \cdot \mathrm{reproj}(i, y))'((\mathrm{proj}(i, m))(y))$ and
partdiff$(h, y, i) = h'(y)((\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(1))$.

(24) Let m be a non empty element of $\mathbb{N}$ and v, w, u be finite sequences of elements of $\mathcal{R}^m$. If $\mathrm{dom}\, v = \mathrm{dom}\, w$ and $u = v + w$, then $\sum u = \sum v + \sum w$.

(25) Let m be a non empty element of $\mathbb{N}$, r be a real number, and w, u be finite sequences of elements of $\mathcal{R}^m$. If $u = r\, w$, then $\sum u = r \cdot \sum w$.

(26) Let n be a non empty element of $\mathbb{N}$ and h, g be finite sequences of elements of $\mathcal{R}^n$. Suppose $\mathrm{len}\, h = \mathrm{len}\, g + 1$ and for every natural number i such that $i \in \mathrm{dom}\, g$ holds $g_i = h_i - h_{i+1}$. Then $h_1 - h_{\mathrm{len}\, h} = \sum g$.

(27) Let n be a non empty element of $\mathbb{N}$ and h, g, j be finite sequences of elements of $\mathcal{R}^n$. Suppose $\mathrm{len}\, h = \mathrm{len}\, j$ and $\mathrm{len}\, g = \mathrm{len}\, j$ and for every natural number i such that $i \in \mathrm{dom}\, j$ holds $j_i = h_i - g_i$. Then $\sum j = \sum h - \sum g$.

(28) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x, y be elements of $\mathcal{R}^m$. Then there exists a finite sequence h of elements of $\mathcal{R}^m$ and there exists a finite sequence g of elements of $\mathcal{R}^n$ such that

(i) $\mathrm{len}\, h = m + 1$,

(ii) $\mathrm{len}\, g = m$,

(iii) for every natural number i such that $i \in \mathrm{dom}\, h$ holds $h_i = (y {\restriction} ((m + 1) -' i)) \frown \langle \underbrace{0, \ldots, 0}_{i -' 1} \rangle$,

(iv) for every natural number i such that $i \in \mathrm{dom}\, g$ holds $g_i = f_{x+h_i} - f_{x+h_{i+1}}$,

(v) for every natural number i and for every element h_1 of $\mathcal{R}^m$ such that $i \in \mathrm{dom}\, h$ and $h_i = h_1$ holds $|h_1| \leq |y|$, and

(vi) $f_{x+y} - f_x = \sum g$.

(29) Let m be a non empty element of $\mathbb{N}$ and f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^1$. Then there exists a partial function f_0 from $\mathcal{R}^m$ to $\mathbb{R}$ such that $f = \langle f_0 \rangle$.

(30) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, f_0 be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of $\mathcal{R}^m$, and x_0 be an element of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $x \in \mathrm{dom}\, f$ and $x = x_0$ and $f = f_0$, then $f_x = (f_0)_{x_0}$.

Let m be a non empty element of $\mathbb{N}$ and let X be a subset of $\mathcal{R}^m$. We say that X is open if and only if:

(Def. 3) There exists a subset X_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $X_0 = X$ and X_0 is open.

The following proposition is true

(31) Let m be a non empty element of $\mathbb{N}$ and X be a subset of $\mathcal{R}^m$. Then X is open if and only if for every element x of $\mathcal{R}^m$ such that $x \in X$ there exists a real number r such that $r > 0$ and $\{y \in \mathcal{R}^m : |y - x| < r\} \subseteq X$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let X be a set. We say that f is partially differentiable on X w.r.t. i if and only if:

(Def. 4) $X \subseteq \operatorname{dom} f$ and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $f \restriction X$ is partially differentiable in x w.r.t. i.

One can prove the following propositions:

(32) Let m, n be non empty elements of $\mathbb{N}$ and f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. Suppose f is partially differentiable on X w.r.t. i. Then X is a subset of $\mathcal{R}^m$.

(33) Let m, n be non empty elements of $\mathbb{N}$, i be an element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and Z be a set. Suppose $f = g$. Then f is partially differentiable on Z w.r.t. i if and only if g is partially differentiable on Z w.r.t. i.

(34) Let m, n be non empty elements of $\mathbb{N}$, i be an element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and Z be a subset of $\mathcal{R}^m$. Suppose Z is open and $1 \leq i \leq m$. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \operatorname{dom} f$ and for every element x of $\mathcal{R}^m$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let us consider X. Let us assume that f is partially differentiable on X w.r.t. i. The functor $f \restriction^i X$ yielding a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$ is defined as follows:

(Def. 5) $\operatorname{dom}(f \restriction^i X) = X$ and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $(f \restriction^i X)_x = \operatorname{partdiff}(f, x, i)$.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x_0 be an element of $\mathcal{R}^m$. We say that f is continuous in x_0 if and only if:

(Def. 6) There exists a point y_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ and there exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $x_0 = y_0$ and $f = g$ and g is continuous in y_0.

The following propositions are true:

(35) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of $\mathcal{R}^m$, and y be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $f = g$ and $x = y$. Then f is continuous in x if and only if g is continuous in y.

(36) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x_0 be an element of $\mathcal{R}^m$. Then f is continuous in x_0 if and

only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every element x_2 of $\mathcal{R}^m$ such that $x_2 \in \operatorname{dom} f$ and $|x_2 - x_0| < s$ holds $|f_{x_2} - f_{x_0}| < r$.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let us consider X. We say that f is continuous on X if and only if:

(Def. 7) $X \subseteq \operatorname{dom} f$ and for every element x_0 of $\mathcal{R}^m$ such that $x_0 \in X$ holds $f{\restriction}X$ is continuous in x_0.

Next we state a number of propositions:

(37) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and X be a set. If $f = g$, then f is continuous on X iff g is continuous on X.

(38) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and X be a set. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every element x_0 of $\mathcal{R}^m$ and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_2 of $\mathcal{R}^m$ such that $x_2 \in X$ and $|x_2 - x_0| < s$ holds $|f_{x_2} - f_{x_0}| < r$.

(39) Let m be a non empty element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and x_1 be a real number. If $1 \leq i \leq m$ and $y = (\mathrm{reproj}(i, x))(x_1)$, then $(\mathrm{proj}(i, m))(y) = x_1$.

(40) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, x, y be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and x_1 be a real number. If $1 \leq i \leq m$ and $y = (\mathrm{reproj}(i, x))(x_1)$, then $\mathrm{reproj}(i, x) = \mathrm{reproj}(i, y)$.

(41) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, g be a partial function from $\mathbb{R}$ to $\mathbb{R}$, x, y be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and x_1 be a real number. If $1 \leq i \leq m$ and $y = (\mathrm{reproj}(i, x))(x_1)$ and $g = f \cdot \mathrm{reproj}(i, x)$, then $g'(x_1) = \mathrm{partdiff}(f, y, i)$.

(42) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, p, q be real numbers, x be an element of $\mathcal{R}^m$, and i be an element of $\mathbb{N}$. Suppose that

(i) $1 \leq i$,

(ii) $i \leq m$,

(iii) $p < q$,

(iv) for every real number h such that $h \in [p, q]$ holds $(\mathrm{reproj}(i, x))(h) \in \operatorname{dom} f$, and

(v) for every real number h such that $h \in [p, q]$ holds f is partially differentiable in $(\mathrm{reproj}(i, x))(h)$ w.r.t. i.

Then there exists a real number r and there exists an element y of $\mathcal{R}^m$ such that $r \in]p, q[$ and $y = (\mathrm{reproj}(i, x))(r)$ and $f_{(\mathrm{reproj}(i,x))(q)} - f_{(\mathrm{reproj}(i,x))(p)} = (q - p) \cdot \mathrm{partdiff}(f, y, i)$.

(43) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, p, q be real numbers, x be an element of $\mathcal{R}^m$, and i be an element of $\mathbb{N}$. Suppose that

(i) $1 \leq i$,

(ii) $i \leq m$,

(iii) $p \leq q$,

(iv) for every real number h such that $h \in [p, q]$ holds $(\mathrm{reproj}(i, x))(h) \in \mathrm{dom}\, f$, and

(v) for every real number h such that $h \in [p, q]$ holds f is partially differentiable in $(\mathrm{reproj}(i, x))(h)$ w.r.t. i.

Then there exists a real number r and there exists an element y of $\mathcal{R}^m$ such that $r \in [p, q]$ and $y = (\mathrm{reproj}(i, x))(r)$ and $f_{(\mathrm{reproj}(i,x))(q)} - f_{(\mathrm{reproj}(i,x))(p)} = (q - p) \cdot \mathrm{partdiff}(f, y, i)$.

(44) Let m be a non empty element of $\mathbb{N}$, x, y, z, w be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and d, p, q, r be real numbers. Suppose $1 \leq i \leq m$ and $|y - x| < d$ and $|z - x| < d$ and $p = (\mathrm{proj}(i, m))(y)$ and $z = (\mathrm{reproj}(i, y))(q)$ and $r \in [p, q]$ and $w = (\mathrm{reproj}(i, y))(r)$. Then $|w - x| < d$.

(45) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, X be a subset of $\mathcal{R}^m$, x, y, z be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and d, p, q be real numbers. Suppose that $1 \leq i \leq m$ and X is open and $x \in X$ and $|y - x| < d$ and $|z - x| < d$ and $X \subseteq \mathrm{dom}\, f$ and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds f is partially differentiable in x w.r.t. i and $0 < d$ and for every element z of $\mathcal{R}^m$ such that $|z - x| < d$ holds $z \in X$ and $z = (\mathrm{reproj}(i, y))(p)$ and $q = (\mathrm{proj}(i, m))(y)$. Then there exists an element w of $\mathcal{R}^m$ such that $|w - x| < d$ and f is partially differentiable in w w.r.t. i and $f_z - f_y = (p - q) \cdot \mathrm{partdiff}(f, w, i)$.

(46) Let m be a non empty element of $\mathbb{N}$, h be a finite sequence of elements of $\mathcal{R}^m$, y, x be elements of $\mathcal{R}^m$, and j be an element of $\mathbb{N}$. Suppose $\mathrm{len}\, h = m + 1$ and $1 \leq j \leq m$ and for every natural number i such that $i \in \mathrm{dom}\, h$ holds $h_i = (y \restriction ((m+1) -' i)) ^\frown \langle \underbrace{0, \ldots, 0}_{i -' 1} \rangle$. Then $x + h_j = (\mathrm{reproj}((m+1) -' j, x + h_{j+1}))((\mathrm{proj}((m+1) -' j, m))(x + y))$.

(47) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^1$, X be a subset of $\mathcal{R}^m$, and x be an element of $\mathcal{R}^m$. Suppose that

(i) X is open,

(ii) $x \in X$, and

(iii) for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X.
Then
(iv) f is differentiable in x, and
(v) for every element h of $\mathcal{R}^m$ there exists a finite sequence w of elements of $\mathcal{R}^1$ such that $\mathrm{dom}\, w = \mathrm{Seg}\, m$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, m$ holds $w(i) = (\mathrm{proj}(i, m))(h) \cdot \mathrm{partdiff}(f, x, i)$ and $f'(x)(h) = \sum w$.

(48) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, X be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose that
(i) X is open,
(ii) $x \in X$, and
(iii) for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X.
Then
(iv) f is differentiable in x, and
(v) for every point h of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ there exists a finite sequence w of elements of $\mathcal{R}^1$ such that $\mathrm{dom}\, w = \mathrm{Seg}\, m$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, m$ holds $w(i) = (\mathrm{partdiff}(f, x, i))(\langle (\mathrm{proj}(i, m))(h) \rangle)$ and $f'(x)(h) = \sum w$.

(49) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and X be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose X is open. Then for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X if and only if f is differentiable on X and $f'_{\restriction X}$ is continuous on X.

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Some Properties of p-Groups and Commutative p-Groups

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Summary. This article describes some properties of p-groups and some properties of commutative p-groups.

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The notation and terminology used here have been introduced in the following papers: [7], [4], [8], [6], [10], [9], [11], [5], [1], [3], [2], and [12].

1. p-Groups

For simplicity, we use the following convention: G is a group, a, b are elements of G, m, n are natural numbers, and p is a prime natural number.

One can prove the following propositions:

(1) If for every natural number r holds $n \neq p^r$, then there exists an element s of $\mathbb{N}$ such that s is prime and $s \mid n$ and $s \neq p$.

(2) For all natural numbers n, m such that $n \mid p^m$ there exists a natural number r such that $n = p^r$ and $r \leq m$.

(3) If $a^n = \mathbf{1}_G$, then $(a^{-1})^n = \mathbf{1}_G$.

(4) If $(a^{-1})^n = \mathbf{1}_G$, then $a^n = \mathbf{1}_G$.

(5) $\mathrm{ord}(a^{-1}) = \mathrm{ord}(a)$.

(6) $\mathrm{ord}(a^b) = \mathrm{ord}(a)$.

(7) Let G be a group, N be a subgroup of G, and a, b be elements of G. Suppose N is normal and $b \in N$. Let given n. Then there exists an element g of G such that $g \in N$ and $(a \cdot b)^n = a^n \cdot g$.

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(8) Let G be a group, N be a normal subgroup of G, a be an element of G, and S be an element of ${}^{G}/_{N}$. If $S = a \cdot N$, then for every n holds $S^n = a^n \cdot N$.

(9) Let G be a group, H be a subgroup of G, and a, b be elements of G. If $a \cdot H = b \cdot H$, then there exists an element h of G such that $a = b \cdot h$ and $h \in H$.

(10) Let G be a finite group and N be a normal subgroup of G. If N is a subgroup of $\mathrm{Z}(G)$ and ${}^{G}/_{N}$ is cyclic, then G is commutative.

(11) Let G be a finite group and N be a normal subgroup of G. If $N = \mathrm{Z}(G)$ and ${}^{G}/_{N}$ is cyclic, then G is commutative.

(12) For every finite group G and for every subgroup H of G such that $\overline{\overline{H}} \neq \overline{\overline{G}}$ there exists an element a of G such that $a \notin H$.

Let p be a natural number, let G be a group, and let a be an element of G. We say that a is p-power if and only if:

(Def. 1) There exists a natural number r such that $\mathrm{ord}(a) = p^r$.

We now state the proposition

(13) $\mathbf{1}_G$ is m-power.

Let us consider G, m. One can verify that there exists an element of G which is m-power.

Let us consider p, G and let a be a p-power element of G. Observe that a^{-1} is p-power.

One can prove the following proposition

(14) If a^b is p-power, then a is p-power.

Let us consider p, G, b and let a be a p-power element of G. One can verify that a^b is p-power.

Let us consider p, let G be a commutative group, and let a, b be p-power elements of G. Observe that $a \cdot b$ is p-power.

Let us consider p and let G be a finite p-group group. One can verify that every element of G is p-power.

The following proposition is true

(15) Let G be a finite group, H be a subgroup of G, and a be an element of G. If H is p-group and $a \in H$, then a is p-power.

Let us consider p and let G be a finite p-group group. One can verify that every subgroup of G is p-group.

We now state the proposition

(16) $\{\mathbf{1}\}_G$ is p-group.

Let us consider p and let G be a group. Note that there exists a subgroup of G which is p-group.

Let us consider p, let G be a finite group, let G_1 be a p-group subgroup of G, and let G_2 be a subgroup of G. One can verify that $G_1 \cap G_2$ is p-group and $G_2 \cap G_1$ is p-group.

Next we state the proposition

(17) For every finite group G such that every element of G is p-power holds G is p-group.

Let us consider p, let G be a finite p-group group, and let N be a normal subgroup of G. Note that ${}^{G}/_{N}$ is p-group.

The following four propositions are true:

(18) Let G be a finite group and N be a normal subgroup of G. If N is p-group and ${}^{G}/_{N}$ is p-group, then G is p-group.

(19) Let G be a finite commutative group and H, H_1, H_2 be subgroups of G. Suppose H_1 is p-group and H_2 is p-group and the carrier of $H = H_1 \cdot H_2$. Then H is p-group.

(20) Let G be a finite group and H, N be subgroups of G. Suppose N is a normal subgroup of G and H is p-group and N is p-group. Then there exists a strict subgroup P of G such that the carrier of $P = H \cdot N$ and P is p-group.

(21) Let G be a finite group and N_1, N_2 be normal subgroups of G. Suppose N_1 is p-group and N_2 is p-group. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and N is p-group.

Let us consider p, let G be a p-group finite group, let H be a finite group, and let g be a homomorphism from G to H. Observe that $\operatorname{Im} g$ is p-group.

The following proposition is true

(22) For all strict groups G, H such that G and H are isomorphic and G is p-group holds H is p-group.

Let p be a prime natural number and let G be a group. Let us assume that G is p-group. The functor $\operatorname{expon}(G, p)$ yields a natural number and is defined by:

(Def. 2) $\overline{\overline{G}} = p^{\operatorname{expon}(G,p)}$.

Let p be a prime natural number and let G be a group. Then $\operatorname{expon}(G, p)$ is an element of $\mathbb{N}$.

Next we state four propositions:

(23) For every finite group G and for every subgroup H of G such that G is p-group holds $\operatorname{expon}(H, p) \leq \operatorname{expon}(G, p)$.

(24) For every strict finite group G such that G is p-group and $\operatorname{expon}(G, p) = 0$ holds $G = \{\mathbf{1}\}_G$.

(25) For every strict finite group G such that G is p-group and $\operatorname{expon}(G, p) = 1$ holds G is cyclic.

(26) Let G be a finite group, p be a prime natural number, and a be an element of G. If G is p-group and $\mathrm{expon}(G, p) = 2$ and $\mathrm{ord}(a) = p^2$, then G is commutative.

2. Commutative p-Groups

Let p be a natural number and let G be a group. We say that G is p-commutative group-like if and only if:

(Def. 3) For all elements a, b of G holds $(a \cdot b)^p = a^p \cdot b^p$.

Let p be a natural number and let G be a group. We say that G is p-commutative group if and only if:

(Def. 4) G is p-group and p-commutative group-like.

Let p be a natural number. Observe that every group which is p-commutative group is also p-group and p-commutative group-like and every group which is p-group and p-commutative group-like is also p-commutative group.

The following proposition is true

(27) $\{\mathbf{1}\}_G$ is p-commutative group-like.

Let us consider p. Note that there exists a group which is p-commutative group, finite, cyclic, and commutative.

Let us consider p and let G be a p-commutative group-like finite group. Note that every subgroup of G is p-commutative group-like.

Let us consider p. Note that every group which is p-group, finite, and commutative is also p-commutative group.

We now state the proposition

(28) For every strict finite group G such that $\overline{\overline{G}} = p$ holds G is p-commutative group.

Let us consider p, G. One can check that there exists a subgroup of G which is p-commutative group and finite.

Let us consider p, let G be a finite group, let H_1 be a p-commutative group-like subgroup of G, and let H_2 be a subgroup of G. One can check that $H_1 \cap H_2$ is p-commutative group-like and $H_2 \cap H_1$ is p-commutative group-like.

Let us consider p, let G be a finite p-commutative group-like group, and let N be a normal subgroup of G. One can verify that ${}^{G}/_{N}$ is p-commutative group-like.

One can prove the following propositions:

(29) Let G be a finite group and a, b be elements of G. Suppose G is p-commutative group-like. Let given n. Then $(a \cdot b)^{p^n} = a^{p^n} \cdot b^{p^n}$.

(30) Let G be a finite commutative group and H, H_1, H_2 be subgroups of G. Suppose H_1 is p-commutative group and H_2 is p-commutative group and the carrier of $H = H_1 \cdot H_2$. Then H is p-commutative group.

(31) Let G be a finite group, H be a subgroup of G, and N be a strict normal subgroup of G. Suppose N is a subgroup of $\mathrm{Z}(G)$ and H is p-commutative group and N is p-commutative group. Then there exists a strict subgroup P of G such that the carrier of $P = H \cdot N$ and P is p-commutative group.

(32) Let G be a finite group and N_1, N_2 be normal subgroups of G. Suppose N_2 is a subgroup of $\mathrm{Z}(G)$ and N_1 is p-commutative group and N_2 is p-commutative group. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and N is p-commutative group.

(33) Let G, H be groups. Suppose G and H are isomorphic and G is p-commutative group-like. Then H is p-commutative group-like.

(34) Let G, H be strict groups. Suppose G and H are isomorphic and G is p-commutative group. Then H is p-commutative group.

Let us consider p, let G be a p-commutative group-like finite group, let H be a finite group, and let g be a homomorphism from G to H. Observe that $\mathrm{Im}\, g$ is p-commutative group-like.

The following propositions are true:

(35) For every strict finite group G such that G is p-group and $\mathrm{expon}(G, p) = 0$ holds G is p-commutative group.

(36) For every strict finite group G such that G is p-group and $\mathrm{expon}(G, p) = 1$ holds G is p-commutative group.

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Riemann Integral of Functions from $\mathbb{R}$ into Real Normed Space

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Summary. In this article, we define the Riemann integral on functions from $\mathbb{R}$ into real normed space and prove the linearity of this operator. As a result, the Riemann integration can be applied to a wider range of functions. The proof method follows the [16].

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The terminology and notation used here have been introduced in the following articles: [2], [3], [4], [5], [7], [10], [8], [9], [1], [14], [6], [13], [15], [11], [19], [17], [12], [18], and [20].

1. Preliminaries

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, and let D be a Division of A. A finite sequence of elements of X is said to be a middle volume of f and D if it satisfies the conditions (Def. 1).

(Def. 1)(i) $\operatorname{len} \mathrm{it} = \operatorname{len} D$, and

(ii) for every natural number i such that $i \in \operatorname{dom} D$ there exists a point c of X such that $c \in \operatorname{rng}(f \restriction \operatorname{divset}(D, i))$ and $\mathrm{it}(i) = \operatorname{vol}(\operatorname{divset}(D, i)) \cdot c$.

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Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, let D be a Division of A, and let F be a middle volume of f and D. The functor middle sum(f, F) yielding a point of X is defined by:

(Def. 2) middle sum$(f, F) = \sum F$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, and let T be a division sequence of A. A function from $\mathbb{N}$ into (the carrier of X)* is said to be a middle volume sequence of f and T if:

(Def. 3) For every element k of $\mathbb{N}$ holds it(k) is a middle volume of f and $T(k)$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, let T be a division sequence of A, let S be a middle volume sequence of f and T, and let k be an element of $\mathbb{N}$. Then $S(k)$ is a middle volume of f and $T(k)$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, let T be a division sequence of A, and let S be a middle volume sequence of f and T. The functor middle sum(f, S) yielding a sequence of X is defined as follows:

(Def. 4) For every element i of $\mathbb{N}$ holds
(middle sum$(f, S))(i) =$ middle sum$(f, S(i))$.

2. Definition of Riemann Integral on Functions from $\mathbb{R}$ into Real Normed Space

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a function from A into the carrier of X. We say that f is integrable if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists a point I of X such that for every division sequence T of A and for every middle volume sequence S of f and T if δ_T is convergent and $\lim(\delta_T) = 0$, then middle sum(f, S) is convergent and lim middle sum$(f, S) = I$.

We now state three propositions:

(1) Let X be a real normed space and R_1, R_2, R_3 be finite sequences of elements of X. If $\operatorname{len} R_1 = \operatorname{len} R_2$ and $R_3 = R_1 + R_2$, then $\sum R_3 = \sum R_1 + \sum R_2$.

(2) Let X be a real normed space and R_1, R_2, R_3 be finite sequences of elements of X. If $\operatorname{len} R_1 = \operatorname{len} R_2$ and $R_3 = R_1 - R_2$, then $\sum R_3 = \sum R_1 - \sum R_2$.

(3) Let X be a real normed space, R_1, R_2 be finite sequences of elements of X, and a be an element of $\mathbb{R}$. If $R_2 = a\, R_1$, then $\sum R_2 = a \cdot \sum R_1$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a function from A into the carrier of X. Let us assume that f is integrable. The functor integral f yields a point of X and is defined by the condition (Def. 6).

(Def. 6) Let T be a division sequence of A and S be a middle volume sequence of f and T. If δ_T is convergent and $\lim(\delta_T) = 0$, then middle sum(f, S) is convergent and lim middle sum$(f, S) =$ integral f.

We now state four propositions:

(4) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, r be a real number, and f, h be functions from A into the carrier of X. If $h = r\,f$ and f is integrable, then h is integrable and integral $h = r \cdot$ integral f.

(5) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, and f, h be functions from A into the carrier of X. If $h = -f$ and f is integrable, then h is integrable and integral $h = -$integral f.

(6) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, and f, g, h be functions from A into the carrier of X. Suppose $h = f + g$ and f is integrable and g is integrable. Then h is integrable and integral $h =$ integral $f +$ integral g.

(7) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, and f, g, h be functions from A into the carrier of X. Suppose $h = f - g$ and f is integrable and g is integrable. Then h is integrable and integral $h =$ integral $f -$ integral g.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{R}$ to the carrier of X. We say that f is integrable on A if and only if:

(Def. 7) There exists a function g from A into the carrier of X such that $g = f{\restriction}A$ and g is integrable.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{R}$ to the carrier of X. Let us assume that $A \subseteq \operatorname{dom} f$. The functor $\int\limits_A f(x)dx$ yields an element of X and is defined as follows:

(Def. 8) There exists a function g from A into the carrier of X such that $g = f{\restriction}A$ and $\int\limits_A f(x)dx =$ integral g.

We now state several propositions:

(8) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to the carrier of X, and g be a function from A into the carrier of X. Suppose $f{\restriction}A = g$. Then f is integrable on A if and only if g is integrable.

(9) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to the carrier of X, and g be a function from A into the carrier of X. If

$A \subseteq \operatorname{dom} f$ and $f \restriction A = g$, then $\int\limits_A f(x)dx = \operatorname{integral} g$.

(10) Let X, Y be non empty sets, V be a real normed space, g, f be partial functions from X to the carrier of V, and g_1, f_1 be partial functions from Y to the carrier of V. If $g = g_1$ and $f = f_1$, then $g_1 + f_1 = g + f$.

(11) Let X, Y be non empty sets, V be a real normed space, g, f be partial functions from X to the carrier of V, and g_1, f_1 be partial functions from Y to the carrier of V. If $g = g_1$ and $f = f_1$, then $g_1 - f_1 = g - f$.

(12) Let r be a real number, X, Y be non empty sets, V be a real normed space, g be a partial function from X to the carrier of V, and g_1 be a partial function from Y to the carrier of V. If $g = g_1$, then $r\, g_1 = r\, g$.

3. Linearity of the Integration Operator

Next we state three propositions:

(13) Let r be a real number, A be a closed-interval subset of $\mathbb{R}$, and f be a partial function from $\mathbb{R}$ to the carrier of X. Suppose $A \subseteq \operatorname{dom} f$ and f is integrable on A. Then $r\, f$ is integrable on A and $\int\limits_A (r\, f)(x)dx = r \cdot \int\limits_A f(x)dx$.

(14) Let A be a closed-interval subset of $\mathbb{R}$ and f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of X. Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \operatorname{dom} f_1$ and $A \subseteq \operatorname{dom} f_2$. Then $f_1 + f_2$ is integrable on A and $\int\limits_A (f_1 + f_2)(x)dx = \int\limits_A f_1(x)dx + \int\limits_A f_2(x)dx$.

(15) Let A be a closed-interval subset of $\mathbb{R}$ and f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of X. Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \operatorname{dom} f_1$ and $A \subseteq \operatorname{dom} f_2$. Then $f_1 - f_2$ is integrable on A and $\int\limits_A (f_1 - f_2)(x)dx = \int\limits_A f_1(x)dx - \int\limits_A f_2(x)dx$.

Let X be a real normed space, let f be a partial function from $\mathbb{R}$ to the carrier of X, and let a, b be real numbers. The functor $\int\limits_a^b f(x)dx$ yielding an element of X is defined as follows:

(Def. 9) $$\int\limits_a^b f(x)dx = \begin{cases} \int\limits_{[a,b]} f(x)dx, & \text{if } a \leq b, \\ -\int\limits_{[b,a]} f(x)dx, & \text{otherwise.} \end{cases}$$

One can prove the following propositions:

(16) Let f be a partial function from $\mathbb{R}$ to the carrier of X, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [a,b]$, then $\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

(17) Let f be a partial function from $\mathbb{R}$ to the carrier of X and A be a closed-interval subset of $\mathbb{R}$. If $\mathrm{vol}(A) = 0$ and $A \subseteq \mathrm{dom}\, f$, then f is integrable on A and $\int\limits_{A} f(x)dx = 0_X$.

(18) Let f be a partial function from $\mathbb{R}$ to the carrier of X, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [b,a]$ and $A \subseteq \mathrm{dom}\, f$, then $-\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

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Normal Subgroup of Product of Groups

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Summary. In [6] it was formalized that the direct product of a family of groups gives a new group. In this article, we formalize that for all $j \in I$, the group $G = \Pi_{i \in I} G_i$ has a normal subgroup isomorphic to G_j. Moreover, we show some relations between a family of groups and its direct product.

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The papers [2], [4], [5], [3], [8], [9], [7], [10], [11], [6], [1], [13], and [12] provide the terminology and notation for this paper.

1. Normal Subgroup of Product of Groups

Let I be a non empty set, let F be a group-like multiplicative magma family of I, and let i be an element of I. Note that $F(i)$ is group-like.

Let I be a non empty set, let F be an associative multiplicative magma family of I, and let i be an element of I. Observe that $F(i)$ is associative.

Let I be a non empty set, let F be a commutative multiplicative magma family of I, and let i be an element of I. Note that $F(i)$ is commutative.

In the sequel I is a non empty set, F is an associative group-like multiplicative magma family of I, and i, j are elements of I.

We now state the proposition

(1) Let x be a function and g be an element of $F(i)$. Then $\operatorname{dom} x = I$ and $x(i) = g$ and for every element j of I such that $j \neq i$ holds $x(j) = \mathbf{1}_{F(j)}$ if and only if $x = \mathbf{1}_{\prod F} +\cdot (i, g)$.

Let I be a non empty set, let F be an associative group-like multiplicative magma family of I, and let i be an element of I. The functor $\operatorname{ProjSet}(F, i)$ yields a subset of $\prod F$ and is defined by:

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(Def. 1) For every set x holds $x \in \mathrm{ProjSet}(F, i)$ iff there exists an element g of $F(i)$ such that $x = \mathbf{1}_{\prod F} +\cdot (i, g)$.

Let I be a non empty set, let F be an associative group-like multiplicative magma family of I, and let i be an element of I. Observe that $\mathrm{ProjSet}(F, i)$ is non empty.

Next we state several propositions:

(2) Let x_0 be a set. Then $x_0 \in \mathrm{ProjSet}(F, i)$ if and only if there exists a function x and there exists an element g of $F(i)$ such that $x = x_0$ and $\mathrm{dom}\, x = I$ and $x(i) = g$ and for every element j of I such that $j \neq i$ holds $x(j) = \mathbf{1}_{F(j)}$.

(3) Let g_1, g_2 be elements of $\prod F$ and z_1, z_2 be elements of $F(i)$. If $g_1 = \mathbf{1}_{\prod F} +\cdot (i, z_1)$ and $g_2 = \mathbf{1}_{\prod F} +\cdot (i, z_2)$, then $g_1 \cdot g_2 = \mathbf{1}_{\prod F} +\cdot (i, z_1 \cdot z_2)$.

(4) For every element g_1 of $\prod F$ and for every element z_1 of $F(i)$ such that $g_1 = \mathbf{1}_{\prod F} +\cdot (i, z_1)$ holds ${g_1}^{-1} = \mathbf{1}_{\prod F} +\cdot (i, {z_1}^{-1})$.

(5) For all elements g_1, g_2 of $\prod F$ such that g_1, $g_2 \in \mathrm{ProjSet}(F, i)$ holds $g_1 \cdot g_2 \in \mathrm{ProjSet}(F, i)$.

(6) For every element g of $\prod F$ such that $g \in \mathrm{ProjSet}(F, i)$ holds $g^{-1} \in \mathrm{ProjSet}(F, i)$.

Let I be a non empty set, let F be an associative group-like multiplicative magma family of I, and let i be an element of I. The functor $\mathrm{ProjGroup}(F, i)$ yields a strict subgroup of $\prod F$ and is defined as follows:

(Def. 2) The carrier of $\mathrm{ProjGroup}(F, i) = \mathrm{ProjSet}(F, i)$.

Let us consider I, F, i. The functor $1\mathrm{ProdHom}(F, i)$ yielding a homomorphism from $F(i)$ to $\mathrm{ProjGroup}(F, i)$ is defined as follows:

(Def. 3) For every element x of $F(i)$ holds $(1\mathrm{ProdHom}(F, i))(x) = \mathbf{1}_{\prod F} +\cdot (i, x)$.

Let us consider I, F, i. Note that $1\mathrm{ProdHom}(F, i)$ is bijective.

Let us consider I, F, i. One can check that $\mathrm{ProjGroup}(F, i)$ is normal.

One can prove the following proposition

(7) For all elements x, y of $\prod F$ such that $i \neq j$ and $x \in \mathrm{ProjGroup}(F, i)$ and $y \in \mathrm{ProjGroup}(F, j)$ holds $x \cdot y = y \cdot x$.

2. Product of Subgroups of a Group

In the sequel n denotes a non empty natural number.

One can prove the following propositions:

(8) Let F be an associative group-like multiplicative magma family of $\mathrm{Seg}\, n$, J be a natural number, and G_1 be a group. Suppose $1 \leq J \leq n$ and $G_1 = F(J)$. Let x be an element of $\prod F$ and s be a finite sequence of elements of $\prod F$. Suppose $\mathrm{len}\, s < J$ and for every element k of $\mathrm{Seg}\, n$

such that $k \in \operatorname{dom} s$ holds $s(k) \in \mathrm{ProjGroup}(F, k)$ and $x = \prod s$. Then $x(J) = \mathbf{1}_{(G_1)}$.

(9) Let F be an associative group-like multiplicative magma family of $\operatorname{Seg} n$, x be an element of $\prod F$, and s be a finite sequence of elements of $\prod F$. Suppose $\operatorname{len} s = n$ and for every element k of $\operatorname{Seg} n$ holds $s(k) \in \mathrm{ProjGroup}(F, k)$ and $x = \prod s$. Let i be a natural number. Suppose $1 \leq i \leq n$. Then there exists an element s_1 of $\prod F$ such that $s_1 = s(i)$ and $x(i) = s_1(i)$.

(10) Let F be an associative group-like multiplicative magma family of $\operatorname{Seg} n$, x be an element of $\prod F$, and s, t be finite sequences of elements of $\prod F$. Suppose that

(i) $\operatorname{len} s = n$,

(ii) for every element k of $\operatorname{Seg} n$ holds $s(k) \in \mathrm{ProjGroup}(F, k)$,

(iii) $x = \prod s$,

(iv) $\operatorname{len} t = n$,

(v) for every element k of $\operatorname{Seg} n$ holds $t(k) \in \mathrm{ProjGroup}(F, k)$, and

(vi) $x = \prod t$.

Then $s = t$.

(11) Let F be an associative group-like multiplicative magma family of $\operatorname{Seg} n$ and x be an element of $\prod F$. Then there exists a finite sequence s of elements of $\prod F$ such that $\operatorname{len} s = n$ and for every element k of $\operatorname{Seg} n$ holds $s(k) \in \mathrm{ProjGroup}(F, k)$ and $x = \prod s$.

(12) Let G be a commutative group and F be an associative group-like multiplicative magma family of $\operatorname{Seg} n$. Suppose that

(i) for every element i of $\operatorname{Seg} n$ holds $F(i)$ is a subgroup of G,

(ii) for every element x of G there exists a finite sequence s of elements of G such that $\operatorname{len} s = n$ and for every element k of $\operatorname{Seg} n$ holds $s(k) \in F(k)$ and $x = \prod s$, and

(iii) for all finite sequences s, t of elements of G such that $\operatorname{len} s = n$ and for every element k of $\operatorname{Seg} n$ holds $s(k) \in F(k)$ and $\operatorname{len} t = n$ and for every element k of $\operatorname{Seg} n$ holds $t(k) \in F(k)$ and $\prod s = \prod t$ holds $s = t$.

Then there exists a homomorphism f from $\prod F$ to G such that

(iv) f is bijective, and

(v) for every element x of $\prod F$ there exists a finite sequence s of elements of G such that $\operatorname{len} s = n$ and for every element k of $\operatorname{Seg} n$ holds $s(k) \in F(k)$ and $s = x$ and $f(x) = \prod s$.

(13) Let G, F be associative commutative group-like multiplicative magma families of $\operatorname{Seg} n$. Suppose that for every element k of $\operatorname{Seg} n$ holds $F(k) = \mathrm{ProjGroup}(G, k)$. Then there exists a homomorphism f from $\prod F$ to $\prod G$ such that

(i) f is bijective, and

(ii) for every element x of $\prod F$ there exists a finite sequence s of elements of $\prod G$ such that $\operatorname{len} s = n$ and for every element k of $\operatorname{Seg} n$ holds $s(k) \in F(k)$ and $s = x$ and $f(x) = \prod s$.

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The Mycielskian of a Graph[1]

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Summary. Let $\omega(G)$ and $\chi(G)$ be the clique number and the chromatic number of a graph G. Mycielski [11] presented a construction that for any n creates a graph M_n which is triangle-free ($\omega(G) = 2$) with $\chi(G) > n$. The starting point is the complete graph of two vertices (K_2). $M_{(n+1)}$ is obtained from M_n through the operation $\mu(G)$ called the Mycielskian of a graph G.

We first define the operation $\mu(G)$ and then show that $\omega(\mu(G)) = \omega(G)$ and $\chi(\mu(G)) = \chi(G) + 1$. This is done for arbitrary graph G, see also [10]. Then we define the sequence of graphs M_n each of exponential size in n and give their clique and chromatic numbers.

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The notation and terminology used here have been introduced in the following papers: [1], [15], [13], [8], [5], [2], [14], [9], [16], [3], [6], [18], [19], [12], [17], [4], and [7].

1. Preliminaries

One can prove the following propositions:

(1) For all real numbers x, y, z such that $0 \leq x$ holds $x \cdot (y -' z) = x \cdot y -' x \cdot z$.

(2) For all natural numbers x, y, z holds $x \in y \setminus z$ iff $z \leq x < y$.

(3) For all sets A, B, C, D, E, X such that $X \subseteq A$ or $X \subseteq B$ or $X \subseteq C$ or $X \subseteq D$ or $X \subseteq E$ holds $X \subseteq A \cup B \cup C \cup D \cup E$.

(4) For all sets A, B, C, D, E, x holds $x \in A \cup B \cup C \cup D \cup E$ iff $x \in A$ or $x \in B$ or $x \in C$ or $x \in D$ or $x \in E$.

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(5) Let R be a symmetric relational structure and x, y be sets. Suppose $x \in$ the carrier of R and $y \in$ the carrier of R and $\langle x, y \rangle \in$ the internal relation of R. Then $\langle y, x \rangle \in$ the internal relation of R.

(6) For every symmetric relational structure R and for all elements x, y of R such that $x \leq y$ holds $y \leq x$.

2. Partitions

One can prove the following proposition

(7) For every set X and for every partition P of X holds $\overline{\overline{P}} \subseteq \overline{\overline{X}}$.

Let X be a set, let P be a partition of X, and let S be a subset of X. The functor $P \restriction S$ yields a partition of S and is defined by:

(Def. 1) $P \restriction S = \{x \cap S; x$ ranges over elements of P: x meets $S\}$.

Let X be a set. Observe that there exists a partition of X which is finite.

Let X be a set, let P be a finite partition of X, and let S be a subset of X. Observe that $P \restriction S$ is finite.

One can prove the following propositions:

(8) For every set X and for every finite partition P of X and for every subset S of X holds $\overline{\overline{P \restriction S}} \leq \overline{\overline{P}}$.

(9) Let X be a set, P be a finite partition of X, and S be a subset of X. Then for every set p such that $p \in P$ holds p meets S if and only if $\overline{\overline{P \restriction S}} = \overline{\overline{P}}$.

(10) Let R be a relational structure, C be a coloring of R, and S be a subset of R. Then $C \restriction S$ is a coloring of $\mathrm{sub}(S)$.

3. Chromatic Number and Clique Cover Number

Let R be a relational structure. We say that R is finitely colorable if and only if:

(Def. 2) There exists a coloring of R which is finite.

One can check that there exists a relational structure which is finitely colorable.

Let us observe that every relational structure which is finite is also finitely colorable.

Let R be a finitely colorable relational structure. Observe that there exists a coloring of R which is finite.

Let R be a finitely colorable relational structure and let S be a subset of R. One can verify that $\mathrm{sub}(S)$ is finitely colorable.

Let R be a finitely colorable relational structure. The functor $\chi(R)$ yielding a natural number is defined by:

(Def. 3) There exists a finite coloring C of R such that $\overline{\overline{C}} = \chi(R)$ and for every finite coloring C of R holds $\chi(R) \leq \overline{\overline{C}}$.

Let R be an empty relational structure. Observe that $\chi(R)$ is empty.

Let R be a non empty finitely colorable relational structure. Observe that $\chi(R)$ is positive.

Let R be a relational structure. We say that R has finite clique cover if and only if:

(Def. 4) There exists a clique-partition of R which is finite.

One can verify that there exists a relational structure which has finite clique cover.

One can verify that every relational structure which is finite has also finite clique cover.

Let R be a relational structure with finite clique cover. Observe that there exists a clique-partition of R which is finite.

Let R be a relational structure with finite clique cover and let S be a subset of R. Observe that $\mathrm{sub}(S)$ has finite clique cover.

Let R be a relational structure with finite clique cover. The functor $\kappa(R)$ yielding a natural number is defined by:

(Def. 5) There exists a finite clique-partition C of R such that $\overline{\overline{C}} = \kappa(R)$ and for every finite clique-partition C of R holds $\kappa(R) \leq \overline{\overline{C}}$.

Let R be an empty relational structure. One can check that $\kappa(R)$ is empty.

Let R be a non empty relational structure with finite clique cover. One can verify that $\kappa(R)$ is positive.

We now state several propositions:

(11) For every finite relational structure R holds $\omega(R) \leq \overline{\overline{\text{the carrier of } R}}$.

(12) For every finite relational structure R holds $\alpha(R) \leq \overline{\overline{\text{the carrier of } R}}$.

(13) For every finite relational structure R holds $\chi(R) \leq \overline{\overline{\text{the carrier of } R}}$.

(14) For every finite relational structure R holds $\kappa(R) \leq \overline{\overline{\text{the carrier of } R}}$.

(15) For every finitely colorable relational structure R with finite clique number holds $\omega(R) \leq \chi(R)$.

(16) For every relational structure R with finite stability number and finite clique cover holds $\alpha(R) \leq \kappa(R)$.

4. COMPLEMENT

The following two propositions are true:

(17) Let R be a relational structure, x, y be elements of R, and a, b be elements of ComplRelStr R. If $x = a$ and $y = b$ and $x \leq y$, then $a \not\leq b$.

(18) Let R be a relational structure, x, y be elements of R, and a, b be elements of ComplRelStr R. If $x = a$ and $y = b$ and $x \neq y$ and $x \in$ the carrier of R and $a \not\leq b$, then $x \leq y$.

Let R be a finite relational structure. Note that ComplRelStr R is finite.

Next we state four propositions:

(19) For every symmetric relational structure R holds every clique of R is a stable set of ComplRelStr R.

(20) For every symmetric relational structure R holds every clique of ComplRelStr R is a stable set of R.

(21) For every relational structure R holds every stable set of R is a clique of ComplRelStr R.

(22) For every relational structure R holds every stable set of ComplRelStr R is a clique of R.

Let R be a relational structure with finite clique number.

One can verify that ComplRelStr R has finite stability number.

Let R be a symmetric relational structure with finite stability number. Observe that ComplRelStr R has finite clique number.

The following propositions are true:

(23) For every symmetric relational structure R with finite clique number holds $\omega(R) = \alpha(\text{ComplRelStr}\, R)$.

(24) For every symmetric relational structure R with finite stability number holds $\alpha(R) = \omega(\text{ComplRelStr}\, R)$.

(25) For every relational structure R holds every coloring of R is a clique-partition of ComplRelStr R.

(26) For every symmetric relational structure R holds every clique-partition of ComplRelStr R is a coloring of R.

(27) For every symmetric relational structure R holds every clique-partition of R is a coloring of ComplRelStr R.

(28) For every relational structure R holds every coloring of ComplRelStr R is a clique-partition of R.

Let R be a finitely colorable relational structure.

Observe that ComplRelStr R has finite clique cover.

Let R be a symmetric relational structure with finite clique cover. One can check that ComplRelStr R is finitely colorable.

The following propositions are true:

(29) For every finitely colorable symmetric relational structure R holds $\chi(R) = \kappa(\text{ComplRelStr}\, R)$.

(30) For every symmetric relational structure R with finite clique cover holds $\kappa(R) = \chi(\text{ComplRelStr}\, R)$.

5. Adjacent Set

Let R be a relational structure and let v be an element of R. The functor $\mathrm{Adjacent}(v)$ yields a subset of R and is defined as follows:

(Def. 6) For every element x of R holds $x \in \mathrm{Adjacent}(v)$ iff $x < v$ or $v < x$.

The following proposition is true

(31) Let R be a finitely colorable relational structure, C be a finite coloring of R, and c be a set. Suppose $c \in C$ and $\overline{\overline{C}} = \chi(R)$. Then there exists an element v of R such that $v \in c$ and for every element d of C such that $d \neq c$ there exists an element w of R such that $w \in \mathrm{Adjacent}(v)$ and $w \in d$.

6. Natural Numbers as Vertices

Let n be a natural number. A strict relational structure is said to be a relational structure of n if:

(Def. 7) The carrier of it $= n$.

Let us observe that every relational structure of 0 is empty.

Let n be a non empty natural number. Note that every relational structure of n is non empty.

Let n be a natural number. Note that every relational structure of n is finite and there exists a relational structure of n which is irreflexive.

Let n be a natural number. The functor $K(n)$ yields a relational structure of n and is defined as follows:

(Def. 8) The internal relation of $K(n) = n \times n \setminus \mathrm{id}_n$.

The following proposition is true

(32) Let n be a natural number and x, y be sets. Suppose x, $y \in n$. Then $\langle x, y\rangle \in$ the internal relation of $K(n)$ if and only if $x \neq y$.

Let n be a natural number. Note that $K(n)$ is irreflexive and symmetric.

Let n be a natural number. Observe that $\Omega_{K(n)}$ is a clique.

The following propositions are true:

(33) For every natural number n holds $\omega(K(n)) = n$.

(34) For every non empty natural number n holds $\alpha(K(n)) = 1$.

(35) For every natural number n holds $\chi(K(n)) = n$.

(36) For every non empty natural number n holds $\kappa(K(n)) = 1$.

7. Mycielskian of a Graph

Let n be a natural number and let R be a relational structure of n. The functor Mycielskian R yields a relational structure of $2 \cdot n + 1$ and is defined by the condition (Def. 9).

(Def. 9) The internal relation of Mycielskian $R =$ (the internal relation of $R) \cup \{\langle x, y + n\rangle; x$ ranges over elements of $\mathbb{N}$, y ranges over elements of $\mathbb{N}$: $\langle x, y\rangle \in$ the internal relation of $R\} \cup \{\langle x + n, y\rangle; x$ ranges over elements of $\mathbb{N}$, y ranges over elements of $\mathbb{N}$: $\langle x, y\rangle \in$ the internal relation of $R\} \cup \{2 \cdot n\} \times (2 \cdot n \setminus n) \cup (2 \cdot n \setminus n) \times \{2 \cdot n\}$.

One can prove the following propositions:

(37) Let n be a natural number and R be a relational structure of n. Then the carrier of $R \subseteq$ the carrier of Mycielskian R.

(38) Let n be a natural number, R be a relational structure of n, and x, y be natural numbers. Suppose $\langle x, y\rangle \in$ the internal relation of Mycielskian R. Then

(i) $x < n$ and $y < n$, or

(ii) $x < n \leq y < 2 \cdot n$, or

(iii) $n \leq x < 2 \cdot n$ and $y < n$, or

(iv) $x = 2 \cdot n$ and $n \leq y < 2 \cdot n$, or

(v) $n \leq x < 2 \cdot n$ and $y = 2 \cdot n$.

(39) Let n be a natural number and R be a relational structure of n. Then the internal relation of $R \subseteq$ the internal relation of Mycielskian R.

(40) Let n be a natural number, R be a relational structure of n, and x, y be sets. Suppose $x, y \in n$ and $\langle x, y\rangle \in$ the internal relation of Mycielskian R. Then $\langle x, y\rangle \in$ the internal relation of R.

(41) Let n be a natural number, R be a relational structure of n, and x, y be natural numbers. Suppose $\langle x, y\rangle \in$ the internal relation of R. Then $\langle x, y + n\rangle \in$ the internal relation of Mycielskian R and $\langle x + n, y\rangle \in$ the internal relation of Mycielskian R.

(42) Let n be a natural number, R be a relational structure of n, and x, y be natural numbers. Suppose $x \in n$ and $\langle x, y + n\rangle \in$ the internal relation of Mycielskian R. Then $\langle x, y\rangle \in$ the internal relation of R.

(43) Let n be a natural number, R be a relational structure of n, and x, y be natural numbers. Suppose $y \in n$ and $\langle x + n, y\rangle \in$ the internal relation of Mycielskian R. Then $\langle x, y\rangle \in$ the internal relation of R.

(44) Let n be a natural number, R be a relational structure of n, and m be a natural number. Suppose $n \leq m < 2 \cdot n$. Then $\langle m, 2 \cdot n\rangle \in$ the internal relation of Mycielskian R and $\langle 2 \cdot n, m\rangle \in$ the internal relation of Mycielskian R.

(45) Let n be a natural number, R be a relational structure of n, and S be a subset of Mycielskian R. If $S = n$, then $R = \mathrm{sub}(S)$.

(46) For every natural number n and for every irreflexive relational structure R of n such that $2 \leq \omega(R)$ holds $\omega(R) = \omega(\text{Mycielskian}\, R)$.

(47) For every finitely colorable relational structure R and for every subset S of R holds $\chi(R) \geq \chi(\mathrm{sub}(S))$.

(48) For every natural number n and for every irreflexive relational structure R of n holds $\chi(\text{Mycielskian}\, R) = 1 + \chi(R)$.

Let n be a natural number. The functor Mycielskian n yielding a relational structure of $3 \cdot 2^n -' 1$ is defined by the condition (Def. 10).

(Def. 10) There exists a function m_1 such that
- (i) Mycielskian $n = m_1(n)$,
- (ii) $\mathrm{dom}\, m_1 = \mathbb{N}$,
- (iii) $m_1(0) = K(2)$, and
- (iv) for every natural number k and for every relational structure R of $3 \cdot 2^k -' 1$ such that $R = m_1(k)$ holds $m_1(k+1) = \text{Mycielskian}\, R$.

The following proposition is true

(49) Mycielskian $0 = K(2)$ and for every natural number k holds Mycielskian$(k+1)$ = Mycielskian Mycielskian k.

Let n be a natural number. One can verify that Mycielskian n is irreflexive.

Let n be a natural number. Observe that Mycielskian n is symmetric.

We now state three propositions:

(50) For every natural number n holds $\omega(\text{Mycielskian}\, n) = 2$ and $\chi(\text{Mycielskian}\, n) = n + 2$.

(51) For every natural number n there exists a finite relational structure R such that $\omega(R) = 2$ and $\chi(R) > n$.

(52) For every natural number n there exists a finite relational structure R such that $\alpha(R) = 2$ and $\kappa(R) > n$.

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Difference and Difference Quotient. Part IV

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Summary. In this article, we give some important theorems of forward difference, backward difference, central difference and difference quotient and forward difference, backward difference, central difference and difference quotient formulas of some special functions.

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The papers [2], [7], [13], [3], [1], [6], [9], [4], [14], [8], [5], [15], [11], [12], and [10] provide the notation and terminology for this paper.

We adopt the following rules: n denotes an element of $\mathbb{N}$, h, k, x, x_0, x_1, x_2, x_3 denote real numbers, and f, g denote functions from $\mathbb{R}$ into $\mathbb{R}$.

Next we state a number of propositions:

(1) If $x_0 > 0$ and $x_1 > 0$, then $\log_e x_0 - \log_e x_1 = \log_e(\frac{x_0}{x_1})$.

(2) If $x_0 > 0$ and $x_1 > 0$, then $\log_e x_0 + \log_e x_1 = \log_e(x_0 \cdot x_1)$.

(3) If $x > 0$, then $\log_e x =$ (the function ln)(x).

(4) If $x_0 > 0$ and $x_1 > 0$, then (the function ln)$(x_0)-$(the function ln)$(x_1) =$ (the function ln)$(\frac{x_0}{x_1})$.

(5) Suppose for every x holds $f(x) = \frac{k}{x^2}$ and $x_0 \neq 0$ and $x_1 \neq 0$ and $x_2 \neq 0$ and $x_3 \neq 0$ and x_0, x_1, x_2, x_3 are mutually different. Then $\Delta[f](x_0, x_1, x_2, x_3) = \frac{k \cdot (\frac{1}{x_1 \cdot x_2 \cdot x_0} \cdot (\frac{1}{x_0} + \frac{1}{x_2} + \frac{1}{x_1}) - \frac{1}{x_2 \cdot x_1 \cdot x_3} \cdot (\frac{1}{x_3} + \frac{1}{x_1} + \frac{1}{x_2}))}{x_0 - x_3}$.

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(6) Suppose $x_0 \in \mathrm{dom}$ (the function cot) and $x_1 \in \mathrm{dom}$ (the function cot). Then Δ[(the function cot) (the function cot)]$(x_0, x_1) = -\frac{(\cos x_1)^2 - (\cos x_0)^2}{(\sin x_0 \cdot \sin x_1)^2 \cdot (x_0 - x_1)}$.

(7) Suppose $x \in \mathrm{dom}$ (the function cot) and $x + h \in \mathrm{dom}$ (the function cot). Then (Δ_h[(the function cot) (the function cot)])$(x) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot (x+h)) - \cos(2 \cdot x))}{(\sin(x+h) \cdot \sin x)^2}$.

(8) Suppose $x \in \mathrm{dom}$ (the function cot) and $x - h \in \mathrm{dom}$ (the function cot). Then (∇_h[(the function cot) (the function cot)])$(x) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot x) - \cos(2 \cdot (h-x)))}{(\sin x \cdot \sin(x-h))^2}$.

(9) Suppose $x + \frac{h}{2} \in \mathrm{dom}$ (the function cot) and $x - \frac{h}{2} \in \mathrm{dom}$ (the function cot). Then (δ_h[(the function cot) (the function cot)])$(x) = \frac{\frac{1}{2} \cdot (\cos(h+2 \cdot x) - \cos(h-2 \cdot x))}{(\sin(x+\frac{h}{2}) \cdot \sin(x-\frac{h}{2}))^2}$.

(10) If x_0, $x_1 \in \mathrm{dom}\,\mathrm{cosec}$, then
$\Delta[\mathrm{cosec}\ \mathrm{cosec}](x_0, x_1) = \frac{4 \cdot (\sin(x_1+x_0) \cdot \sin(x_1-x_0))}{(\cos(x_0+x_1) - \cos(x_0-x_1))^2 \cdot (x_0-x_1)}$.

(11) If x, $x+h \in \mathrm{dom}\,\mathrm{cosec}$, then $(\Delta_h[\mathrm{cosec}\ \mathrm{cosec}])(x) = -\frac{4 \cdot \sin(2 \cdot x+h) \cdot \sin h}{(\cos(2 \cdot x+h) - \cos h)^2}$.

(12) If x, $x-h \in \mathrm{dom}\,\mathrm{cosec}$, then $(\nabla_h[\mathrm{cosec}\ \mathrm{cosec}])(x) = -\frac{4 \cdot \sin(2 \cdot x-h) \cdot \sin h}{(\cos(2 \cdot x-h) - \cos h)^2}$.

(13) If $x+\frac{h}{2}$, $x-\frac{h}{2} \in \mathrm{dom}\,\mathrm{cosec}$, then $(\delta_h[\mathrm{cosec}\ \mathrm{cosec}])(x) = -\frac{4 \cdot \sin(2 \cdot x) \cdot \sin h}{(\cos(2 \cdot x) - \cos h)^2}$.

(14) If x_0, $x_1 \in \mathrm{dom}\,\mathrm{sec}$, then
$\Delta[\mathrm{sec}\ \mathrm{sec}](x_0, x_1) = \frac{4 \cdot (\sin(x_0+x_1) \cdot \sin(x_0-x_1))}{(\cos(x_0+x_1) + \cos(x_0-x_1))^2 \cdot (x_0-x_1)}$.

(15) If x, $x+h \in \mathrm{dom}\,\mathrm{sec}$, then $(\Delta_h[\mathrm{sec}\ \mathrm{sec}])(x) = \frac{4 \cdot \sin(2 \cdot x+h) \cdot \sin h}{(\cos(2 \cdot x+h) + \cos h)^2}$.

(16) If x, $x-h \in \mathrm{dom}\,\mathrm{sec}$, then $(\nabla_h[\mathrm{sec}\ \mathrm{sec}])(x) = \frac{4 \cdot \sin(2 \cdot x-h) \cdot \sin h}{(\cos(2 \cdot x-h) + \cos h)^2}$.

(17) If $x+\frac{h}{2}$, $x-\frac{h}{2} \in \mathrm{dom}\,\mathrm{sec}$, then $(\delta_h[\mathrm{sec}\ \mathrm{sec}])(x) = \frac{4 \cdot \sin(2 \cdot x) \cdot \sin h}{(\cos(2 \cdot x) + \cos h)^2}$.

(18) If x_0, $x_1 \in \mathrm{dom}\,\mathrm{cosec} \cap \mathrm{dom}\,\mathrm{sec}$, then $\Delta[\mathrm{cosec}\ \mathrm{sec}](x_0, x_1) = \frac{\frac{4 \cdot (\cos(x_1+x_0) \cdot \sin(x_1-x_0))}{\sin(2 \cdot x_0) \cdot \sin(2 \cdot x_1)}}{x_0-x_1}$.

(19) If $x+h$, $x \in \mathrm{dom}\,\mathrm{cosec} \cap \mathrm{dom}\,\mathrm{sec}$, then $(\Delta_h[\mathrm{cosec}\ \mathrm{sec}])(x) = -4 \cdot \frac{\cos(2 \cdot x+h) \cdot \sin h}{\sin(2 \cdot (x+h)) \cdot \sin(2 \cdot x)}$.

(20) If $x-h$, $x \in \mathrm{dom}\,\mathrm{cosec} \cap \mathrm{dom}\,\mathrm{sec}$, then $(\nabla_h[\mathrm{cosec}\ \mathrm{sec}])(x) = -4 \cdot \frac{\cos(2 \cdot x-h) \cdot \sin h}{\sin(2 \cdot x) \cdot \sin(2 \cdot (x-h))}$.

(21) If $x+\frac{h}{2}$, $x-\frac{h}{2} \in \mathrm{dom}\,\mathrm{cosec} \cap \mathrm{dom}\,\mathrm{sec}$, then $(\delta_h[\mathrm{cosec}\ \mathrm{sec}])(x) = -4 \cdot \frac{\cos(2 \cdot x) \cdot \sin h}{\sin(2 \cdot x+h) \cdot \sin(2 \cdot x-h)}$.

(22) Suppose $x_0 \in \mathrm{dom}$ (the function tan) and $x_1 \in \mathrm{dom}$ (the function tan). Then Δ[(the function tan) (the function tan) (the function cos)]$(x_0, x_1) =$ Δ[(the function tan) (the function sin)](x_0, x_1).

(23) Suppose $x \in \mathrm{dom}$ (the function tan) and $x+h \in \mathrm{dom}$ (the function tan). Then (Δ_h[(the function tan) (the function tan) (the function cos)])$(x) =$

((the function tan) (the function sin))$(x+h)$ − ((the function tan) (the function sin))(x).

(24) Suppose $x \in$ dom (the function tan) and $x-h \in$ dom (the function tan). Then $(\nabla_h$[(the function tan) (the function tan) (the function cos)]$)(x)$ = ((the function tan) (the function sin))(x) − ((the function tan) (the function sin))$(x-h)$.

(25) Suppose $x+\frac{h}{2} \in$ dom (the function tan) and $x-\frac{h}{2} \in$ dom (the function tan). Then $(\delta_h$[(the function tan) (the function tan) (the function cos)]$)(x)$ = ((the function tan) (the function sin))$(x+\frac{h}{2})$ − ((the function tan) (the function sin))$(x-\frac{h}{2})$.

(26) Suppose $x_0 \in$ dom (the function cot) and $x_1 \in$ dom (the function cot). Then Δ[(the function cot) (the function cot) (the function sin)](x_0, x_1) = Δ[(the function cot) (the function cos)](x_0, x_1).

(27) Suppose $x \in$ dom (the function cot) and $x+h \in$ dom (the function cot). Then $(\Delta_h$[(the function cot) (the function cot) (the function sin)]$)(x)$ = ((the function cot) (the function cos))$(x+h)$ − ((the function cot) (the function cos))(x).

(28) Suppose $x \in$ dom (the function cot) and $x-h \in$ dom (the function cot). Then $(\nabla_h$[(the function cot) (the function cot) (the function sin)]$)(x)$ = ((the function cot) (the function cos))(x) − ((the function cot) (the function cos))$(x-h)$.

(29) Suppose $x+\frac{h}{2} \in$ dom (the function cot) and $x-\frac{h}{2} \in$ dom (the function cot). Then $(\delta_h$[(the function cot) (the function cot) (the function sin)]$)(x)$ = ((the function cot) (the function cos))$(x+\frac{h}{2})$ − ((the function cot) (the function cos))$(x-\frac{h}{2})$.

(30) If $x_0 > 0$ and $x_1 > 0$, then Δ[the function ln]$(x_0, x_1) = \frac{(\text{the function ln})(\frac{x_0}{x_1})}{x_0 - x_1}$.

(31) If $x > 0$ and $x+h > 0$, then $(\Delta_h$[the function ln]$)(x)$ = (the function ln)$(1+\frac{h}{x})$.

(32) If $x > 0$ and $x-h > 0$, then $(\nabla_h$[the function ln]$)(x)$ = (the function ln)$(1+\frac{h}{x-h})$.

(33) If $x+\frac{h}{2} > 0$ and $x-\frac{h}{2} > 0$, then $(\delta_h$[the function ln]$)(x)$ = (the function ln)$(1+\frac{h}{x-\frac{h}{2}})$.

(34) For all real numbers h, k holds $\exp(h-k) = \frac{\exp h}{\exp k}$.

(35) $(\Delta_h[f])(x) = (\mathrm{Shift}(f, h))(x) - f(x)$.

(36) If for every x holds $f(x) = (\Delta_h[g])(x)$, then $\Delta[f](x_0, x_1) = \Delta[g](x_0 + h, x_1 + h) - \Delta[g](x_0, x_1)$.

(37) $(\Delta_h[\Delta_h[f]])(x) = (\Delta_{2 \cdot h}[f])(x) - 2 \cdot (\Delta_h[f])(x)$.

(38) $(\nabla_h[\Delta_h[f]])(x) = (\Delta_h[f])(x) - (\nabla_h[f])(x)$.

(39) $(\delta_h[\Delta_h[f]])(x) = (\Delta_h[f])(x + \frac{h}{2}) - (\delta_h[f])(x)$.

(40) $(\vec{\Delta}_h[f])(1)(x) = (\vec{\Delta}_h[f])(0)(x + h) - (\vec{\Delta}_h[f])(0)(x)$.

(41) $(\vec{\Delta}_h[f])(n + 1)(x) = (\vec{\Delta}_h[f])(n)(x + h) - (\vec{\Delta}_h[f])(n)(x)$.

(42) $(\nabla_h[f])(x) = f(x) - (\mathrm{Shift}(f, -h))(x)$.

(43) If for every x holds $f(x) = (\nabla_h[g])(x)$, then $\Delta[f](x_0, x_1) = \Delta[g](x_0, x_1) - \Delta[g](x_0 - h, x_1 - h)$.

(44) $(\Delta_h[\nabla_h[f]])(x) = (\Delta_h[f])(x) - (\nabla_h[f])(x)$.

(45) $(\nabla_h[\nabla_h[f]])(x) = 2 \cdot (\nabla_h[f])(x) - (\nabla_{2 \cdot h}[f])(x)$.

(46) $(\delta_h[\nabla_h[f]])(x) = (\delta_h[f])(x) - (\nabla_h[f])(x - \frac{h}{2})$.

(47) $(\vec{\nabla}_h[f])(1)(x) = (\vec{\nabla}_h[f])(0)(x) - (\vec{\nabla}_h[f])(0)(x - h)$.

(48) $(\vec{\nabla}_h[f])(n + 1)(x) = (\vec{\nabla}_h[f])(n)(x) - (\vec{\nabla}_h[f])(n)(x - h)$.

(49) $(\delta_h[f])(x) = (\mathrm{Shift}(f, \frac{h}{2}))(x) - (\mathrm{Shift}(f, -\frac{h}{2}))(x)$.

(50) If for every x holds $f(x) = (\delta_h[g])(x)$, then $\Delta[f](x_0, x_1) = \Delta[g](x_0 + \frac{h}{2}, x_1 + \frac{h}{2}) - \Delta[g](x_0 - \frac{h}{2}, x_1 - \frac{h}{2})$.

(51) $(\Delta_h[\delta_h[f]])(x) = (\Delta_h[f])(x + \frac{h}{2}) - (\delta_h[f])(x)$.

(52) $(\nabla_h[\delta_h[f]])(x) = (\delta_h[f])(x) - (\nabla_h[f])(x - \frac{h}{2})$.

(53) $(\delta_h[\delta_h[f]])(x) = (\Delta_h[f])(x) - (\nabla_h[f])(x)$.

(54) $(\vec{\delta}_h[f])(1)(x) = (\vec{\delta}_h[f])(0)(x + \frac{h}{2}) - (\vec{\delta}_h[f])(0)(x - \frac{h}{2})$.

(55) $(\vec{\delta}_h[f])(n + 1)(x) = (\vec{\delta}_h[f])(n)(x + \frac{h}{2}) - (\vec{\delta}_h[f])(n)(x - \frac{h}{2})$.

(56) Suppose $x_0 \in \mathrm{dom}$ (the function tan) and $x_1 \in \mathrm{dom}$ (the function tan). Then Δ[(the function tan) (the function tan) (the function sin)]$(x_0, x_1) = \frac{(\sin x_0)^3 \cdot (\cos x_1)^{\mathbf{2}} - (\sin x_1)^3 \cdot (\cos x_0)^{\mathbf{2}}}{(\cos x_0)^{\mathbf{2}} \cdot (\cos x_1)^{\mathbf{2}} \cdot (x_0 - x_1)}$.

(57) Suppose $x \in \mathrm{dom}$ (the function tan) and $x + h \in \mathrm{dom}$ (the function tan). Then $(\Delta_h$[(the function tan) (the function tan) (the function sin)]$)(x)$ = (the function sin)$(x + h)^3 \cdot ($(the function cos)$(x + h)^{-1})^2 -$ (the function sin)$(x)^3 \cdot ($(the function cos)$(x)^{-1})^2$.

(58) Suppose $x \in \mathrm{dom}$ (the function tan) and $x - h \in \mathrm{dom}$ (the function tan). Then $(\nabla_h$[(the function tan) (the function tan) (the function sin)]$)(x)$ = (the function sin)$(x)^3 \cdot ($(the function cos)$(x)^{-1})^2 -$ (the function sin)$(x - h)^3 \cdot ($(the function cos)$(x - h)^{-1})^2$.

(59) Suppose $x + \frac{h}{2} \in \mathrm{dom}$ (the function tan) and $x - \frac{h}{2} \in \mathrm{dom}$ (the function tan). Then $(\delta_h$[(the function tan) (the function tan) (the function sin)]$)(x)$ = (the function sin)$(x + \frac{h}{2})^3 \cdot ($(the function cos)$(x + \frac{h}{2})^{-1})^2 -$ (the function sin)$(x - \frac{h}{2})^3 \cdot ($(the function cos)$(x - \frac{h}{2})^{-1})^2$.

(60) Suppose $x_0 \in \mathrm{dom}$ (the function cot) and $x_1 \in \mathrm{dom}$ (the function cot). Then Δ[(the function cot) (the function cot) (the function cos)]$(x_0, x_1) = \frac{(\cos x_0)^3 \cdot (\sin x_1)^{\mathbf{2}} - (\cos x_1)^3 \cdot (\sin x_0)^{\mathbf{2}}}{(\sin x_0)^{\mathbf{2}} \cdot (\sin x_1)^{\mathbf{2}} \cdot (x_0 - x_1)}$.

(61) Suppose $x \in$ dom (the function cot) and $x + h \in$ dom (the function cot). Then $(\Delta_h$[(the function cot) (the function cot) (the function cos)]$)(x) =$ (the function cos)$(x+h)^3 \cdot (($the function sin$)(x+h)^{-1})^2 -$ (the function cos)$(x)^3 \cdot (($the function sin$)(x)^{-1})^2$.

(62) Suppose $x \in$ dom (the function cot) and $x - h \in$ dom (the function cot). Then $(\nabla_h$[(the function cot) (the function cot) (the function cos)]$)(x) =$ (the function cos)$(x)^3 \cdot (($the function sin$)(x)^{-1})^2 -$ (the function cos)$(x-h)^3 \cdot (($the function sin$)(x-h)^{-1})^2$.

(63) Suppose $x + \frac{h}{2} \in$ dom (the function cot) and $x - \frac{h}{2} \in$ dom (the function cot). Then $(\delta_h$[(the function cot) (the function cot) (the function cos)]$)(x) =$ (the function cos)$(x+\frac{h}{2})^3 \cdot (($the function sin$)(x+\frac{h}{2})^{-1})^2 -$ (the function cos)$(x-\frac{h}{2})^3 \cdot (($the function sin$)(x-\frac{h}{2})^{-1})^2$.

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The Definition of Topological Manifolds

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Summary. This article introduces the definition of n-locally Euclidean topological spaces and topological manifolds [13].

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The papers [8], [1], [6], [15], [7], [18], [3], [4], [17], [2], [16], [9], [19], [20], [11], [12], [10], [14], and [5] provide the terminology and notation for this paper.

1. Preliminaries

Let x, y be sets. Observe that $\{\langle x, y\rangle\}$ is one-to-one.

In the sequel n denotes a natural number.

One can prove the following two propositions:

(1) For every non empty topological space T holds T and $T \restriction \Omega_T$ are homeomorphic.

(2) Let X be a non empty subspace of $\mathcal{E}^n_{\mathrm{T}}$ and f be a function from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f is continuous. Then there exists a function g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that

(i) for every point a of X and for every point b of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number r such that $a = b$ and $f(a) = r$ holds $g(b) = r \cdot b$, and

(ii) g is continuous.

Let us consider n and let S be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that S is ball if and only if:

(Def. 1) There exists a point p of $\mathcal{E}^n_{\mathrm{T}}$ and there exists a real number r such that $S = \mathrm{Ball}(p, r)$.

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Let us consider n. Observe that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is ball and every subset of $\mathcal{E}^n_{\mathrm{T}}$ which is ball is also open.

Let us consider n. One can verify that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is non empty and ball.

In the sequel p denotes a point of $\mathcal{E}^n_{\mathrm{T}}$ and r denotes a real number.

The following proposition is true

(3) For every open subset S of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in S$ there exists ball subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that $B \subseteq S$ and $p \in B$.

Let us consider n, p, r. The functor $\mathbb{B}_r(p)$ yields a subspace of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 2) $\mathbb{B}_r(p) = \mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{Ball}(p, r)$.

Let us consider n. The functor $\mathbb{B}^n$ yields a subspace of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 3) $\mathbb{B}^n = \mathbb{B}_1(0_{\mathcal{E}^n_{\mathrm{T}}})$.

Let us consider n. One can verify that $\mathbb{B}^n$ is non empty. Let us consider p and let s be a positive real number. Observe that $\mathbb{B}_s(p)$ is non empty.

The following propositions are true:

(4) The carrier of $\mathbb{B}_r(p) = \mathrm{Ball}(p, r)$.

(5) If $n \neq 0$ and p is a point of $\mathbb{B}^n$, then $|p| < 1$.

(6) Let f be a function from $\mathbb{B}^n$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose $n \neq 0$ and for every point a of $\mathbb{B}^n$ and for every point b of $\mathcal{E}^n_{\mathrm{T}}$ such that $a = b$ holds $f(a) = \frac{1}{1-|b| \cdot |b|} \cdot b$. Then f is homeomorphism.

(7) Let r be a positive real number and f be a function from $\mathbb{B}^n$ into $\mathbb{B}_r(p)$. Suppose $n \neq 0$ and for every point a of $\mathbb{B}^n$ and for every point b of $\mathcal{E}^n_{\mathrm{T}}$ such that $a = b$ holds $f(a) = r \cdot b + p$. Then f is homeomorphism.

(8) $\mathbb{B}^n$ and $\mathcal{E}^n_{\mathrm{T}}$ are homeomorphic.

In the sequel q denotes a point of $\mathcal{E}^n_{\mathrm{T}}$.

We now state three propositions:

(9) For all positive real numbers r, s holds $\mathbb{B}_r(p)$ and $\mathbb{B}_s(q)$ are homeomorphic.

(10) For every non empty ball subset B of $\mathcal{E}^n_{\mathrm{T}}$ holds B and $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ are homeomorphic.

(11) Let M, N be non empty topological spaces, p be a point of M, U be a neighbourhood of p, and B be an open subset of N. Suppose U and B are homeomorphic. Then there exists an open subset V of M and there exists an open subset S of N such that $V \subseteq U$ and $p \in V$ and V and S are homeomorphic.

2. Manifold

In the sequel M is a non empty topological space.

Let us consider n, M. We say that M is n-locally Euclidean if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let p be a point of M. Then there exists a neighbourhood U of p and there exists an open subset S of $\mathcal{E}^n_{\mathrm{T}}$ such that U and S are homeomorphic.

Let us consider n. Observe that $\mathcal{E}^n_{\mathrm{T}}$ is n-locally Euclidean.

Let us consider n. Observe that there exists a non empty topological space which is n-locally Euclidean.

We now state two propositions:

(12) M is n-locally Euclidean if and only if for every point p of M there exists a neighbourhood U of p and there exists ball subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that U and B are homeomorphic.

(13) M is n-locally Euclidean if and only if for every point p of M there exists a neighbourhood U of p such that U and $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ are homeomorphic.

Let us consider n. Observe that every non empty topological space which is n-locally Euclidean is also first-countable.

Let us note that every non empty topological space which is 0-locally Euclidean is also discrete and every non empty topological space which is discrete is also 0-locally Euclidean.

Let us consider n. One can verify that $\mathcal{E}^n_{\mathrm{T}}$ is second-countable.

Let us consider n. Note that there exists a non empty topological space which is second-countable, Hausdorff, and n-locally Euclidean.

Let us consider n, M. We say that M is n-manifold if and only if:

(Def. 5) M is second-countable, Hausdorff, and n-locally Euclidean.

Let us consider M. We say that M is manifold-like if and only if:

(Def. 6) There exists n such that M is n-manifold.

Let us consider n. Observe that there exists a non empty topological space which is n-manifold.

Let us consider n. One can check the following observations:

* every non empty topological space which is n-manifold is also second-countable, Hausdorff, and n-locally Euclidean,
* every non empty topological space which is second-countable, Hausdorff, and n-locally Euclidean is also n-manifold, and
* every non empty topological space which is n-manifold is also manifold-like.

Let us note that every non empty topological space which is second-countable and discrete is also 0-manifold.

Let us consider n and let M be an n-manifold non empty topological space. One can verify that every non empty subspace of M which is open is also n-manifold.

Let us note that there exists a non empty topological space which is manifold-like.

A manifold is a manifold-like non empty topological space.

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More on Continuous Functions on Normed Linear Spaces

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Summary. In this article we formalize the definition and some facts about continuous functions from $\mathbb{R}$ into normed linear spaces [14].

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The terminology and notation used in this paper have been introduced in the following papers: [2], [12], [3], [4], [10], [11], [1], [5], [13], [7], [17], [18], [15], [9], [8], [16], [19], and [6].

1. Preliminaries

For simplicity, we adopt the following rules: n denotes an element of $\mathbb{N}$, X, X_1 denote sets, r, p denote real numbers, s, x_0, x_1, x_2 denote real numbers, S, T denote real normed spaces, f, f_1, f_2 denote partial functions from $\mathbb{R}$ to the carrier of S, s_1 denotes a sequence of real numbers, and Y denotes a subset of $\mathbb{R}$.

The following propositions are true:

(1) Let s_2 be a sequence of real numbers and h be a partial function from $\mathbb{R}$ to the carrier of S. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$, then $s_2(n) \in \operatorname{dom} h$.

(2) Let h_1, h_2 be partial functions from $\mathbb{R}$ to the carrier of S and s_2 be a sequence of real numbers. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1+h_2)_* s_2 = (h_{1*} s_2) + (h_{2*} s_2)$ and $(h_1 - h_2)_* s_2 = (h_{1*} s_2) - (h_{2*} s_2)$.

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(3) For every sequence h of S and for every real number r holds $r\,h = r \cdot h$.

(4) Let h be a partial function from $\mathbb{R}$ to the carrier of S, s_2 be a sequence of real numbers, and r be a real number. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$, then $r\,h_* s_2 = r \cdot (h_* s_2)$.

(5) Let h be a partial function from $\mathbb{R}$ to the carrier of S and s_2 be a sequence of real numbers. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$, then $\|h_* s_2\| = \|h\|_* s_2$ and $-(h_* s_2) = -h_* s_2$.

2. Continuous Real Functions into Normed Linear Spaces

Let us consider S, f, x_0. We say that f is continuous in x_0 if and only if:

(Def. 1) $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f_* s_1$ is convergent and $f_{x_0} = \lim(f_* s_1)$.

Next we state a number of propositions:

(6) If $x_0 \in X$ and f is continuous in x_0, then $f{\restriction}X$ is continuous in x_0.

(7) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ and for every n holds $s_1(n) \neq x_0$ holds $f_* s_1$ is convergent and $f_{x_0} = \lim(f_* s_1)$.

(8) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(9) Let given S, f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_1 of f_{x_0} there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_1$.

(10) Let given S, f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_1 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^{\circ} N \subseteq N_1$.

(11) If there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$, then f is continuous in x_0.

(12) If $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(13) If f is continuous in x_0, then $r\,f$ is continuous in x_0.

(14) If $x_0 \in \operatorname{dom} f$ and f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

(15) Let f_1 be a partial function from $\mathbb{R}$ to the carrier of S and f_2 be a partial function from the carrier of S to the carrier of T. Suppose $x_0 \in \operatorname{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0.

Let us consider S, f. We say that f is continuous if and only if:

(Def. 2) For every x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0.

Next we state two propositions:

(16) Let given X, f. Suppose $X \subseteq \operatorname{dom} f$. Then $f \restriction X$ is continuous if and only if for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds $f_* s_1$ is convergent and $f_{\lim s_1} = \lim(f_* s_1)$.

(17) Suppose $X \subseteq \operatorname{dom} f$. Then $f \restriction X$ is continuous if and only if for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

Let us consider S. One can check that every partial function from $\mathbb{R}$ to the carrier of S which is constant is also continuous.

Let us consider S. Note that there exists a partial function from $\mathbb{R}$ to the carrier of S which is continuous.

Let us consider S, let f be a continuous partial function from $\mathbb{R}$ to the carrier of S, and let X be a set. Observe that $f \restriction X$ is continuous.

Next we state the proposition

(18) If $f \restriction X$ is continuous and $X_1 \subseteq X$, then $f \restriction X_1$ is continuous.

Let us consider S. Observe that every partial function from $\mathbb{R}$ to the carrier of S which is empty is also continuous.

Let us consider S, f and let X be a trivial set. Observe that $f \restriction X$ is continuous.

Let us consider S and let f_1, f_2 be continuous partial functions from $\mathbb{R}$ to the carrier of S. Observe that $f_1 + f_2$ is continuous and $f_1 - f_2$ is continuous.

The following two propositions are true:

(19) Let given X, f_1, f_2. Suppose $X \subseteq \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $f_1 \restriction X$ is continuous and $f_2 \restriction X$ is continuous. Then $(f_1 + f_2) \restriction X$ is continuous and $(f_1 - f_2) \restriction X$ is continuous.

(20) Let given X, X_1, f_1, f_2. Suppose $X \subseteq \operatorname{dom} f_1$ and $X_1 \subseteq \operatorname{dom} f_2$ and $f_1 \restriction X$ is continuous and $f_2 \restriction X_1$ is continuous. Then $(f_1 + f_2) \restriction (X \cap X_1)$ is continuous and $(f_1 - f_2) \restriction (X \cap X_1)$ is continuous.

Let us consider S, let f be a continuous partial function from $\mathbb{R}$ to the carrier of S, and let us consider r. One can check that $r\,f$ is continuous.

We now state several propositions:

(21) If $X \subseteq \operatorname{dom} f$ and $f \restriction X$ is continuous, then $(r\,f) \restriction X$ is continuous.

(22) If $X \subseteq \operatorname{dom} f$ and $f{\restriction}X$ is continuous, then $\|f\|{\restriction}X$ is continuous and $(-f){\restriction}X$ is continuous.

(23) If f is total and for all x_1, x_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0, then $f{\restriction}\mathbb{R}$ is continuous.

(24) If $\operatorname{dom} f$ is compact and $f{\restriction}\operatorname{dom} f$ is continuous, then $\operatorname{rng} f$ is compact.

(25) If $Y \subseteq \operatorname{dom} f$ and Y is compact and $f{\restriction}Y$ is continuous, then $f^{\circ}Y$ is compact.

3. Lipschitz Continuity

Let us consider S, f. We say that f is Lipschitzian if and only if:

(Def. 3) There exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$.

The following proposition is true

(26) $f{\restriction}X$ is Lipschitzian if and only if there exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \operatorname{dom}(f{\restriction}X)$ holds $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$.

Let us consider S. Observe that every partial function from $\mathbb{R}$ to the carrier of S which is empty is also Lipschitzian.

Let us consider S. One can verify that there exists a partial function from $\mathbb{R}$ to the carrier of S which is empty.

Let us consider S, let f be a Lipschitzian partial function from $\mathbb{R}$ to the carrier of S, and let X be a set. One can check that $f{\restriction}X$ is Lipschitzian.

The following proposition is true

(27) If $f{\restriction}X$ is Lipschitzian and $X_1 \subseteq X$, then $f{\restriction}X_1$ is Lipschitzian.

Let us consider S and let f_1, f_2 be Lipschitzian partial functions from $\mathbb{R}$ to the carrier of S. One can check that $f_1 + f_2$ is Lipschitzian and $f_1 - f_2$ is Lipschitzian.

One can prove the following propositions:

(28) If $f_1{\restriction}X$ is Lipschitzian and $f_2{\restriction}X_1$ is Lipschitzian, then $(f_1+f_2){\restriction}(X \cap X_1)$ is Lipschitzian.

(29) If $f_1{\restriction}X$ is Lipschitzian and $f_2{\restriction}X_1$ is Lipschitzian, then $(f_1-f_2){\restriction}(X \cap X_1)$ is Lipschitzian.

Let us consider S, let f be a Lipschitzian partial function from $\mathbb{R}$ to the carrier of S, and let us consider p. Note that $p\,f$ is Lipschitzian.

Next we state the proposition

(30) If $f{\restriction}X$ is Lipschitzian and $X \subseteq \operatorname{dom} f$, then $(p\,f){\restriction}X$ is Lipschitzian.

Let us consider S and let f be a Lipschitzian partial function from $\mathbb{R}$ to the carrier of S. Note that $\|f\|$ is Lipschitzian.

One can prove the following proposition

(31) If $f\restriction X$ is Lipschitzian, then $-f\restriction X$ is Lipschitzian and $(-f)\restriction X$ is Lipschitzian and $\|f\|\restriction X$ is Lipschitzian.

Let us consider S. One can verify that every partial function from $\mathbb{R}$ to the carrier of S which is constant is also Lipschitzian.

Let us consider S. Observe that every partial function from $\mathbb{R}$ to the carrier of S which is Lipschitzian is also continuous.

Next we state two propositions:

(32) If there exists a point r of S such that $\operatorname{rng} f = \{r\}$, then f is continuous.

(33) For all points r, p of S such that for every x_0 such that $x_0 \in X$ holds $f_{x_0} = x_0 \cdot r + p$ holds $f\restriction X$ is continuous.

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Cartesian Products of Family of Real Linear Spaces

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Summary. In this article we introduced the isomorphism mapping between cartesian products of family of linear spaces [4]. Those products had been formalized by two different ways, i.e., the way using the functor [:X,Y:] and ones using the functor "product". By the same way, the isomorphism mapping was defined between Cartesian products of family of linear normed spaces also.

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The notation and terminology used in this paper are introduced in the following articles: [5], [1], [16], [11], [3], [6], [17], [7], [8], [15], [14], [2], [13], [12], [20], [18], [10], [19], and [9].

1. Preliminaries

One can prove the following propositions:

(1) Let D, E, F, G be non empty sets. Then there exists a function I from $D \times E \times (F \times G)$ into $D \times F \times (E \times G)$ such that

(i) I is one-to-one and onto, and

(ii) for all sets d, e, f, g such that $d \in D$ and $e \in E$ and $f \in F$ and $g \in G$ holds $I(\langle\langle d, e\rangle, \langle f, g\rangle\rangle) = \langle\langle\langle d, f\rangle, \langle e, g\rangle\rangle\rangle$.

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(2) Let X be a non empty set and D be a function. Suppose $\mathrm{dom}\, D = \{1\}$ and $D(1) = X$. Then there exists a function I from X into $\prod D$ such that I is one-to-one and onto and for every set x such that $x \in X$ holds $I(x) = \langle x \rangle$.

(3) Let X, Y be non empty sets and D be a function. Suppose $\mathrm{dom}\, D = \{1, 2\}$ and $D(1) = X$ and $D(2) = Y$. Then there exists a function I from $X \times Y$ into $\prod D$ such that I is one-to-one and onto and for all sets x, y such that $x \in X$ and $y \in Y$ holds $I(x, y) = \langle x, y \rangle$.

(4) Let X be a non empty set. Then there exists a function I from X into $\prod \langle X \rangle$ such that I is one-to-one and onto and for every set x such that $x \in X$ holds $I(x) = \langle x \rangle$.

Let X, Y be non-empty non empty finite sequences. Observe that $X \frown Y$ is non-empty.

We now state two propositions:

(5) Let X, Y be non empty sets. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that I is one-to-one and onto and for all sets x, y such that $x \in X$ and $y \in Y$ holds $I(x, y) = \langle x, y \rangle$.

(6) Let X, Y be non-empty non empty finite sequences. Then there exists a function I from $\prod X \times \prod Y$ into $\prod (X \frown Y)$ such that I is one-to-one and onto and for all finite sequences x, y such that $x \in \prod X$ and $y \in \prod Y$ holds $I(x, y) = x \frown y$.

Let G, F be non empty additive loop structures. The functor $\mathrm{prodadd}(G, F)$ yielding a binary operation on (the carrier of G) $\times$ (the carrier of F) is defined by:

(Def. 1) For all points g_1, g_2 of G and for all points f_1, f_2 of F holds $(\mathrm{prodadd}(G, F))(\langle g_1, f_1 \rangle, \langle g_2, f_2 \rangle) = \langle g_1 + g_2, f_1 + f_2 \rangle$.

Let G, F be non empty RLS structures. The functor $\mathrm{prodmlt}(G, F)$ yielding a function from $\mathbb{R} \times$ ((the carrier of G) $\times$ (the carrier of F)) into (the carrier of G) $\times$ (the carrier of F) is defined by:

(Def. 2) For every element r of $\mathbb{R}$ and for every point g of G and for every point f of F holds $(\mathrm{prodmlt}(G, F))(r, \langle g, f \rangle) = \langle r \cdot g, r \cdot f \rangle$.

Let G, F be non empty additive loop structures. The functor $\mathrm{prodzero}(G, F)$ yields an element of (the carrier of G) $\times$ (the carrier of F) and is defined by:

(Def. 3) $\mathrm{prodzero}(G, F) = \langle 0_G, 0_F \rangle$.

Let G, F be non empty additive loop structures. The functor $G \times F$ yielding a strict non empty additive loop structure is defined by:

(Def. 4) $G \times F = \langle$(the carrier of G) $\times$ (the carrier of F), $\mathrm{prodadd}(G, F)$, $\mathrm{prodzero}(G, F)\rangle$.

Let G, F be Abelian non empty additive loop structures. Observe that $G \times F$ is Abelian.

Let G, F be add-associative non empty additive loop structures. Note that $G \times F$ is add-associative.

Let G, F be right zeroed non empty additive loop structures. Note that $G \times F$ is right zeroed.

Let G, F be right complementable non empty additive loop structures. Note that $G \times F$ is right complementable.

Next we state two propositions:

(7) Let G, F be non empty additive loop structures. Then

(i) for every set x holds x is a point of $G \times F$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,

(ii) for all points x, y of $G \times F$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$, and

(iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$.

(8) Let G, F be add-associative right zeroed right complementable non empty additive loop structures, x be a point of $G \times F$, x_1 be a point of G, and x_2 be a point of F. If $x = \langle x_1, x_2 \rangle$, then $-x = \langle -x_1, -x_2 \rangle$.

Let G, F be Abelian add-associative right zeroed right complementable strict non empty additive loop structures. One can check that $G \times F$ is strict, Abelian, add-associative, right zeroed, and right complementable.

Let G, F be non empty RLS structures. The functor $G \times F$ yields a strict non empty RLS structure and is defined by:

(Def. 5) $G \times F = \langle$(the carrier of G) $\times$ (the carrier of F), $\mathrm{prodzero}(G, F)$, $\mathrm{prodadd}(G, F), \mathrm{prodmlt}(G, F)\rangle$.

Let G, F be Abelian non empty RLS structures. Observe that $G \times F$ is Abelian.

Let G, F be add-associative non empty RLS structures. Note that $G \times F$ is add-associative.

Let G, F be right zeroed non empty RLS structures. Note that $G \times F$ is right zeroed.

Let G, F be right complementable non empty RLS structures. One can check that $G \times F$ is right complementable.

Next we state two propositions:

(9) Let G, F be non empty RLS structures. Then

(i) for every set x holds x is a point of $G \times F$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,

(ii) for all points x, y of $G \times F$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,

(iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$, and

(iv) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$.

(10) Let G, F be add-associative right zeroed right complementable non empty RLS structures, x be a point of $G \times F$, x_1 be a point of G, and x_2 be a point of F. If $x = \langle x_1, x_2 \rangle$, then $-x = \langle -x_1, -x_2 \rangle$.

Let G, F be vector distributive non empty RLS structures. Note that $G \times F$ is vector distributive.

Let G, F be scalar distributive non empty RLS structures. Note that $G \times F$ is scalar distributive.

Let G, F be scalar associative non empty RLS structures. Observe that $G \times F$ is scalar associative.

Let G, F be scalar unital non empty RLS structures. One can verify that $G \times F$ is scalar unital.

Let G be an Abelian add-associative right zeroed right complementable scalar distributive vector distributive scalar associative scalar unital non empty RLS structure. Note that $\langle G \rangle$ is real-linear-space-yielding.

Let G, F be Abelian add-associative right zeroed right complementable scalar distributive vector distributive scalar associative scalar unital non empty RLS structures. Note that $\langle G, F \rangle$ is real-linear-space-yielding.

2. Cartesian Products of Real Linear Spaces

One can prove the following proposition

(11) Let X be a real linear space. Then there exists a function I from X into $\prod \langle X \rangle$ such that

(i) I is one-to-one and onto,

(ii) for every point x of X holds $I(x) = \langle x \rangle$,

(iii) for all points v, w of X holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of X and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_X) = 0_{\prod \langle X \rangle}$.

Let G, F be non empty real-linear-space-yielding finite sequences. Observe that $G ^\frown F$ is real-linear-space-yielding.

We now state three propositions:

(12) Let X, Y be real linear spaces. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that

(i) I is one-to-one and onto,

(ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, y \rangle$,

(iii) for all points v, w of $X \times Y$ holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of $X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{X \times Y}) = 0_{\prod \langle X, Y \rangle}$.

(13) Let X, Y be non empty real linear space-sequences. Then there exists a function I from $\prod X \times \prod Y$ into $\prod (X \frown Y)$ such that

(i) I is one-to-one and onto,

(ii) for every point x of $\prod X$ and for every point y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(x, y) = x_1 \frown y_1$,

(iii) for all points v, w of $\prod X \times \prod Y$ holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of $\prod X \times \prod Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{\prod X \times \prod Y}) = 0_{\prod (X \frown Y)}$.

(14) Let G, F be real linear spaces. Then

(i) for every set x holds x is a point of $\prod \langle G, F \rangle$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,

(ii) for all points x, y of $\prod \langle G, F \rangle$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,

(iii) $0_{\prod \langle G, F \rangle} = \langle 0_G, 0_F \rangle$,

(iv) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$, and

(v) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$.

3. Cartesian Products of Real Normed Linear Spaces

Let G, F be non empty normed structures. The functor prodnorm(G, F) yields a function from (the carrier of G) $\times$ (the carrier of F) into $\mathbb{R}$ and is defined by:

(Def. 6) For every point g of G and for every point f of F there exists an element v of $\mathcal{R}^2$ such that $v = \langle \|g\|, \|f\| \rangle$ and $(\text{prodnorm}(G, F))(g, f) = |v|$.

Let G, F be non empty normed structures. The functor $G \times F$ yielding a strict non empty normed structure is defined as follows:

(Def. 7) $G \times F = \langle$(the carrier of G) $\times$ (the carrier of F), prodzero(G, F), prodadd(G, F), prodmlt(G, F), prodnorm$(G, F)\rangle$.

Let G, F be real normed spaces. Observe that $G \times F$ is reflexive, discernible, and real normed space-like.

Let G, F be reflexive discernible real normed space-like scalar distributive vector distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty normed structures. One can verify that $G \times F$ is strict, reflexive, discernible, real normed space-like, scalar distributive, vector distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Let G be a reflexive discernible real normed space-like scalar distributive vector distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty normed structure. One can verify that $\langle G \rangle$ is real-norm-space-yielding.

Let G, F be reflexive discernible real normed space-like scalar distributive vector distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty normed structures. Observe that $\langle G, F \rangle$ is real-norm-space-yielding.

One can prove the following propositions:

(15) Let X, Y be real normed spaces. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that
- (i) I is one-to-one and onto,
- (ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, y \rangle$,
- (iii) for all points v, w of $X \times Y$ holds $I(v + w) = I(v) + I(w)$,
- (iv) for every point v of $X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,
- (v) $0_{\prod \langle X, Y \rangle} = I(0_{X \times Y})$, and
- (vi) for every point v of $X \times Y$ holds $\|I(v)\| = \|v\|$.

(16) Let X be a real normed space. Then there exists a function I from X into $\prod \langle X \rangle$ such that
- (i) I is one-to-one and onto,
- (ii) for every point x of X holds $I(x) = \langle x \rangle$,
- (iii) for all points v, w of X holds $I(v + w) = I(v) + I(w)$,
- (iv) for every point v of X and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,
- (v) $0_{\prod \langle X \rangle} = I(0_X)$, and
- (vi) for every point v of X holds $\|I(v)\| = \|v\|$.

Let G, F be non empty real-norm-space-yielding finite sequences. One can check that $G \frown F$ is non empty and real-norm-space-yielding.

One can prove the following propositions:

(17) Let X, Y be non empty real norm space-sequences. Then there exists a function I from $\prod X \times \prod Y$ into $\prod (X \frown Y)$ such that
- (i) I is one-to-one and onto,
- (ii) for every point x of $\prod X$ and for every point y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(x, y) = x_1 \frown y_1$,
- (iii) for all points v, w of $\prod X \times \prod Y$ holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of $\prod X \times \prod Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,

(v) $I(0_{\prod X \times \prod Y}) = 0_{\prod (X ^\frown Y)}$, and

(vi) for every point v of $\prod X \times \prod Y$ holds $\|I(v)\| = \|v\|$.

(18) Let G, F be real normed spaces. Then

(i) for every set x holds x is a point of $G \times F$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,

(ii) for all points x, y of $G \times F$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,

(iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$,

(iv) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$,

(v) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$, and

(vi) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ there exists an element w of $\mathcal{R}^2$ such that $w = \langle \|x_1\|, \|x_2\| \rangle$ and $\|x\| = |w|$.

(19) Let G, F be real normed spaces. Then

(i) for every set x holds x is a point of $\prod \langle G, F \rangle$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,

(ii) for all points x, y of $\prod \langle G, F \rangle$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,

(iii) $0_{\prod \langle G, F \rangle} = \langle 0_G, 0_F \rangle$,

(iv) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$,

(v) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$, and

(vi) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ there exists an element w of $\mathcal{R}^2$ such that $w = \langle \|x_1\|, \|x_2\| \rangle$ and $\|x\| = |w|$.

Let X, Y be complete real normed spaces. Observe that $X \times Y$ is complete.

We now state several propositions:

(20) Let X, Y be non empty real norm space-sequences. Then there exists a function I from $\prod \langle \prod X, \prod Y \rangle$ into $\prod (X ^\frown Y)$ such that

(i) I is one-to-one and onto,

(ii) for every point x of $\prod X$ and for every point y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(\langle x, y \rangle) = x_1 ^\frown y_1$,

(iii) for all points v, w of $\prod\langle\prod X, \prod Y\rangle$ holds $I(v+w) = I(v) + I(w)$,

(iv) for every point v of $\prod\langle\prod X, \prod Y\rangle$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,

(v) $I(0_{\prod\langle\prod X, \prod Y\rangle}) = 0_{\prod(X ^\frown Y)}$, and

(vi) for every point v of $\prod\langle\prod X, \prod Y\rangle$ holds $\|I(v)\| = \|v\|$.

(21) Let X, Y be non empty real linear spaces. Then there exists a function I from $X \times Y$ into $X \times \prod\langle Y\rangle$ such that

(i) I is one-to-one and onto,

(ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle\!\langle x, \langle y\rangle\rangle\!\rangle$,

(iii) for all points v, w of $X \times Y$ holds $I(v+w) = I(v) + I(w)$,

(iv) for every point v of $X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{X \times Y}) = 0_{X \times \prod\langle Y\rangle}$.

(22) Let X be a non empty real linear space-sequence and Y be a real linear space. Then there exists a function I from $\prod X \times Y$ into $\prod(X ^\frown \langle Y\rangle)$ such that

(i) I is one-to-one and onto,

(ii) for every point x of $\prod X$ and for every point y of Y there exist finite sequences x_1, y_1 such that $x = x_1$ and $\langle y\rangle = y_1$ and $I(x, y) = x_1 ^\frown y_1$,

(iii) for all points v, w of $\prod X \times Y$ holds $I(v+w) = I(v) + I(w)$,

(iv) for every point v of $\prod X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{\prod X \times Y}) = 0_{\prod(X ^\frown \langle Y\rangle)}$.

(23) Let X, Y be non empty real normed spaces. Then there exists a function I from $X \times Y$ into $X \times \prod\langle Y\rangle$ such that

(i) I is one-to-one and onto,

(ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle\!\langle x, \langle y\rangle\rangle\!\rangle$,

(iii) for all points v, w of $X \times Y$ holds $I(v+w) = I(v) + I(w)$,

(iv) for every point v of $X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,

(v) $I(0_{X \times Y}) = 0_{X \times \prod\langle Y\rangle}$, and

(vi) for every point v of $X \times Y$ holds $\|I(v)\| = \|v\|$.

(24) Let X be a non empty real norm space-sequence and Y be a real normed space. Then there exists a function I from $\prod X \times Y$ into $\prod(X ^\frown \langle Y\rangle)$ such that

(i) I is one-to-one and onto,

(ii) for every point x of $\prod X$ and for every point y of Y there exist finite sequences x_1, y_1 such that $x = x_1$ and $\langle y\rangle = y_1$ and $I(x, y) = x_1 ^\frown y_1$,

(iii) for all points v, w of $\prod X \times Y$ holds $I(v+w) = I(v) + I(w)$,

(iv) for every point v of $\prod X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,

(v) $I(0_{\prod X \times Y}) = 0_{\prod (X^\frown \langle Y \rangle)}$, and

(vi) for every point v of $\prod X \times Y$ holds $\|I(v)\| = \|v\|$.

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Formalization of Integral Linear Space[1]

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Summary. In this article, we formalize integral linear spaces, that is a linear space with integer coefficients. Integral linear spaces are necessary for lattice problems, LLL (Lenstra-Lenstra-Lovász) base reduction algorithm that outputs short lattice base and cryptographic systems with lattice [8].

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The notation and terminology used here have been introduced in the following papers: [1], [10], [3], [9], [11], [2], [4], [6], [16], [14], [13], [12], [5], [7], [15], and [17].

1. Preliminaries

The following propositions are true:

(1) Let X be a real linear space and R_1, R_2 be finite sequences of elements of X. If $\operatorname{len} R_1 = \operatorname{len} R_2$, then $\sum (R_1 + R_2) = \sum R_1 + \sum R_2$.

(2) Let X be a real linear space and R_1, R_2, R_3 be finite sequences of elements of X. If $\operatorname{len} R_1 = \operatorname{len} R_2$ and $R_3 = R_1 - R_2$, then $\sum R_3 = \sum R_1 - \sum R_2$.

(3) Let X be a real linear space, R_1, R_2 be finite sequences of elements of X, and a be an element of $\mathbb{R}$. If $R_2 = a\, R_1$, then $\sum R_2 = a \cdot \sum R_1$.

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2. Integral Linear Space

For simplicity, we use the following convention: x denotes a set, a denotes a real number, i denotes an integer, V denotes a real linear space, v, v_1, v_2, v_3, u, w, w_1, w_2, w_3 denote vectors of V, A, B denote subsets of V, L denotes a linear combination of V, and l, l_1, l_2 denote linear combinations of A.

Let us consider V, i, L. The functor $i \cdot L$ yielding a linear combination of V is defined as follows:

(Def. 1) For every v holds $(i \cdot L)(v) = i \cdot L(v)$.

Let us consider V, A. The functor $\mathrm{Lin}_{\mathbb{Z}}\, A$ yielding a subset of V is defined by:

(Def. 2) $\mathrm{Lin}_{\mathbb{Z}}\, A = \{\sum l : \mathrm{rng}\, l \subseteq \mathbb{Z}\}$.

One can prove the following propositions:

(4) $(i) \cdot l = i \cdot l$.

(5) If $\mathrm{rng}\, l_1 \subseteq \mathbb{Z}$ and $\mathrm{rng}\, l_2 \subseteq \mathbb{Z}$, then $\mathrm{rng}(l_1 + l_2) \subseteq \mathbb{Z}$.

(6) If $\mathrm{rng}\, l \subseteq \mathbb{Z}$, then $\mathrm{rng}(i \cdot l) \subseteq \mathbb{Z}$.

(7) $\mathrm{rng}(\mathbf{0}_{\mathrm{LC}_V}) \subseteq \mathbb{Z}$.

(8) $\mathrm{Lin}_{\mathbb{Z}}\, A \subseteq$ the carrier of $\mathrm{Lin}(A)$.

(9) If v, $u \in \mathrm{Lin}_{\mathbb{Z}}\, A$, then $v + u \in \mathrm{Lin}_{\mathbb{Z}}\, A$.

(10) If $v \in \mathrm{Lin}_{\mathbb{Z}}\, A$, then $i \cdot v \in \mathrm{Lin}_{\mathbb{Z}}\, A$.

(11) $0_V \in \mathrm{Lin}_{\mathbb{Z}}\, A$.

(12) If $x \in A$, then $x \in \mathrm{Lin}_{\mathbb{Z}}\, A$.

(13) If $A \subseteq B$, then $\mathrm{Lin}_{\mathbb{Z}}\, A \subseteq \mathrm{Lin}_{\mathbb{Z}}\, B$.

(14) $\mathrm{Lin}_{\mathbb{Z}}(A \cup B) = (\mathrm{Lin}_{\mathbb{Z}}\, A) + \mathrm{Lin}_{\mathbb{Z}}\, B$.

(15) $\mathrm{Lin}_{\mathbb{Z}}(A \cap B) \subseteq (\mathrm{Lin}_{\mathbb{Z}}\, A) \cap \mathrm{Lin}_{\mathbb{Z}}\, B$.

(16) $x \in \mathrm{Lin}_{\mathbb{Z}}\{v\}$ iff there exists an integer a such that $x = a \cdot v$.

(17) $v \in \mathrm{Lin}_{\mathbb{Z}}\{v\}$.

(18) $x \in v + \mathrm{Lin}_{\mathbb{Z}}\{w\}$ iff there exists an integer a such that $x = v + a \cdot w$.

(19) $x \in \mathrm{Lin}_{\mathbb{Z}}\{w_1, w_2\}$ iff there exist integers a, b such that $x = a \cdot w_1 + b \cdot w_2$.

(20) $w_1 \in \mathrm{Lin}_{\mathbb{Z}}\{w_1, w_2\}$.

(21) $x \in v + \mathrm{Lin}_{\mathbb{Z}}\{w_1, w_2\}$ iff there exist integers a, b such that $x = v + a \cdot w_1 + b \cdot w_2$.

(22) $x \in \mathrm{Lin}_{\mathbb{Z}}\{v_1, v_2, v_3\}$ iff there exist integers a, b, c such that $x = a \cdot v_1 + b \cdot v_2 + c \cdot v_3$.

(23) w_1, w_2, $w_3 \in \mathrm{Lin}_{\mathbb{Z}}\{w_1, w_2, w_3\}$.

(24) $x \in v + \mathrm{Lin}_{\mathbb{Z}}\{w_1, w_2, w_3\}$ iff there exist integers a, b, c such that $x = v + a \cdot w_1 + b \cdot w_2 + c \cdot w_3$.

(25) Let x be a set. Then $x \in \mathrm{Lin}_{\mathbb{Z}} A$ if and only if there exist finite sequences g_1, h_1 of elements of V and there exists an integer-valued finite sequence a_1 such that $x = \sum h_1$ and $\mathrm{rng}\, g_1 \subseteq A$ and $\mathrm{len}\, g_1 = \mathrm{len}\, h_1$ and $\mathrm{len}\, g_1 = \mathrm{len}\, a_1$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, g_1$ holds $(h_1)_i = a_1(i) \cdot (g_1)_i$.

Let R_4 be a real linear space and let f be a finite sequence of elements of R_4. The functor $\mathrm{Lin}_{\mathbb{Z}} f$ yielding a subset of R_4 is defined by the condition (Def. 3).

(Def. 3) $\mathrm{Lin}_{\mathbb{Z}} f = \{\sum g; g$ ranges over $\mathrm{len}\, f$-element finite sequences of elements of R_4: $\bigvee_{a:\ \mathrm{len}\, f\text{-element integer-valued finite sequence}} \bigwedge_{i:\,\text{natural number}} (i \in \mathrm{Seg}\,\mathrm{len}\, f \Rightarrow g_i = a(i) \cdot f_i)\}$.

One can prove the following propositions:

(26) Let R_4 be a real linear space, f be a finite sequence of elements of R_4, and x be a set. Then $x \in \mathrm{Lin}_{\mathbb{Z}} f$ if and only if there exists a $\mathrm{len}\, f$-element finite sequence g of elements of R_4 and there exists a $\mathrm{len}\, f$-element integer-valued finite sequence a such that $x = \sum g$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, f$ holds $g_i = a(i) \cdot f_i$.

(27) Let R_4 be a real linear space, f be a finite sequence of elements of R_4, x, y be elements of R_4, and a, b be elements of $\mathbb{Z}$. If x, $y \in \mathrm{Lin}_{\mathbb{Z}} f$, then $a \cdot x + b \cdot y \in \mathrm{Lin}_{\mathbb{Z}} f$.

(28) For every real linear space R_4 and for every finite sequence f of elements of R_4 such that $f = \mathrm{Seg}\,\mathrm{len}\, f \longmapsto 0_{(R_4)}$ holds $\sum f = 0_{(R_4)}$.

(29) Let R_4 be a real linear space, f be a finite sequence of elements of R_4, v be an element of R_4, and i be a natural number. If $i \in \mathrm{Seg}\,\mathrm{len}\, f$ and $f = (\mathrm{Seg}\,\mathrm{len}\, f \longmapsto 0_{(R_4)}) {+\cdot} (\{i\} \longmapsto v)$, then $\sum f = v$.

(30) Let R_4 be a real linear space, f be a finite sequence of elements of R_4, and i be a natural number. If $i \in \mathrm{Seg}\,\mathrm{len}\, f$, then $f_i \in \mathrm{Lin}_{\mathbb{Z}} f$.

(31) For every real linear space R_4 and for every finite sequence f of elements of R_4 holds $\mathrm{rng}\, f \subseteq \mathrm{Lin}_{\mathbb{Z}} f$.

(32) Let R_4 be a real linear space, f be a non empty finite sequence of elements of R_4, g, h be finite sequences of elements of R_4, and s be an integer-valued finite sequence. Suppose $\mathrm{rng}\, g \subseteq \mathrm{Lin}_{\mathbb{Z}} f$ and $\mathrm{len}\, g = \mathrm{len}\, s$ and $\mathrm{len}\, g = \mathrm{len}\, h$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, g$ holds $h_i = s(i) \cdot g_i$. Then $\sum h \in \mathrm{Lin}_{\mathbb{Z}} f$.

(33) For every real linear space R_4 and for every non empty finite sequence f of elements of R_4 holds $\mathrm{Lin}_{\mathbb{Z}}\, \mathrm{rng}\, f = \mathrm{Lin}_{\mathbb{Z}} f$.

(34) $\mathrm{Lin}(\mathrm{Lin}_{\mathbb{Z}} A) = \mathrm{Lin}(A)$.

(35) Let x be a set, g_1, h_1 be finite sequences of elements of V, and a_1 be an integer-valued finite sequence. Suppose $x = \sum h_1$ and $\mathrm{rng}\, g_1 \subseteq \mathrm{Lin}_{\mathbb{Z}} A$ and $\mathrm{len}\, g_1 = \mathrm{len}\, h_1$ and $\mathrm{len}\, g_1 = \mathrm{len}\, a_1$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, g_1$ holds $(h_1)_i = a_1(i) \cdot (g_1)_i$. Then $x \in \mathrm{Lin}_{\mathbb{Z}} A$.

(36) $\mathrm{Lin}_{\mathbb{Z}}\,\mathrm{Lin}_{\mathbb{Z}}\,A = \mathrm{Lin}_{\mathbb{Z}}\,A$.

(37) If $\mathrm{Lin}_{\mathbb{Z}}\,A = \mathrm{Lin}_{\mathbb{Z}}\,B$, then $\mathrm{Lin}(A) = \mathrm{Lin}(B)$.

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