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(a computer assisted approach)

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Contents

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Combinatorial Grassmannians

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Summary. In the paper I construct the configuration G which is a partial linear space. It consists of k-element subsets of some base set as points and $(k+1)$-element subsets as lines. The incidence is given by inclusion. I also introduce automorphisms of partial linear spaces and show that automorphisms of G are generated by permutations of the base set.

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The articles [15], [17], [3], [14], [7], [11], [13], [8], [18], [19], [4], [12], [16], [9], [5], [6], [10], [2], and [1] provide the notation and terminology for this paper.

1. Preliminaries

We follow the rules: k, n denote elements of $\mathbb{N}$ and X, Y, Z denote sets.

One can prove the following propositions:

(1) For all sets a, b such that $a \neq b$ and $\overline{\overline{a}} = n$ and $\overline{\overline{b}} = n$ holds $\overline{\overline{a \cap b}} < n$ and $n + 1 \leq \overline{\overline{a \cup b}}$.

(2) For all sets a, b such that $\overline{\overline{a}} = n + k$ and $\overline{\overline{b}} = n + k$ holds $\overline{\overline{a \cap b}} = n$ iff $\overline{\overline{a \cup b}} = n + 2 \cdot k$.

(3) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ iff there exists a function f such that f is one-to-one and $X \subseteq \operatorname{dom} f$ and $f^{\circ}X \subseteq Y$.

(4) For every function f such that f is one-to-one and $X \subseteq \operatorname{dom} f$ holds $\overline{\overline{f^{\circ}X}} = \overline{\overline{X}}$.

(5) If $X \setminus Y = X \setminus Z$ and $Y \subseteq X$ and $Z \subseteq X$, then $Y = Z$.

(6) Let Y be a non empty set and p be a function from X into Y. Suppose p is one-to-one. Let x_1, x_2 be subsets of X. If $x_1 \neq x_2$, then $p^{\circ}x_1 \neq p^{\circ}x_2$.

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(7) Let a, b, c be sets such that $\overline{\overline{a}} = n-1$ and $\overline{\overline{b}} = n-1$ and $\overline{\overline{c}} = n-1$ and $\overline{\overline{a \cap b}} = n-2$ and $\overline{\overline{a \cap c}} = n-2$ and $\overline{\overline{b \cap c}} = n-2$ and $2 \leq n$. Then

(i) if $3 \leq n$, then $\overline{\overline{a \cap b \cap c}} = n-2$ and $\overline{\overline{a \cup b \cup c}} = n+1$ or $\overline{\overline{a \cap b \cap c}} = n-3$ and $\overline{\overline{a \cup b \cup c}} = n$, and

(ii) if $n = 2$, then $\overline{\overline{a \cap b \cap c}} = n-2$ and $\overline{\overline{a \cup b \cup c}} = n+1$.

(8) Let P_1, P_2 be projective incidence structures. Suppose the projective incidence structure of P_1 = the projective incidence structure of P_2. Let A_1 be a point of P_1 and A_2 be a point of P_2. Suppose $A_1 = A_2$. Let L_1 be a line of P_1 and L_2 be a line of P_2. If $L_1 = L_2$, then if A_1 lies on L_1, then A_2 lies on L_2.

(9) Let P_1, P_2 be projective incidence structures. Suppose the projective incidence structure of P_1 = the projective incidence structure of P_2. Let A_1 be a subset of the points of P_1 and A_2 be a subset of the points of P_2. Suppose $A_1 = A_2$. Let L_1 be a line of P_1 and L_2 be a line of P_2. If $L_1 = L_2$, then if A_1 lies on L_1, then A_2 lies on L_2.

Let us note that there exists a projective incidence structure which is linear, up-2-rank, and strict and has non-trivial-lines.

2. Configuration G

A partial linear space is an up-2-rank projective incidence structure with non-trivial-lines.

Let k be an element of $\mathbb{N}$ and let X be a non empty set. Let us assume that $0 < k$ and $k+1 \leq \overline{\overline{X}}$. The functor $\mathrm{G}_k(X)$ yields a strict partial linear space and is defined by the conditions (Def. 1).

(Def. 1)(i) The points of $\mathrm{G}_k(X) = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k\}$,

(ii) the lines of $\mathrm{G}_k(X) = \{L; L$ ranges over subsets of X: $\overline{\overline{L}} = k+1\}$, and

(iii) the incidence of $\mathrm{G}_k(X) = \subseteq_{2^X} \cap [\!:$ the points of $\mathrm{G}_k(X)$, the lines of $\mathrm{G}_k(X) :\!]$.

One can prove the following four propositions:

(10) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let A be a point of $\mathrm{G}_k(X)$ and L be a line of $\mathrm{G}_k(X)$. Then A lies on L if and only if $A \subseteq L$.

(11) For every element k of $\mathbb{N}$ and for every non empty set X such that $0 < k$ and $k+1 \leq \overline{\overline{X}}$ holds $\mathrm{G}_k(X)$ is Vebleian.

(12) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let A_1, A_2, A_3, A_4, A_5, A_6 be points of $\mathrm{G}_k(X)$ and L_1, L_2, L_3, L_4 be lines of $\mathrm{G}_k(X)$. Suppose that A_1 lies on L_1 and A_2 lies on L_1 and A_3 lies on L_2 and A_4 lies on L_2 and A_5 lies on L_1 and A_5 lies on

L_2 and A_1 lies on L_3 and A_3 lies on L_3 and A_2 lies on L_4 and A_4 lies on L_4 and A_5 does not lie on L_3 and A_5 does not lie on L_4 and $L_1 \neq L_2$ and $L_3 \neq L_4$. Then there exists a point A_6 of $\mathrm{G}_k(X)$ such that A_6 lies on L_3 and A_6 lies on L_4 and $A_6 = A_1 \cap A_2 \cup A_3 \cap A_4$.

(13) For every element k of $\mathbb{N}$ and for every non empty set X such that $0 < k$ and $k + 1 \leq \overline{\overline{X}}$ holds $\mathrm{G}_k(X)$ is Desarguesian.

Let S be a projective incidence structure and let K be a subset of the points of S. We say that K is a clique if and only if:

(Def. 2) For all points A, B of S such that $A \in K$ and $B \in K$ there exists a line L of S such that $\{A, B\}$ lies on L.

Let S be a projective incidence structure and let K be a subset of the points of S. We say that K is a maximal-clique if and only if:

(Def. 3) K is a clique and for every subset U of the points of S such that U is a clique and $K \subseteq U$ holds $U = K$.

Let k be an element of $\mathbb{N}$, let X be a non empty set, and let T be a subset of the points of $\mathrm{G}_k(X)$. We say that T is a star if and only if:

(Def. 4) There exists a subset S of X such that $\overline{\overline{S}} = k - 1$ and $T = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k \ \wedge \ S \subseteq A\}$.

We say that T is a top if and only if:

(Def. 5) There exists a subset S of X such that $\overline{\overline{S}} = k + 1$ and $T = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k \ \wedge \ A \subseteq S\}$.

Next we state two propositions:

(14) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k + 2 \leq \overline{\overline{X}}$. Let K be a subset of the points of $\mathrm{G}_k(X)$. If K is a star or a top, then K is a maximal-clique.

(15) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k + 2 \leq \overline{\overline{X}}$. Let K be a subset of the points of $\mathrm{G}_k(X)$. If K is a maximal-clique, then K is a star or a top.

3. Automorphisms

Let S_1, S_2 be projective incidence structures. We consider maps between projective spaces S_1 and S_2 as systems

$\langle$ a point-map, a line-map $\rangle$,

where the point-map is a function from the points of S_1 into the points of S_2 and the line-map is a function from the lines of S_1 into the lines of S_2.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2, and let a be a point of S_1. The functor $F(a)$ yields a point of S_2 and is defined as follows:

(Def. 6) $F(a) =$ (the point-map of F)(a).

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2, and let L be a line of S_1. The functor $F(L)$ yields a line of S_2 and is defined by:

(Def. 7) $F(L) =$ (the line-map of F)(L).

Next we state the proposition

(16) Let S_1, S_2 be projective incidence structures and F_1, F_2 be maps between projective spaces S_1 and S_2. Suppose for every point A of S_1 holds $F_1(A) = F_2(A)$ and for every line L of S_1 holds $F_1(L) = F_2(L)$. Then the map of $F_1 =$ the map of F_2.

Let S_1, S_2 be projective incidence structures and let F be a map between projective spaces S_1 and S_2. We say that F preserves incidence strongly if and only if:

(Def. 8) For every point A_1 of S_1 and for every line L_1 of S_1 holds A_1 lies on L_1 iff $F(A_1)$ lies on $F(L_1)$.

The following proposition is true

(17) Let S_1, S_2 be projective incidence structures and F_1, F_2 be maps between projective spaces S_1 and S_2. Suppose the map of $F_1 =$ the map of F_2. If F_1 preserves incidence strongly, then F_2 preserves incidence strongly.

Let S be a projective incidence structure and let F be a map between projective spaces S and S. We say that F is automorphism if and only if:

(Def. 9) The line-map of F is bijective and the point-map of F is bijective and F preserves incidence strongly.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2, and let K be a subset of the points of S_1. The functor $F^\circ K$ yielding a subset of the points of S_2 is defined by:

(Def. 10) $F^\circ K =$ (the point-map of F)$^\circ K$.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2, and let K be a subset of the points of S_2. The functor $F^{-1}(K)$ yielding a subset of the points of S_1 is defined as follows:

(Def. 11) $F^{-1}(K) =$ (the point-map of F)$^{-1}(K)$.

Let X be a set and let A be a finite set. The functor $\uparrow(A, X)$ yielding a subset of 2^X is defined as follows:

(Def. 12) $\uparrow(A, X) = \{B; B$ ranges over subsets of X: $\overline{\overline{B}} = \mathrm{card}\, A + 1 \ \wedge\ A \subseteq B\}$.

Let k be an element of $\mathbb{N}$ and let X be a non empty set. Let us assume that $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. Let A be a finite set. Let us assume that $\overline{\overline{A}} = k - 1$ and $A \subseteq X$. The functor $\uparrow(A, X, k)$ yields a subset of the points of $\mathrm{G}_k(X)$ and is defined as follows:

(Def. 13) $\uparrow(A, X, k) = \uparrow(A, X)$.

The following propositions are true:

(18) Let S_1, S_2 be projective incidence structures, F be a map between projective spaces S_1 and S_2, and K be a subset of the points of S_1. Then $F^\circ K = \{B; B$ ranges over points of S_2: $\bigvee_{A\,:\,\text{point of }S_1} (A \in K \ \wedge\ F(A) = B)\}$.

(19) Let S_1, S_2 be projective incidence structures, F be a map between projective spaces S_1 and S_2, and K be a subset of the points of S_2. Then $F^{-1}(K) = \{A; A$ ranges over points of S_1: $\bigvee_{B\,:\,\text{point of }S_2} (B \in K \ \wedge\ F(A) = B)\}$.

(20) Let S be a projective incidence structure, F be a map between projective spaces S and S, and K be a subset of the points of S. If F preserves incidence strongly and K is a clique, then $F^\circ K$ is a clique.

(21) Let S be a projective incidence structure, F be a map between projective spaces S and S, and K be a subset of the points of S. Suppose F preserves incidence strongly and the line-map of F is onto and K is a clique. Then $F^{-1}(K)$ is a clique.

(22) Let S be a projective incidence structure, F be a map between projective spaces S and S, and K be a subset of the points of S. Suppose F is automorphism and K is a maximal-clique. Then $F^\circ K$ is a maximal-clique and $F^{-1}(K)$ is a maximal-clique.

(23) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k+2 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose F is automorphism. Let K be a subset of the points of $\mathrm{G}_k(X)$. If K is a star, then $F^\circ K$ is a star and $F^{-1}(K)$ is a star.

Let k be an element of $\mathbb{N}$ and let X be a non empty set. Let us assume that $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let s be a permutation of X. The functor $\mathrm{incprojmap}(k, s)$ yielding a strict map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$ is defined as follows:

(Def. 14) For every point A of $\mathrm{G}_k(X)$ holds $(\mathrm{incprojmap}(k, s))(A) = s^\circ A$ and for every line L of $\mathrm{G}_k(X)$ holds $(\mathrm{incprojmap}(k, s))(L) = s^\circ L$.

One can prove the following propositions:

(24) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $k = 1$ and $k + 1 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose F is automorphism. Then there exists a permutation s of X such that the map of $F = \mathrm{incprojmap}(k, s)$.

(25) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $1 < k$ and $\overline{\overline{X}} = k + 1$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose F is automorphism. Then there exists a permutation s of X such that the map of $F = \mathrm{incprojmap}(k, s)$.

(26) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$

and $k+1 \leq \overline{\overline{X}}$. Let T be a subset of the points of $\mathrm{G}_k(X)$ and S be a subset of X. If $\overline{\overline{S}} = k-1$ and $T = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k \ \wedge \ S \subseteq A\}$, then $S = \bigcap T$.

(27) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let T be a subset of the points of $\mathrm{G}_k(X)$. Suppose T is a star. Let S be a subset of X. If $S = \bigcap T$, then $\overline{\overline{S}} = k-1$ and $T = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k \ \wedge \ S \subseteq A\}$.

(28) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let T_1, T_2 be subsets of the points of $\mathrm{G}_k(X)$. If T_1 is a star and T_2 is a star and $\bigcap T_1 = \bigcap T_2$, then $T_1 = T_2$.

(29) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let A be a finite subset of X. If $\overline{\overline{A}} = k-1$, then $\uparrow(A, X, k)$ is a star.

(30) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let A be a finite subset of X. If $\overline{\overline{A}} = k-1$, then $\bigcap \uparrow(A, X, k) = A$.

(31) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+3 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_{(k+1)}(X)$ and $\mathrm{G}_{(k+1)}(X)$. Suppose F is automorphism. Then there exists a map H between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$ such that
   (i) H is automorphism,
   (ii) the line-map of $H =$ the point-map of F, and
   (iii) for every point A of $\mathrm{G}_k(X)$ and for every finite set B such that $B = A$ holds $H(A) = \bigcap(F^\circ \uparrow(B, X, k+1))$.

(32) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+3 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_{(k+1)}(X)$ and $\mathrm{G}_{(k+1)}(X)$. Suppose F is automorphism. Let H be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose that
   (i) H is automorphism,
   (ii) the line-map of $H =$ the point-map of F, and
   (iii) for every point A of $\mathrm{G}_k(X)$ and for every finite set B such that $B = A$ holds $H(A) = \bigcap(F^\circ \uparrow(B, X, k+1))$.

   Let f be a permutation of X. If the map of $H = \mathrm{incprojmap}(k, f)$, then the map of $F = \mathrm{incprojmap}(k+1, f)$.

(33) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k+2 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose F is automorphism. Then there exists a permutation s of X such that the map of $F = \mathrm{incprojmap}(k, s)$.

(34) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$

and $k+1 \leq \overline{\overline{X}}$. Let s be a permutation of X. Then incprojmap(k,s) is automorphism.

(35) Let X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Then F is automorphism if and only if there exists a permutation s of X such that the map of $F = \text{incprojmap}(k,s)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[10] Wojciech Leończuk and Krzysztof Prażmowski. Incidence projective spaces. *Formalized Mathematics*, 2(**2**):225–232, 1991.
[11] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[13] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Wojciech A. Trybulec. Axioms of incidency. *Formalized Mathematics*, 1(**1**):205–213, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[19] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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The Jordan-Hölder Theorem

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Summary. The goal of this article is to formalize the Jordan-Hölder theorem in the context of group with operators as in the book [5]. Accordingly, the article introduces the structure of group with operators and reformulates some theorems on a group already present in the Mizar Mathematical Library. Next, the article formalizes the Zassenhaus butterfly lemma and the Schreier refinement theorem, and defines the composition series.

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The terminology and notation used here are introduced in the following articles: [17], [25], [3], [26], [7], [27], [8], [9], [4], [10], [1], [12], [18], [2], [6], [21], [20], [22], [19], [15], [23], [11], [14], [16], [13], and [24].

1. Actions and Groups with Operators

Let O, E be sets. An action of O on E is a function from O into E^E.

Let O, E be sets, let A be an action of O on E, and let I_1 be a set. We say that I_1 is stable under the action of A if and only if:

(Def. 1) For every element o of O and for every function f from E into E such that $o \in O$ and $f = A(o)$ holds $f^\circ I_1 \subseteq I_1$.

Let O, E be sets, let A be an action of O on E, and let X be a subset of E. The stable subset generated by X yields a subset of E and is defined by the conditions (Def. 2).

(Def. 2)(i) $X \subseteq$ the stable subset generated by X,

(ii) the stable subset generated by X is stable under the action of A, and

(iii) for every subset Y of E such that Y is stable under the action of A and $X \subseteq Y$ holds the stable subset generated by $X \subseteq Y$.

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Let O, E be sets, let A be an action of O on E, and let F be a finite sequence of elements of O. The functor $\mathrm{Product}(F, A)$ yields a function from E into E and is defined by:

(Def. 3)(i) $\mathrm{Product}(F, A) = \mathrm{id}_E$ if $\mathrm{len}\, F = 0$,

(ii) there exists a finite sequence P_1 of elements of E^E such that $\mathrm{Product}(F, A) = P_1(\mathrm{len}\, F)$ and $\mathrm{len}\, P_1 = \mathrm{len}\, F$ and $P_1(1) = A(F(1))$ and for every natural number n such that $n \neq 0$ and $n < \mathrm{len}\, F$ there exist functions f, g from E into E such that $f = P_1(n)$ and $g = A(F(n + 1))$ and $P_1(n + 1) = f \cdot g$, otherwise.

Let O be a set, let G be a group, and let I_1 be an action of O on the carrier of G. We say that I_1 is distributive if and only if:

(Def. 4) For every element o of O such that $o \in O$ holds $I_1(o)$ is a homomorphism from G to G.

Let O be a set. We consider group structures with operators in O as extensions of groupoid as systems

$\langle$ a carrier, a multiplication, an action $\rangle$,

where the carrier is a set, the multiplication is a binary operation on the carrier, and the action is an action of O on the carrier.

Let O be a set. Observe that there exists a group structure with operators in O which is non empty.

Let O be a set and let I_1 be a non empty group structure with operators in O. We say that I_1 is distributive if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let G be a group and a be an action of O on the carrier of G. Suppose $a =$ the action of I_1 and the groupoid of $G =$ the groupoid of I_1. Then a is distributive.

Let O be a set. Observe that there exists a non empty group structure with operators in O which is strict, distributive, group-like, and associative.

Let O be a set. A group with operators in O is a distributive group-like associative non empty group structure with operators in O.

Let O be a set, let G be a group with operators in O, and let o be an element of O. The functor $G \frown o$ yields a homomorphism from G to G and is defined as follows:

(Def. 6) $G \frown o = \begin{cases} (\text{the action of } G)(o), & \text{if } o \in O, \\ \mathrm{id}_{\text{the carrier of } G}, & \text{otherwise.} \end{cases}$

Let O be a set and let G be a group with operators in O. A distributive group-like associative non empty group structure with operators in O is said to be a stable subgroup of G if:

(Def. 7) It is a subgroup of G and for every element o of O holds $\mathrm{it} \frown o = (G \frown o) \restriction$ the carrier of it.

Let O be a set and let G be a group with operators in O. Note that there exists a stable subgroup of G which is strict.

Let O be a set and let G be a group with operators in O. The functor $\{\mathbf{1}\}_G$ yields a strict stable subgroup of G and is defined by:

(Def. 8) The carrier of $\{\mathbf{1}\}_G = \{\mathbf{1}_G\}$.

Let O be a set and let G be a group with operators in O. The functor Ω_G yielding a strict stable subgroup of G is defined as follows:

(Def. 9) $\Omega_G =$ the group structure with operators of G.

Let O be a set, let G be a group with operators in O, and let I_1 be a stable subgroup of G. We say that I_1 is normal if and only if:

(Def. 10) For every strict subgroup H of G such that $H =$ the groupoid of I_1 holds H is normal.

Let O be a set and let G be a group with operators in O. Note that there exists a stable subgroup of G which is strict and normal.

Let O be a set, let G be a group with operators in O, and let H be a stable subgroup of G. Observe that there exists a stable subgroup of H which is normal.

Let O be a set and let G be a group with operators in O. Note that $\{\mathbf{1}\}_G$ is normal and Ω_G is normal.

Let O be a set and let G be a group with operators in O. The stable subgroups of G yields a set and is defined as follows:

(Def. 11) For every set x holds $x \in$ the stable subgroups of G iff x is a strict stable subgroup of G.

Let O be a set and let G be a group with operators in O. Observe that the stable subgroups of G is non empty.

Let I_1 be a group. We say that I_1 is simple if and only if:

(Def. 12) I_1 is not trivial and it is not true that there exists a strict normal subgroup H of I_1 such that $H \neq \Omega_{(I_1)}$ and $H \neq \{\mathbf{1}\}_{(I_1)}$.

Let us note that there exists a group which is strict and simple.

Let O be a set and let I_1 be a group with operators in O. We say that I_1 is simple if and only if:

(Def. 13) I_1 is not trivial and it is not true that there exists a strict normal stable subgroup H of I_1 such that $H \neq \Omega_{(I_1)}$ and $H \neq \{\mathbf{1}\}_{(I_1)}$.

Let O be a set. Observe that there exists a group with operators in O which is strict and simple.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The functor $\mathrm{Cosets}\, N$ yields a set and is defined by:

(Def. 14) For every strict normal subgroup H of G such that $H =$ the groupoid of N holds $\mathrm{Cosets}\, N = \mathrm{Cosets}\, H$.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The functor $\mathrm{CosOp}\, N$ yielding a binary operation on $\mathrm{Cosets}\, N$ is defined by:

(Def. 15) For every strict normal subgroup H of G such that $H =$ the groupoid of N holds $\operatorname{CosOp} N = \operatorname{CosOp} H$.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The functor $\operatorname{CosAc} N$ yielding an action of O on $\operatorname{Cosets} N$ is defined as follows:

(Def. 16)(i) For every element o of O holds $(\operatorname{CosAc} N)(o) = \{\langle A, B\rangle; A$ ranges over elements of $\operatorname{Cosets} N, B$ ranges over elements of $\operatorname{Cosets} N$: $\bigvee_{g,h:\text{element of } G} (g \in A \wedge h \in B \wedge h = (G \frown o)(g))\}$ if O is not empty,

(ii) $\operatorname{CosAc} N = [\emptyset, \{\mathrm{id}_{\operatorname{Cosets} N}\}]$, otherwise.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The functor ${}^{G}/_{N}$ yields a group structure with operators in O and is defined as follows:

(Def. 17) ${}^{G}/_{N} = \langle \operatorname{Cosets} N, \operatorname{CosOp} N, \operatorname{CosAc} N\rangle$.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. Note that ${}^{G}/_{N}$ is non empty and ${}^{G}/_{N}$ is distributive, group-like, and associative.

Let O be a set, let G, H be groups with operators in O, and let f be a function from G into H. We say that f is homomorphic if and only if:

(Def. 18) For every element o of O and for every element g of G holds $f((G \frown o)(g)) = (H \frown o)(f(g))$.

Let O be a set and let G, H be groups with operators in O. One can check that there exists a function from G into H which is multiplicative and homomorphic.

Let O be a set and let G, H be groups with operators in O. A homomorphism from G to H is a multiplicative homomorphic function from G into H.

Let O be a set, let G, H, I be groups with operators in O, let h be a homomorphism from G to H, and let h_1 be a homomorphism from H to I. Then $h_1 \cdot h$ is a homomorphism from G to I.

Let O be a set, let G, H be groups with operators in O, and let h be a homomorphism from G to H. We say that h is monomorphism if and only if:

(Def. 19) h is one-to-one.

We say that h is epimorphism if and only if:

(Def. 20) $\operatorname{rng} h =$ the carrier of H.

Let O be a set, let G, H be groups with operators in O, and let h be a homomorphism from G to H. We say that h is isomorphism if and only if:

(Def. 21) h is an epimorphism and a monomorphism.

Let O be a set and let G, H be groups with operators in O. We say that G and H are isomorphic if and only if:

(Def. 22) There exists a homomorphism from G to H which is an isomorphism.

Let us note that the predicate G and H are isomorphic is reflexive.

Let O be a set and let G, H be groups with operators in O. Let us note that the predicate G and H are isomorphic is symmetric.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The canonical homomorphism onto cosets of N yields a homomorphism from G to ${}^{G}/_{N}$ and is defined by the condition (Def. 23).

(Def. 23) Let H be a strict normal subgroup of G. Suppose $H =$ the groupoid of N. Then the canonical homomorphism onto cosets of $N =$ the canonical homomorphism onto cosets of H.

Let O be a set, let G, H be groups with operators in O, and let g be a homomorphism from G to H. The functor $\operatorname{Ker} g$ yields a strict stable subgroup of G and is defined as follows:

(Def. 24) The carrier of $\operatorname{Ker} g = \{a; a$ ranges over elements of G: $g(a) = \mathbf{1}_H\}$.

Let O be a set, let G, H be groups with operators in O, and let g be a homomorphism from G to H. Observe that $\operatorname{Ker} g$ is normal.

Let O be a set, let G, H be groups with operators in O, and let g be a homomorphism from G to H. The functor $\operatorname{Im} g$ yielding a strict stable subgroup of H is defined by:

(Def. 25) The carrier of $\operatorname{Im} g = g^{\circ}$(the carrier of G).

Let O be a set, let G be a group with operators in O, and let H be a stable subgroup of G. The functor $\overline{H}$ yielding a subset of G is defined as follows:

(Def. 26) $\overline{H} =$ the carrier of H.

Let O be a set, let G be a group with operators in O, and let H_1, H_2 be stable subgroups of G. The functor $H_1 \cdot H_2$ yields a subset of G and is defined as follows:

(Def. 27) $H_1 \cdot H_2 = \overline{H_1} \cdot \overline{H_2}$.

Let O be a set, let G be a group with operators in O, and let H_1, H_2 be stable subgroups of G. The functor $H_1 \cap H_2$ yielding a strict stable subgroup of G is defined by:

(Def. 28) The carrier of $H_1 \cap H_2 = \overline{H_1} \cap \overline{H_2}$.

Let us note that the functor $H_1 \cap H_2$ is commutative.

Let O be a set, let G be a group with operators in O, and let A be a subset of G. The stable subgroup of A yielding a strict stable subgroup of G is defined by the conditions (Def. 29).

(Def. 29)(i) $A \subseteq$ the carrier of the stable subgroup of A, and

(ii) for every strict stable subgroup H of G such that $A \subseteq$ the carrier of H holds the stable subgroup of A is a stable subgroup of H.

Let O be a set, let G be a group with operators in O, and let H_1, H_2 be stable subgroups of G. The functor $H_1 \sqcup H_2$ yielding a strict stable subgroup of G is defined as follows:

(Def. 30) $H_1 \sqcup H_2 =$ the stable subgroup of $\overline{H_1} \cup \overline{H_2}$.

2. Some Theorems on Groups Reformulated for Groups with Operators

For simplicity, we follow the rules: x, O are sets, o is an element of O, G, H, I are groups with operators in O, A, B are subsets of G, N is a normal stable subgroup of G, H_1, H_2, H_3 are stable subgroups of G, g_1, g_2 are elements of G, h_1, h_2 are elements of H_1, and h is a homomorphism from G to H.

One can prove the following propositions:

(1) If $x \in H_1$, then $x \in G$.

(2) h_1 is an element of G.

(3) If $h_1 = g_1$ and $h_2 = g_2$, then $h_1 \cdot h_2 = g_1 \cdot g_2$.

(4) $\mathbf{1}_G = \mathbf{1}_{(H_1)}$.

(5) $\mathbf{1}_G \in H_1$.

(6) If $h_1 = g_1$, then ${h_1}^{-1} = {g_1}^{-1}$.

(7) If $g_1 \in H_1$ and $g_2 \in H_1$, then $g_1 \cdot g_2 \in H_1$.

(8) If $g_1 \in H_1$, then ${g_1}^{-1} \in H_1$.

(9) Suppose that

(i) $A \neq \emptyset$,

(ii) for all g_1, g_2 such that $g_1 \in A$ and $g_2 \in A$ holds $g_1 \cdot g_2 \in A$,

(iii) for every g_1 such that $g_1 \in A$ holds ${g_1}^{-1} \in A$, and

(iv) for all o, g_1 such that $g_1 \in A$ holds $(G \frown o)(g_1) \in A$.

Then there exists a strict stable subgroup H of G such that the carrier of $H = A$.

(10) G is a stable subgroup of G.

(11) Let G_1, G_2, G_3 be groups with operators in O. Suppose G_1 is a stable subgroup of G_2 and G_2 is a stable subgroup of G_3. Then G_1 is a stable subgroup of G_3.

(12) If the carrier of $H_1 \subseteq$ the carrier of H_2, then H_1 is a stable subgroup of H_2.

(13) If for every element g of G such that $g \in H_1$ holds $g \in H_2$, then H_1 is a stable subgroup of H_2.

(14) For all strict stable subgroups H_1, H_2 of G such that the carrier of $H_1 =$ the carrier of H_2 holds $H_1 = H_2$.

(15) $\{\mathbf{1}\}_G = \{\mathbf{1}\}_{(H_1)}$.

(16) $\{\mathbf{1}\}_G$ is a stable subgroup of H_1.

(17) If $\overline{H_1} \cdot \overline{H_2} = \overline{H_2} \cdot \overline{H_1}$, then there exists a strict stable subgroup H of G such that the carrier of $H = \overline{H_1} \cdot \overline{H_2}$.

(18)(i) For every stable subgroup H of G such that $H = H_1 \cap H_2$ holds the carrier of $H =$ (the carrier of H_1) $\cap$ (the carrier of H_2), and

(ii) for every strict stable subgroup H of G such that the carrier of $H =$ (the carrier of H_1) $\cap$ (the carrier of H_2) holds $H = H_1 \cap H_2$.

(19) For every strict stable subgroup H of G holds $H \cap H = H$.

(20) $(H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)$.

(21) $\{\mathbf{1}\}_G \cap H_1 = \{\mathbf{1}\}_G$ and $H_1 \cap \{\mathbf{1}\}_G = \{\mathbf{1}\}_G$.

(22) $\bigcup \mathrm{Cosets}\, N =$ the carrier of G.

(23) Let N_1, N_2 be strict normal stable subgroups of G. Then there exists a strict normal stable subgroup N of G such that the carrier of $N = \overline{N_1} \cdot \overline{N_2}$.

(24) $g_1 \in$ the stable subgroup of A if and only if there exists a finite sequence F of elements of the carrier of G and there exists a finite sequence I of elements of $\mathbb{Z}$ and there exists a subset C of G such that $C =$ the stable subset generated by A and $\mathrm{len}\, F = \mathrm{len}\, I$ and $\mathrm{rng}\, F \subseteq C$ and $\prod(F^I) = g_1$.

(25) For every strict stable subgroup H of G holds the stable subgroup of $\overline{H} = H$.

(26) If $A \subseteq B$, then the stable subgroup of A is a stable subgroup of the stable subgroup of B.

The scheme *MeetSbgWOpEx* deals with a set $\mathcal{A}$, a group $\mathcal{B}$ with operators in $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a strict stable subgroup H of $\mathcal{B}$ such that the carrier of $H = \bigcap\{A; A$ ranges over subsets of $\mathcal{B}$: $\bigvee_{K\,:\,\text{strict stable subgroup of } \mathcal{B}} (A = \text{the carrier of } K \ \wedge\ \mathcal{P}[K])\}$

provided the parameters meet the following requirement:

- There exists a strict stable subgroup H of $\mathcal{B}$ such that $\mathcal{P}[H]$.

The following propositions are true:

(27) The carrier of the stable subgroup of $A = \bigcap\{B; B$ ranges over subsets of G: $\bigvee_{H\,:\,\text{strict stable subgroup of } G} (B = \text{the carrier of } H \ \wedge\ A \subseteq \overline{H})\}$.

(28) For all strict normal stable subgroups N_1, N_2 of G holds $N_1 \cdot N_2 = N_2 \cdot N_1$.

(29) $H_1 \sqcup H_2 =$ the stable subgroup of $H_1 \cdot H_2$.

(30) If $H_1 \cdot H_2 = H_2 \cdot H_1$, then the carrier of $H_1 \sqcup H_2 = H_1 \cdot H_2$.

(31) For all strict normal stable subgroups N_1, N_2 of G holds the carrier of $N_1 \sqcup N_2 = N_1 \cdot N_2$.

(32) For all strict normal stable subgroups N_1, N_2 of G holds $N_1 \sqcup N_2$ is a normal stable subgroup of G.

(33) For every strict stable subgroup H of G holds $\{\mathbf{1}\}_G \sqcup H = H$ and $H \sqcup \{\mathbf{1}\}_G = H$.

(34) $\Omega_G \sqcup H_1 = \Omega_G$ and $H_1 \sqcup \Omega_G = \Omega_G$.

(35) H_1 is a stable subgroup of $H_1 \sqcup H_2$ and H_2 is a stable subgroup of $H_1 \sqcup H_2$.

(36) For every strict stable subgroup H_2 of G holds H_1 is a stable subgroup of H_2 iff $H_1 \sqcup H_2 = H_2$.

(37) Let H_3 be a strict stable subgroup of G. Suppose H_1 is a stable subgroup of H_3 and H_2 is a stable subgroup of H_3. Then $H_1 \sqcup H_2$ is a stable subgroup of H_3.

(38) Let H_2, H_3 be strict stable subgroups of G. Suppose H_1 is a stable subgroup of H_2. Then $H_1 \sqcup H_3$ is a stable subgroup of $H_2 \sqcup H_3$.

(39) For all stable subgroups X, Y of H_1 and for all stable subgroups X', Y' of G such that $X = X'$ and $Y = Y'$ holds $X' \cap Y' = X \cap Y$.

(40) If N is a stable subgroup of H_1, then N is a normal stable subgroup of H_1.

(41) $H_1 \cap N$ is a normal stable subgroup of H_1 and $N \cap H_1$ is a normal stable subgroup of H_1.

(42) For every strict group G with operators in O such that G is trivial holds $\{\mathbf{1}\}_G = G$.

(43) $\mathbf{1}_{{}^{G}/_{N}} = \overline{N}$.

(44) Let M, N be strict normal stable subgroups of G and M_1 be a normal stable subgroup of N. Suppose $M_1 = M$ and M is a stable subgroup of N. Then ${}^{N}/_{M_1}$ is a normal stable subgroup of ${}^{G}/_{M}$.

(45) $h(\mathbf{1}_G) = \mathbf{1}_H$.

(46) $h({g_1}^{-1}) = h(g_1)^{-1}$.

(47) $g_1 \in \operatorname{Ker} h$ iff $h(g_1) = \mathbf{1}_H$.

(48) For every strict normal stable subgroup N of G holds Ker (the canonical homomorphism onto cosets of N) $= N$.

(49) $\operatorname{rng} h =$ the carrier of $\operatorname{Im} h$.

(50) Im (the canonical homomorphism onto cosets of N) $= {}^{G}/_{N}$.

(51) Let H be a strict group with operators in O and h be a homomorphism from G to H. Then h is an epimorphism if and only if $\operatorname{Im} h = H$.

(52) Let H be a strict group with operators in O and h be a homomorphism from G to H. Suppose h is an epimorphism. Let c be an element of H. Then there exists an element a of G such that $h(a) = c$.

(53) The canonical homomorphism onto cosets of N is an epimorphism.

(54) The canonical homomorphism onto cosets of $\{\mathbf{1}\}_G$ is an isomorphism.

(55) If G and H are isomorphic and H and I are isomorphic, then G and I are isomorphic.

(56) For every strict group G with operators in O holds G and ${}^{G}/_{\{\mathbf{1}\}_G}$ are isomorphic.

(57) For every strict group G with operators in O holds ${}^{G}/_{\Omega_G}$ is trivial.

(58) Let G, H be strict groups with operators in O. If G and H are isomorphic and G is trivial, then H is trivial.

(59) ${}^{G}/_{\mathrm{Ker}\,h}$ and $\mathrm{Im}\,h$ are isomorphic.

(60) Let H, F_1, F_2 be strict stable subgroups of G. Suppose F_1 is a normal stable subgroup of F_2. Then $H \cap F_1$ is a normal stable subgroup of $H \cap F_2$.

3. Others Theorems on Actions and Groups with Operators

In the sequel E is a set, A is an action of O on E, C is a subset of G, and N_1 is a normal stable subgroup of H_1.

One can prove the following propositions:

(61) Ω_E is stable under the action of A.

(62) $[: O, \{\mathrm{id}_E\} :]$ is an action of O on E.

(63) Let O be a non empty set, E be a set, o be an element of O, and A be an action of O on E. Then $\mathrm{Product}(\langle o \rangle, A) = A(o)$.

(64) Let O be a non empty set, E be a set, F_1, F_2 be finite sequences of elements of O, and A be an action of O on E. Then $\mathrm{Product}(F_1 \frown F_2, A) = \mathrm{Product}(F_1, A) \cdot \mathrm{Product}(F_2, A)$.

(65) Let F be a finite sequence of elements of O and Y be a subset of E. If Y is stable under the action of A, then $(\mathrm{Product}(F, A))^{\circ}Y \subseteq Y$.

(66) Let E be a non empty set, A be an action of O on E, X be a subset of E, and a be an element of E. Suppose X is not empty. Then $a \in$ the stable subset generated by X if and only if there exists a finite sequence F of elements of O and there exists an element x of X such that $(\mathrm{Product}(F, A))(x) = a$.

(67) For every strict group G there exists a strict group H with operators in O such that $G =$ the groupoid of H.

(68) The groupoid of H_1 is a strict subgroup of G.

(69) The groupoid of N is a strict normal subgroup of G.

(70) If $g_1 \in H_1$, then $(G \frown o)(g_1) \in H_1$.

(71) Let O be a set, G, H be groups with operators in O, G' be a strict stable subgroup of G, and f be a homomorphism from G to H. Then there exists a strict stable subgroup H' of H such that the carrier of $H' = f^{\circ}$(the carrier of G').

(72) If B is empty, then the stable subgroup of $B = \{\mathbf{1}\}_G$.

(73) If $B =$ the carrier of $\mathrm{gr}(C)$, then the stable subgroup of $C =$ the stable subgroup of B.

(74) Let N' be a normal subgroup of G. Suppose N' = the groupoid of N. Then ${}^{G}/_{N'}$ = the groupoid of ${}^{G}/_{N}$ and $\mathbf{1}_{{}^{G}/_{N'}} = \mathbf{1}_{{}^{G}/_{N}}$.

(75) Suppose the carrier of H_1 = the carrier of H_2. Then the group structure with operators of H_1 = the group structure with operators of H_2.

(76) Suppose ${}^{H_1}/_{N_1}$ is trivial. Then the group structure with operators of H_1 = the group structure with operators of N_1.

(77) If the carrier of H_1 = the carrier of N_1, then ${}^{H_1}/_{N_1}$ is trivial.

(78) Let G, H be groups with operators in O, N be a stable subgroup of G, H' be a strict stable subgroup of H, and f be a homomorphism from G to H. Suppose $N = \operatorname{Ker} f$. Then there exists a strict stable subgroup G' of G such that

(i) the carrier of $G' = f^{-1}$(the carrier of H'), and

(ii) if H' is normal, then N is a normal stable subgroup of G' and G' is normal.

(79) Let G, H be groups with operators in O, N be a stable subgroup of G, G' be a strict stable subgroup of G, and f be a homomorphism from G to H. Suppose $N = \operatorname{Ker} f$. Then there exists a strict stable subgroup H' of H such that

(i) the carrier of $H' = f^{\circ}$(the carrier of G'),

(ii) f^{-1}(the carrier of H') = the carrier of $G' \sqcup N$, and

(iii) if f is an epimorphism and G' is normal, then H' is normal.

(80) Let G be a strict group with operators in O, N be a strict normal stable subgroup of G, and H be a strict stable subgroup of ${}^{G}/_{N}$. Suppose the carrier of G = (the canonical homomorphism onto cosets of $N)^{-1}$(the carrier of H). Then $H = \Omega_{{}^{G}/_{N}}$.

(81) Let G be a strict group with operators in O, N be a strict normal stable subgroup of G, and H be a strict stable subgroup of ${}^{G}/_{N}$. Suppose the carrier of N = (the canonical homomorphism onto cosets of $N)^{-1}$(the carrier of H). Then $H = \{\mathbf{1}\}_{{}^{G}/_{N}}$.

(82) Let G, H be strict groups with operators in O. If G and H are isomorphic and G is simple, then H is simple.

(83) Let G be a group with operators in O, H be a stable subgroup of G, F_3 be a finite sequence of elements of the carrier of G, F_4 be a finite sequence of elements of the carrier of H, and I be a finite sequence of elements of $\mathbb{Z}$. If $F_3 = F_4$ and $\operatorname{len} F_3 = \operatorname{len} I$, then $\prod({F_3}^{I}) = \prod({F_4}^{I})$.

(84) Let O, E_1, E_2 be sets, A_1 be an action of O on E_1, A_2 be an action of O on E_2, and F be a finite sequence of elements of O. Suppose that

(i) $E_1 \subseteq E_2$, and

(ii) for every element o of O and for every function f_1 from E_1 into E_1 and for every function f_2 from E_2 into E_2 such that $f_1 = A_1(o)$ and $f_2 = A_2(o)$

holds $f_1 = f_2 \restriction E_1$.

Then $\mathrm{Product}(F, A_1) = \mathrm{Product}(F, A_2) \restriction E_1$.

(85) Let N_1, N_2 be strict stable subgroups of H_1 and N_1', N_2' be strict stable subgroups of G. If $N_1 = N_1'$ and $N_2 = N_2'$, then $N_1' \cdot N_2' = N_1 \cdot N_2$.

(86) Let N_1, N_2 be strict stable subgroups of H_1 and N_1', N_2' be strict stable subgroups of G. If $N_1 = N_1'$ and $N_2 = N_2'$, then $N_1' \sqcup N_2' = N_1 \sqcup N_2$.

(87) Let N_1, N_2 be strict stable subgroups of G. Suppose N_1 is a normal stable subgroup of H_1 and N_2 is a normal stable subgroup of H_1. Then $N_1 \sqcup N_2$ is a normal stable subgroup of H_1.

(88) Let f be a homomorphism from G to H and g be a homomorphism from H to I. Then the carrier of $\mathrm{Ker}(g \cdot f) = f^{-1}$(the carrier of $\mathrm{Ker}\, g$).

(89) Let G' be a stable subgroup of G, H' be a stable subgroup of H, and f be a homomorphism from G to H. Suppose the carrier of $H' = f^\circ$(the carrier of G') or the carrier of $G' = f^{-1}$(the carrier of H'). Then $f\restriction$the carrier of G' is a homomorphism from G' to H'.

(90) Let G, H be strict groups with operators in O, N, L, G' be strict stable subgroups of G, and f be a homomorphism from G to H. Suppose $N = \mathrm{Ker}\, f$ and L is a strict normal stable subgroup of G'. Then

(i) $L \sqcup G' \cap N$ is a normal stable subgroup of G',

(ii) $L \sqcup N$ is a normal stable subgroup of $G' \sqcup N$, and

(iii) for every strict normal stable subgroup N_1 of $G' \sqcup N$ and for every strict normal stable subgroup N_2 of G' such that $N_1 = L \sqcup N$ and $N_2 = L \sqcup G' \cap N$ holds ${}^{(G' \sqcup N)}/_{N_1}$ and ${}^{G'}/_{N_2}$ are isomorphic.

4. The Zassenhaus Butterfly Lemma

The following propositions are true:

(91) Let H, K, H', K' be strict stable subgroups of G, J_1 be a normal stable subgroup of $H' \sqcup H \cap K$, and H_4 be a normal stable subgroup of $H \cap K$. Suppose H' is a normal stable subgroup of H and K' is a normal stable subgroup of K and $J_1 = H' \sqcup H \cap K'$ and $H_4 = H' \cap K \sqcup K' \cap H$. Then ${}^{(H' \sqcup H \cap K)}/_{J_1}$ and ${}^{(H \cap K)}/_{H_4}$ are isomorphic.

(92) Let H, K, H', K' be strict stable subgroups of G. Suppose H' is a normal stable subgroup of H and K' is a normal stable subgroup of K. Then $H' \sqcup H \cap K'$ is a normal stable subgroup of $H' \sqcup H \cap K$.

(93) Let H, K, H', K' be strict stable subgroups of G, J_1 be a normal stable subgroup of $H' \sqcup H \cap K$, and J_2 be a normal stable subgroup of $K' \sqcup K \cap H$. Suppose $J_1 = H' \sqcup H \cap K'$ and $J_2 = K' \sqcup K \cap H'$ and H' is a normal stable subgroup of H and K' is a normal stable subgroup of K. Then ${}^{(H' \sqcup H \cap K)}/_{J_1}$ and ${}^{(K' \sqcup K \cap H)}/_{J_2}$ are isomorphic.

5. Composition Series

Let O be a set, let G be a group with operators in O, and let I_1 be a finite sequence of elements of the stable subgroups of G. We say that I_1 is composition series if and only if the conditions (Def. 31) are satisfied.

(Def. 31)(i) $I_1(1) = \Omega_G$,

(ii) $I_1(\operatorname{len} I_1) = \{\mathbf{1}\}_G$, and

(iii) for every natural number i such that $i \in \operatorname{dom} I_1$ and $i+1 \in \operatorname{dom} I_1$ and for all stable subgroups H_1, H_2 of G such that $H_1 = I_1(i)$ and $H_2 = I_1(i+1)$ holds H_2 is a normal stable subgroup of H_1.

Let O be a set and let G be a group with operators in O. One can verify that there exists a finite sequence of elements of the stable subgroups of G which is composition series.

Let O be a set and let G be a group with operators in O. A composition series of G is a composition series finite sequence of elements of the stable subgroups of G.

Let O be a set, let G be a group with operators in O, and let s_1, s_2 be composition series of G. We say that s_1 is finer than s_2 if and only if:

(Def. 32) There exists a set x such that $x \subseteq \operatorname{dom} s_1$ and $s_2 = s_1 \cdot \operatorname{Sgm} x$.

Let us note that the predicate s_1 is finer than s_2 is reflexive.

Let O be a set, let G be a group with operators in O, and let I_1 be a composition series of G. We say that I_1 is strictly decreasing if and only if the condition (Def. 33) is satisfied.

(Def. 33) Let i be a natural number. Suppose $i \in \operatorname{dom} I_1$ and $i+1 \in \operatorname{dom} I_1$. Let H be a stable subgroup of G and N be a normal stable subgroup of H. If $H = I_1(i)$ and $N = I_1(i+1)$, then ${}^{H}/_{N}$ is not trivial.

Let O be a set, let G be a group with operators in O, and let I_1 be a composition series of G. We say that I_1 is Jordan-Hölder if and only if the conditions (Def. 34) are satisfied.

(Def. 34)(i) I_1 is strictly decreasing, and

(ii) it is not true that there exists a composition series s of G such that $s \neq I_1$ and s is strictly decreasing and finer than I_1.

Let O be a set, let G_1, G_2 be groups with operators in O, let s_1 be a composition series of G_1, and let s_2 be a composition series of G_2. We say that s_1 is equivalent with s_2 if and only if the conditions (Def. 35) are satisfied.

(Def. 35)(i) $\operatorname{len} s_1 = \operatorname{len} s_2$, and

(ii) for every natural number n such that $n+1 = \operatorname{len} s_1$ there exists a permutation p of $\operatorname{Seg} n$ such that for every stable subgroup H_1 of G_1 and for every stable subgroup H_2 of G_2 and for every normal stable subgroup N_1 of H_1 and for every normal stable subgroup N_2 of H_2 and for all natural numbers i, j such that $1 \leq i$ and $i \leq n$ and $j = p(i)$ and $H_1 = s_1(i)$ and

$H_2 = s_2(j)$ and $N_1 = s_1(i+1)$ and $N_2 = s_2(j+1)$ holds ${}^{H_1}/_{N_1}$ and ${}^{H_2}/_{N_2}$ are isomorphic.

Let O be a set, let G be a group with operators in O, and let s be a composition series of G. The series of quotients of s yielding a finite sequence is defined as follows:

(Def. 36)(i) $\operatorname{len} s = \operatorname{len}(\text{the series of quotients of } s) + 1$ and for every natural number i such that $i \in \operatorname{dom}(\text{the series of quotients of } s)$ and for every stable subgroup H of G and for every normal stable subgroup N of H such that $H = s(i)$ and $N = s(i+1)$ holds (the series of quotients of $s)(i) = {}^{H}/_{N}$ if $\operatorname{len} s > 1$,

(ii) the series of quotients of $s = \emptyset$, otherwise.

Let O be a set, let f_1, f_2 be finite sequences, and let p be a permutation of $\operatorname{dom} f_1$. We say that f_1 and f_2 are equivalent under p in O if and only if the conditions (Def. 37) are satisfied.

(Def. 37)(i) $\operatorname{len} f_1 = \operatorname{len} f_2$, and

(ii) for all groups H_1, H_2 with operators in O and for all natural numbers i, j such that $i \in \operatorname{dom} f_1$ and $j = p^{-1}(i)$ and $H_1 = f_1(i)$ and $H_2 = f_2(j)$ holds H_1 and H_2 are isomorphic.

For simplicity, we follow the rules: y is a set, s_1, s_1', s_2, s_2' are composition series of G, f_3 is a finite sequence of elements of the stable subgroups of G, f_1, f_2 are finite sequences, and i, j, n are natural numbers.

We now state a number of propositions:

(94) If $i \in \operatorname{dom} s_1$ and $i+1 \in \operatorname{dom} s_1$ and $s_1(i) = s_1(i+1)$ and $f_3 = (s_1)_{\restriction i}$, then f_3 is composition series.

(95) If s_1 is finer than s_2, then there exists n such that $\operatorname{len} s_1 = \operatorname{len} s_2 + n$.

(96) If $\operatorname{len} s_2 = \operatorname{len} s_1$ and s_2 is finer than s_1, then $s_1 = s_2$.

(97) If s_1 is not empty and s_2 is finer than s_1, then s_2 is not empty.

(98) If s_1 is finer than s_2 and Jordan-Hölder and s_2 is Jordan-Hölder, then $s_1 = s_2$.

(99) If $i \in \operatorname{dom} s_1$ and $i+1 \in \operatorname{dom} s_1$ and $s_1(i) = s_1(i+1)$ and $s_1' = (s_1)_{\restriction i}$ and s_2 is Jordan-Hölder and s_1 is finer than s_2, then s_1' is finer than s_2.

(100) Suppose $\operatorname{len} s_1 > 1$ and $s_2 \neq s_1$ and s_2 is strictly decreasing and finer than s_1. Then there exist i, j such that $i \in \operatorname{dom} s_1$ and $i \in \operatorname{dom} s_2$ and $i+1 \in \operatorname{dom} s_1$ and $i+1 \in \operatorname{dom} s_2$ and $j \in \operatorname{dom} s_2$ and $i+1 < j$ and $s_1(i) = s_2(i)$ and $s_1(i+1) \neq s_2(i+1)$ and $s_1(i+1) = s_2(j)$.

(101) If $i \in \operatorname{dom} s_1$ and $j \in \operatorname{dom} s_1$ and $i \leq j$ and $H_1 = s_1(i)$ and $H_2 = s_1(j)$, then H_2 is a stable subgroup of H_1.

(102) If $y \in \operatorname{rng}(\text{the series of quotients of } s_1)$, then y is a strict group with operators in O.

(103) Suppose $i \in \operatorname{dom}$ (the series of quotients of s_1) and for every H such that $H =$ (the series of quotients of s_1)(i) holds H is trivial. Then $i \in \operatorname{dom} s_1$ and $i + 1 \in \operatorname{dom} s_1$ and $s_1(i) = s_1(i + 1)$.

(104) Suppose $i \in \operatorname{dom} s_1$ and $i + 1 \in \operatorname{dom} s_1$ and $s_1(i) = s_1(i + 1)$ and $s_2 = (s_1)_{\restriction i}$. Then the series of quotients of $s_2 =$ (the series of quotients of s_1)$_{\restriction i}$.

(105) Suppose $f_1 =$ the series of quotients of s_1 and $i \in \operatorname{dom} f_1$ and for every H such that $H = f_1(i)$ holds H is trivial. Then $(s_1)_{\restriction i}$ is a composition series of G and for every s_2 such that $s_2 = (s_1)_{\restriction i}$ holds the series of quotients of $s_2 = (f_1)_{\restriction i}$.

(106) Suppose that

(i) $f_1 =$ the series of quotients of s_1,

(ii) $f_2 =$ the series of quotients of s_2,

(iii) $i \in \operatorname{dom} f_1$,

(iv) for every H such that $H = f_1(i)$ holds H is trivial, and

(v) there exists a permutation p of $\operatorname{dom} f_1$ such that f_1 and f_2 are equivalent under p in O and $j = p^{-1}(i)$.

Then there exists a permutation p' of $\operatorname{dom}((f_1)_{\restriction i})$ such that $(f_1)_{\restriction i}$ and $(f_2)_{\restriction j}$ are equivalent under p' in O.

(107) Let G_1, G_2 be groups with operators in O, s_1 be a composition series of G_1, and s_2 be a composition series of G_2. If s_1 is empty and s_2 is empty, then s_1 is equivalent with s_2.

(108) Let G_1, G_2 be groups with operators in O, s_1 be a composition series of G_1, and s_2 be a composition series of G_2. Suppose s_1 is not empty and s_2 is not empty. Then s_1 is equivalent with s_2 if and only if there exists a permutation p of dom (the series of quotients of s_1) such that the series of quotients of s_1 and the series of quotients of s_2 are equivalent under p in O.

(109) Suppose s_1 is finer than s_2 and s_2 is Jordan-Hölder and $\operatorname{len} s_1 > \operatorname{len} s_2$. Then there exists i such that $i \in \operatorname{dom}$ (the series of quotients of s_1) and for every H such that $H =$ (the series of quotients of s_1)(i) holds H is trivial.

(110) Suppose $\operatorname{len} s_1 > 1$. Then s_1 is Jordan-Hölder if and only if for every i such that $i \in \operatorname{dom}$ (the series of quotients of s_1) holds (the series of quotients of s_1)(i) is a strict simple group with operators in O.

(111) Suppose $1 \leq i$ and $i \leq \operatorname{len} s_1 - 1$. Then $s_1(i)$ is a strict stable subgroup of G and $s_1(i + 1)$ is a strict stable subgroup of G.

(112) If $1 \leq i$ and $i \leq \operatorname{len} s_1 - 1$ and $H_1 = s_1(i)$ and $H_2 = s_1(i + 1)$, then H_2 is a normal stable subgroup of H_1.

(113) s_1 is equivalent with s_1.

(114) If $\operatorname{len} s_1 \leq 1$ or $\operatorname{len} s_2 \leq 1$ and if $\operatorname{len} s_1 \leq \operatorname{len} s_2$, then s_2 is finer than s_1.

(115) If s_1 is equivalent with s_2 and Jordan-Hölder, then s_2 is Jordan-Hölder.

6. The Schreier Refinement Theorem

Let us consider O, G, s_1, s_2. Let us assume that $\operatorname{len} s_1 > 1$ and $\operatorname{len} s_2 > 1$. The Schreier series of s_1 and s_2 yielding a composition series of G is defined by the condition (Def. 38).

(Def. 38) Let k, i, j be natural numbers and H_1, H_2, H_3 be stable subgroups of G. Then

- (i) if $k = (i-1) \cdot (\operatorname{len} s_2 - 1) + j$ and $1 \leq i$ and $i \leq \operatorname{len} s_1 - 1$ and $1 \leq j$ and $j \leq \operatorname{len} s_2 - 1$ and $H_1 = s_1(i+1)$ and $H_2 = s_1(i)$ and $H_3 = s_2(j)$, then (the Schreier series of s_1 and $s_2)(k) = H_1 \sqcup H_2 \cap H_3$,
- (ii) if $k = (\operatorname{len} s_1 - 1) \cdot (\operatorname{len} s_2 - 1) + 1$, then (the Schreier series of s_1 and $s_2)(k) = \{\mathbf{1}\}_G$, and
- (iii) len (the Schreier series of s_1 and $s_2) = (\operatorname{len} s_1 - 1) \cdot (\operatorname{len} s_2 - 1) + 1$.

Next we state three propositions:

(116) If $\operatorname{len} s_1 > 1$ and $\operatorname{len} s_2 > 1$, then the Schreier series of s_1 and s_2 is finer than s_1.

(117) If $\operatorname{len} s_1 > 1$ and $\operatorname{len} s_2 > 1$, then the Schreier series of s_1 and s_2 is equivalent with the Schreier series of s_2 and s_1.

(118) There exist s_1', s_2' such that s_1' is finer than s_1 and s_2' is finer than s_2 and s_1' is equivalent with s_2'.

7. The Jordan-Hölder Theorem

One can prove the following proposition

(119) If s_1 is Jordan-Hölder and s_2 is Jordan-Hölder, then s_1 is equivalent with s_2.

8. Appendix

Next we state several propositions:

(120) For all binary relations P, R holds $P = \operatorname{rng} P \restriction R$ iff $P^{\smile} = R^{\smile} \restriction \operatorname{dom}(P^{\smile})$.

(121) For every set X and for all binary relations P, R holds $P \cdot (R \restriction X) = (X \restriction P) \cdot R$.

(122) Let n be a natural number, X be a set, and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. If $X \subseteq \operatorname{Seg} n$ and $X \subseteq \operatorname{dom} f$ and f is increasing on X and $f^{\circ} X \subseteq \mathbb{N} \setminus \{0\}$, then $\operatorname{Sgm}(f^{\circ} X) = f \cdot \operatorname{Sgm} X$.

(123) Let y be a set and i, n be natural numbers. Suppose $y \subseteq \mathrm{Seg}(n+1)$ and $i \in \mathrm{Seg}(n+1)$ and $i \notin y$. Then there exists x such that $\mathrm{Sgm}\, x = (\mathrm{Sgm}(\mathrm{Seg}(n+1) \setminus \{i\}))^{-1} \cdot \mathrm{Sgm}\, y$ and $x \subseteq \mathrm{Seg}\, n$.

(124) Let D be a non empty set, f be a finite sequence of elements of D, p be an element of D, and n be an element of $\mathbb{N}$. If $n \in \mathrm{dom}\, f$, then $f = (\mathrm{Ins}(f, n, p))\restriction_{n+1}$.

(125) Let G, H be groups, F_1 be a finite sequence of elements of the carrier of G, F_2 be a finite sequence of elements of the carrier of H, I be a finite sequence of elements of $\mathbb{Z}$, and f be a homomorphism from G to H. Suppose for every element k of $\mathbb{N}$ such that $k \in \mathrm{Seg}\,\mathrm{len}\, F_1$ holds $F_2(k) = f(F_1(k))$ and $\mathrm{len}\, F_1 = \mathrm{len}\, I$ and $\mathrm{len}\, F_2 = \mathrm{len}\, I$. Then $f(\prod(F_1{}^I)) = \prod(F_2{}^I)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Nicolas Bourbaki. *Elements of Mathematics. Algebra I. Chapters 1-3.* Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1989.
[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[11] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(**2**):241–245, 1996.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Formalized Mathematics*, 2(**5**):711–717, 1991.
[14] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.
[15] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[16] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[18] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[19] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Formalized Mathematics*, 1(**5**):955–962, 1990.
[20] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[21] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.
[22] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(**5**):855–864, 1990.
[23] Wojciech A. Trybulec. Lattice of subgroups of a group. Frattini subgroup. *Formalized Mathematics*, 2(**1**):41–47, 1991.
[24] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.
[25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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Regular Expression Quantifiers – m to n Occurrences

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Summary. This article includes proofs of several facts that are supplemental to the theorems proved in [10]. Next, it builds upon that theory to extend the framework for proving facts about formal languages in general and regular expression operators in particular. In this article, two quantifiers are defined and their properties are shown: m to n occurrences (or the union of a range of powers) and optional occurrence. Although optional occurrence is a special case of the previous operator (0 to 1 occurrences), it is often defined in regex applications as a separate operator – hence its explicit definition and properties in the article. Notation and terminology were taken from [13].

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The articles [9], [4], [11], [7], [8], [2], [14], [3], [1], [5], [12], [6], and [10] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: E, x denote sets, A, B, C denote subsets of E^ω, a, b denote elements of E^ω, and i, k, l, k_1, m, n, m_1 denote natural numbers.

We now state four propositions:

(1) If $m + k \leq i$ and $i \leq n + k$, then there exists m_1 such that $m_1 + k = i$ and $m \leq m_1$ and $m_1 \leq n$.

(2) If $m \leq n$ and $k \leq l$ and $m + k \leq i$ and $i \leq n + l$, then there exist m_1, k_1 such that $m_1 + k_1 = i$ and $m \leq m_1$ and $m_1 \leq n$ and $k \leq k_1$ and $k_1 \leq l$.

(3) If $m < n$, then there exists k such that $m + k = n$ and $k > 0$.

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(4) If $a \frown b = a$ or $b \frown a = a$, then $b = \emptyset$.

2. Addenda to [10]

One can prove the following propositions:

(5) If $x \in A$ or $x \in B$ and if $x \neq \langle\rangle_E$, then $A \frown B \neq \{\langle\rangle_E\}$.

(6) $\langle x\rangle \in A \frown B$ iff $\langle\rangle_E \in A$ and $\langle x\rangle \in B$ or $\langle x\rangle \in A$ and $\langle\rangle_E \in B$.

(7) If $x \in A$ and $x \neq \langle\rangle_E$ and $n > 0$, then $A^n \neq \{\langle\rangle_E\}$.

(8) $\langle\rangle_E \in A^n$ iff $n = 0$ or $\langle\rangle_E \in A$.

(9) $\langle x\rangle \in A^n$ iff $\langle x\rangle \in A$ but $\langle\rangle_E \in A$ and $n > 1$ or $n = 1$.

(10) If $m \neq n$ and $A^m = \{x\}$ and $A^n = \{x\}$, then $x = \langle\rangle_E$.

(11) $(A^m)^n = (A^n)^m$.

(12) $(A^m) \frown A^n = (A^n) \frown A^m$.

(13) If $\langle\rangle_E \in B$, then $A \subseteq A \frown B^l$ and $A \subseteq (B^l) \frown A$.

(14) If $A \subseteq C^k$ and $B \subseteq C^l$, then $A \frown B \subseteq C^{k+l}$.

(15) If $x \in A$ and $x \neq \langle\rangle_E$, then $A^* \neq \{\langle\rangle_E\}$.

(16) If $\langle\rangle_E \in A$ and $n > 0$, then $(A^n)^* = A^*$.

(17) If $\langle\rangle_E \in A$, then $(A^n)^* = (A^*)^n$.

(18) $A \subseteq A \frown B^*$ and $A \subseteq (B^*) \frown A$.

3. Union of a Range of Powers

Let us consider E, A and let us consider m, n. The functor $A^{m,n}$ yields a subset of E^ω and is defined as follows:

(Def. 1) $A^{m,n} = \bigcup\{B : \bigvee_k (m \leq k \ \wedge\ k \leq n \ \wedge\ B = A^k)\}$.

One can prove the following propositions:

(19) $x \in A^{m,n}$ iff there exists k such that $m \leq k$ and $k \leq n$ and $x \in A^k$.

(20) If $m \leq k$ and $k \leq n$, then $A^k \subseteq A^{m,n}$.

(21) $A^{m,n} = \emptyset$ iff $m > n$ or $m > 0$ and $A = \emptyset$.

(22) $A^{m,m} = A^m$.

(23) If $m \leq k$ and $l \leq n$, then $A^{k,l} \subseteq A^{m,n}$.

(24) If $m \leq k$ and $k \leq n$, then $A^{m,n} = A^{m,k} \cup A^{k,n}$.

(25) If $m \leq k$ and $k \leq n$, then $A^{m,n} = A^{m,k} \cup A^{k+1,n}$.

(26) If $m \leq n + 1$, then $A^{m,n+1} = A^{m,n} \cup A^{n+1}$.

(27) If $m \leq n$, then $A^{m,n} = A^m \cup A^{m+1,n}$.

(28) $A^{n,n+1} = A^n \cup A^{n+1}$.

(29) If $A \subseteq B$, then $A^{m,n} \subseteq B^{m,n}$.

(30) If $x \in A$ and if $x \neq \langle\rangle_E$ and if $m > 0$ or $n > 0$, then $A^{m,n} \neq \{\langle\rangle_E\}$.

(31) $A^{m,n} = \{\langle\rangle_E\}$ iff $m \leq n$ and $A = \{\langle\rangle_E\}$ or $m = 0$ and $n = 0$ or $m = 0$ and $A = \emptyset$.

(32) $A^{m,n} \subseteq A^*$.

(33) $\langle\rangle_E \in A^{m,n}$ iff $m = 0$ or $m \leq n$ and $\langle\rangle_E \in A$.

(34) If $\langle\rangle_E \in A$ and $m \leq n$, then $A^{m,n} = A^n$.

(35) $(A^{m,n}) \frown A^k = (A^k) \frown A^{m,n}$.

(36) $(A^{m,n}) \frown A = A \frown A^{m,n}$.

(37) If $m \leq n$ and $k \leq l$, then $(A^{m,n}) \frown A^{k,l} = A^{m+k,n+l}$.

(38) $A^{m+1,n+1} = (A^{m,n}) \frown A$.

(39) $(A^{m,n}) \frown A^{k,l} = (A^{k,l}) \frown A^{m,n}$.

(40) $(A^{m,n})^k = A^{m \cdot k, n \cdot k}$.

(41) $(A^{k+1})^{m,n} \subseteq ((A^k)^{m,n}) \frown A^{m,n}$.

(42) $(A^k)^{m,n} \subseteq A^{k \cdot m, k \cdot n}$.

(43) $(A^k)^{m,n} \subseteq (A^{m,n})^k$.

(44) $(A^{k+l})^{m,n} \subseteq ((A^k)^{m,n}) \frown (A^l)^{m,n}$.

(45) $A^{0,0} = \{\langle\rangle_E\}$.

(46) $A^{0,1} = \{\langle\rangle_E\} \cup A$.

(47) $A^{1,1} = A$.

(48) $A^{0,2} = \{\langle\rangle_E\} \cup A \cup A \frown A$.

(49) $A^{1,2} = A \cup A \frown A$.

(50) $A^{2,2} = A \frown A$.

(51) If $m > 0$ and $m \neq n$ and $A^{m,n} = \{x\}$, then for every m_1 such that $m \leq m_1$ and $m_1 \leq n$ holds $A^{m_1} = \{x\}$.

(52) If $m \neq n$ and $A^{m,n} = \{x\}$, then $x = \langle\rangle_E$.

(53) $\langle x \rangle \in A^{m,n}$ iff $\langle x \rangle \in A$ but $m \leq n$ but $\langle\rangle_E \in A$ and $n > 0$ or $m \leq 1$ and $1 \leq n$.

(54) $(A \cap B)^{m,n} \subseteq A^{m,n} \cap B^{m,n}$.

(55) $A^{m,n} \cup B^{m,n} \subseteq (A \cup B)^{m,n}$.

(56) $(A^{m,n})^{k,l} \subseteq A^{m \cdot k, n \cdot l}$.

(57) If $m \leq n$ and $\langle\rangle_E \in B$, then $A \subseteq A \frown B^{m,n}$ and $A \subseteq (B^{m,n}) \frown A$.

(58) If $m \leq n$ and $k \leq l$ and $A \subseteq C^{m,n}$ and $B \subseteq C^{k,l}$, then $A \frown B \subseteq C^{m+k,n+l}$.

(59) $(A^{m,n})^* \subseteq A^*$.

(60) $(A^*)^{m,n} \subseteq A^*$.

(61) If $m \leq n$ and $n > 0$, then $(A^*)^{m,n} = A^*$.

(62) If $m \leq n$ and $n > 0$ and $\langle\rangle_E \in A$, then $(A^{m,n})^* = A^*$.

(63) If $m \leq n$ and $\langle\rangle_E \in A$, then $(A^{m,n})^* = (A^*)^{m,n}$.

(64) If $A \subseteq B^*$, then $A^{m,n} \subseteq B^*$.

(65) If $A \subseteq B^*$, then $B^* = (B \cup A^{m,n})^*$.

(66) $(A^{m,n}) \frown A^* = (A^*) \frown A^{m,n}$.

(67) If $\langle\rangle_E \in A$ and $m \leq n$, then $A^* = (A^*) \frown A^{m,n}$.

(68) $(A^{m,n})^k \subseteq A^*$.

(69) $(A^k)^{m,n} \subseteq A^*$.

(70) If $m \leq n$, then $(A^m)^* \subseteq (A^{m,n})^*$.

(71) $(A^{m,n})^{k,l} \subseteq A^*$.

(72) If $\langle\rangle_E \in A$ and $k \leq n$ and $l \leq n$, then $A^{k,n} = A^{l,n}$.

4. Optional Occurrence

Let us consider E, A. The functor $A?$ yields a subset of E^ω and is defined by:

(Def. 2) $A? = \bigcup\{B : \bigvee_k \ (k \leq 1 \ \wedge \ B = A^k)\}$.

One can prove the following propositions:

(73) $x \in A?$ iff there exists k such that $k \leq 1$ and $x \in A^k$.

(74) If $n \leq 1$, then $A^n \subseteq A?$.

(75) $A? = A^0 \cup A^1$.

(76) $A? = \{\langle\rangle_E\} \cup A$.

(77) $A \subseteq A?$.

(78) $x \in A?$ iff $x = \langle\rangle_E$ or $x \in A$.

(79) $A? = A^{0,1}$.

(80) $A? = A$ iff $\langle\rangle_E \in A$.

Let us consider E, A. One can check that $A?$ is non empty.

We now state a number of propositions:

(81) $A?? = A?$.

(82) If $A \subseteq B$, then $A? \subseteq B?$.

(83) If $x \in A$ and $x \neq \langle\rangle_E$, then $A? \neq \{\langle\rangle_E\}$.

(84) $A? = \{\langle\rangle_E\}$ iff $A = \emptyset$ or $A = \{\langle\rangle_E\}$.

(85) $A^*? = A^*$ and $A?^* = A^*$.

(86) $A? \subseteq A^*$.

(87) $(A \cap B)? = A? \cap B?$.

(88) $A? \cup B? = (A \cup B)?$.

(89) If $A? = \{x\}$, then $x = \langle\rangle_E$.

(90) $\langle x\rangle \in A?$ iff $\langle x\rangle \in A$.

(91) $A? \frown A = A \frown A?$.

(92) $A? \frown A = A^{1,2}$.

(93) $A? \frown A? = A^{0,2}$.

(94) $A?^k = A?^{0,k}$.

(95) $A?^k = A^{0,k}$.

(96) If $m \leq n$, then $A?^{m,n} = A?^{0,n}$.

(97) $A?^{0,n} = A^{0,n}$.

(98) If $m \leq n$, then $A?^{m,n} = A^{0,n}$.

(99) $A^{1,n}? = A^{0,n}$.

(100) If $\langle\rangle_E \in A$ and $\langle\rangle_E \in B$, then $A? \subseteq A \frown B$ and $A? \subseteq B \frown A$.

(101) $A \subseteq A \frown B?$ and $A \subseteq B? \frown A$.

(102) If $A \subseteq C?$ and $B \subseteq C?$, then $A \frown B \subseteq C^{0,2}$.

(103) If $\langle\rangle_E \in A$ and $n > 0$, then $A? \subseteq A^n$.

(104) $A? \frown A^k = (A^k) \frown A?$.

(105) If $A \subseteq B^*$, then $A? \subseteq B^*$.

(106) If $A \subseteq B^*$, then $B^* = (B \cup A?)^*$.

(107) $A? \frown A^* = (A^*) \frown A?$.

(108) $A? \frown A^* = A^*$.

(109) $A?^k \subseteq A^*$.

(110) $A^k? \subseteq A^*$.

(111) $A? \frown A^{m,n} = (A^{m,n}) \frown A?$.

(112) $A? \frown A^k = A^{k,k+1}$.

(113) $A?^{m,n} \subseteq A^*$.

(114) $A^{m,n}? \subseteq A^*$.

(115) $A? = (A \setminus \{\langle\rangle_E\})?$.

(116) If $A \subseteq B?$, then $A? \subseteq B?$.

(117) If $A \subseteq B?$, then $B? = (B \cup A)?$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[5] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[6] Karol Pąk. The Catalan numbers. Part II. *Formalized Mathematics*, 14(**4**):153–159, 2006.

[7] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[8] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[10] Michał Trybulec. Formal languages – concatenation and closure. *Formalized Mathematics*, 15(**1**):11–15, 2007.
[11] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[12] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(**4**):825–829, 2001.
[13] Larry Wall, Tom Christiansen, and Jon Orwant. *Programming Perl, Third Edition.* O'Reilly Media, 2000.
[14] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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Riemann Indefinite Integral of Functions of Real Variable[1]

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Summary. In this article we define the Riemann indefinite integral of functions of real variable and prove the linearity of that [1]. And we give some examples of the indefinite integral of some elementary functions. Furthermore, also the theorem about integral operation and uniform convergent sequence of functions is proved.

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The papers [24], [25], [3], [23], [5], [13], [2], [26], [7], [21], [8], [10], [4], [17], [16], [15], [14], [19], [20], [6], [9], [11], [18], [12], [27], and [22] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: a, b, r are real numbers, A is a non empty set, X, x are sets, f, g, F, G are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and n is an element of $\mathbb{N}$.

Next we state a number of propositions:

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(1) Let f, g be functions from A into $\mathbb{R}$. Suppose $\operatorname{rng} f$ is upper bounded and $\operatorname{rng} g$ is upper bounded and for every set x such that $x \in A$ holds $|f(x)-g(x)| \leq a$. Then $\sup \operatorname{rng} f - \sup \operatorname{rng} g \leq a$ and $\sup \operatorname{rng} g - \sup \operatorname{rng} f \leq a$.

(2) Let f, g be functions from A into $\mathbb{R}$. Suppose $\operatorname{rng} f$ is lower bounded and $\operatorname{rng} g$ is lower bounded and for every set x such that $x \in A$ holds $|f(x)-g(x)| \leq a$. Then $\inf \operatorname{rng} f - \inf \operatorname{rng} g \leq a$ and $\inf \operatorname{rng} g - \inf \operatorname{rng} f \leq a$.

(3) If $f \restriction X$ is bounded on X, then f is bounded on X.

(4) For every real number x such that $x \in X$ and $f \restriction X$ is differentiable in x holds f is differentiable in x.

(5) If $f \restriction X$ is differentiable on X, then f is differentiable on X.

(6) Suppose f is differentiable on X and g is differentiable on X. Then $f+g$ is differentiable on X and $f-g$ is differentiable on X and $f\,g$ is differentiable on X.

(7) If f is differentiable on X, then $r\,f$ is differentiable on X.

(8) Suppose for every set x such that $x \in X$ holds $g(x) \neq 0$ and f is differentiable on X and g is differentiable on X. Then $\frac{f}{g}$ is differentiable on X.

(9) If for every set x such that $x \in X$ holds $f(x) \neq 0$ and f is differentiable on X, then $\frac{1}{f}$ is differentiable on X.

(10) Suppose $a \leq b$ and $['a, b'] \subseteq X$ and F is differentiable on X and $F'_{\restriction X}$ is integrable on $['a, b']$ and $F'_{\restriction X}$ is bounded on $['a, b']$. Then $F(b) = \int\limits_{a}^{b} (F'_{\restriction X})(x)dx + F(a)$.

2. The Definition of Indefinite Integral

Let X be a set and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\mathrm{IntegralFuncs}(f, X)$ yields a set and is defined by the condition (Def. 1).

(Def. 1) $x \in \mathrm{IntegralFuncs}(f, X)$ if and only if there exists a partial function F from $\mathbb{R}$ to $\mathbb{R}$ such that $x = F$ and F is differentiable on X and $F'_{\restriction X} = f \restriction X$.

Let X be a set and let F, f be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We say that F is an integral of f on X if and only if:

(Def. 2) $F \in \mathrm{IntegralFuncs}(f, X)$.

The following propositions are true:

(11) If F is an integral of f on X, then $X \subseteq \operatorname{dom} F$.

(12) Suppose F is an integral of f on X and G is an integral of g on X. Then $F+G$ is an integral of $f+g$ on X and $F-G$ is an integral of $f-g$ on X.

(13) If F is an integral of f on X, then $r\,F$ is an integral of $r\,f$ on X.

(14) If F is an integral of f on X and G is an integral of g on X, then $F\,G$ is an integral of $f\,G + F\,g$ on X.

(15) Suppose for every set x such that $x \in X$ holds $G(x) \neq 0$ and F is an integral of f on X and G is an integral of g on X. Then $\frac{F}{G}$ is an integral of $\frac{f\,G - F\,g}{G\,G}$ on X.

(16) Suppose that
 (i) $a \leq b$,
 (ii) $['a, b'] \subseteq \operatorname{dom} f$,
 (iii) f is continuous on $['a, b']$,
 (iv) $]a, b[\subseteq \operatorname{dom} F$, and
 (v) for every real number x such that $x \in]a, b[$ holds $F(x) = \int\limits_{a}^{x} f(x)dx + F(a)$.

Then F is an integral of f on $]a, b[$.

(17) Let x, x_0 be real numbers. Suppose f is continuous on $[a, b]$ and $x \in]a, b[$ and $x_0 \in]a, b[$ and F is an integral of f on $]a, b[$. Then $F(x) = \int\limits_{x_0}^{x} f(x)dx + F(x_0)$.

(18) Suppose $a \leq b$ and $['a, b'] \subseteq X$ and F is an integral of f on X and f is integrable on $['a, b']$ and f is bounded on $['a, b']$. Then $F(b) = \int\limits_{a}^{b} f(x)dx + F(a)$.

(19) Suppose $a \leq b$ and $[a, b] \subseteq X$ and f is continuous on X. Then f is continuous on $['a, b']$ and f is integrable on $['a, b']$ and f is bounded on $['a, b']$.

(20) If $a \leq b$ and $[a, b] \subseteq X$ and f is continuous on X and F is an integral of f on X, then $F(b) = \int\limits_{a}^{b} f(x)dx + F(a)$.

(21) Suppose that $b \leq a$ and $['b, a'] \subseteq X$ and f is integrable on $['b, a']$ and g is integrable on $['b, a']$ and f is bounded on $['b, a']$ and g is bounded on $['b, a']$ and $X \subseteq \operatorname{dom} f$ and $X \subseteq \operatorname{dom} g$ and F is an integral of f on X and G is an integral of g on X. Then $F(a) \cdot G(a) - F(b) \cdot G(b) = \int\limits_{b}^{a} (f\,G)(x)dx + \int\limits_{b}^{a} (F\,g)(x)dx$.

(22) Suppose that $b \leq a$ and $[b, a] \subseteq X$ and $X \subseteq \operatorname{dom} f$ and $X \subseteq \operatorname{dom} g$ and

f is continuous on X and g is continuous on X and F is an integral of f on X and G is an integral of g on X. Then $F(a) \cdot G(a) - F(b) \cdot G(b) = \int\limits_b^a (f\,G)(x)dx + \int\limits_b^a (F\,g)(x)dx$.

3. Examples of Indefinite Integral

We now state several propositions:

(23) The function sin is an integral of the function cos on $\mathbb{R}$.

(24) (The function sin)$(b)-$(the function sin)$(a) = \int\limits_a^b$ (the function cos)$(x)dx$.

(25) (-1) (the function cos) is an integral of the function sin on $\mathbb{R}$.

(26) (The function cos)$(a)-$(the function cos)$(b) = \int\limits_a^b$ (the function sin)$(x)dx$.

(27) The function exp is an integral of the function exp on $\mathbb{R}$.

(28) (The function exp)$(b)-$(the function exp)$(a) = \int\limits_a^b$ (the function exp)$(x)dx$.

(29) $\frac{n+1}{\mathbb{Z}}$ is an integral of $(n+1)\,\frac{n}{\mathbb{Z}}$ on $\mathbb{R}$.

(30) $(\frac{n+1}{\mathbb{Z}})(b) - (\frac{n+1}{\mathbb{Z}})(a) = \int\limits_a^b ((n+1)\,\frac{n}{\mathbb{Z}})(x)dx$.

4. Uniform Convergent Functional Sequence

We now state the proposition

(31) Let H be a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ and r_1 be a sequence of real numbers. Suppose that

(i) $a < b$,

(ii) for every element n of $\mathbb{N}$ holds $H(n)$ is integrable on $['a, b']$ and $H(n)$ is bounded on $['a, b']$ and $r_1(n) = \int\limits_a^b H(n)(x)dx$, and

(iii) H is uniform-convergent on $['a, b']$.

Then $\lim_{['a,b']} H$ is bounded on $['a, b']$ and $\lim_{['a,b']} H$ is integrable on $['a, b']$ and r_1 is convergent and $\lim r_1 = \int\limits_a^b \lim\limits_{['a,b']} H(x)dx$.

References

[1] Tom M. Apostol. *Mathematical Analysis.* Addison-Wesley, 1969.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[10] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.
[11] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(**1**):93–102, 1999.
[12] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from $\mathbb{R}$ to $\mathbb{R}$ and integrability for continuous functions. *Formalized Mathematics*, 9(**2**):281–284, 2001.
[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[14] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[15] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[16] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.
[17] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[18] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(**1**):17–21, 1992.
[19] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.
[20] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.
[21] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[22] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(**2**):195–200, 2004.
[23] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[24] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[26] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[27] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

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Partial Differentiation on Normed Linear Spaces $\mathcal{R}^n$

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Summary. In this article, we define the partial differentiation of functions of real variable and prove the linearity of this operator [18].

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The notation and terminology used here are introduced in the following papers: [21], [24], [25], [5], [26], [7], [6], [15], [13], [3], [1], [20], [11], [22], [23], [14], [8], [2], [4], [27], [28], [16], [9], [19], [17], [12], and [10].

1. Preliminaries

Let i, n be elements of $\mathbb{N}$. The functor $\mathrm{proj}(i,n)$ yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined by:

(Def. 1) For every element x of $\mathcal{R}^n$ holds $(\mathrm{proj}(i,n))(x) = x(i)$.

Next we state two propositions:

(1) $\mathrm{dom}\,\mathrm{proj}(1,1) = \mathcal{R}^1$ and $\mathrm{rng}\,\mathrm{proj}(1,1) = \mathbb{R}$ and for every element x of $\mathbb{R}$ holds $(\mathrm{proj}(1,1))(\langle x \rangle) = x$ and $(\mathrm{proj}(1,1))^{-1}(x) = \langle x \rangle$.

(2)(i) $(\mathrm{proj}(1,1))^{-1}$ is a function from $\mathbb{R}$ into $\mathcal{R}^1$,

(ii) $(\mathrm{proj}(1,1))^{-1}$ is one-to-one,

(iii) $\mathrm{dom}((\mathrm{proj}(1,1))^{-1}) = \mathbb{R}$,

(iv) $\mathrm{rng}((\mathrm{proj}(1,1))^{-1}) = \mathcal{R}^1$, and

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(v) there exists a function g from $\mathbb{R}$ into $\mathcal{R}^1$ such that g is bijective and $(\mathrm{proj}(1,1))^{-1} = g$.

One can check that $\mathrm{proj}(1,1)$ is bijective.

Let g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\langle g \rangle$ yields a partial function from $\mathcal{R}^1$ to $\mathcal{R}^1$ and is defined as follows:

(Def. 2) $\langle g \rangle = (\mathrm{proj}(1,1))^{-1} \cdot g \cdot \mathrm{proj}(1,1)$.

Let n be an element of $\mathbb{N}$ and let g be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$. The functor $\langle g \rangle$ yielding a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$ is defined as follows:

(Def. 3) $\langle g \rangle = (\mathrm{proj}(1,1))^{-1} \cdot g$.

Let i, n be elements of $\mathbb{N}$. The functor $\mathrm{Proj}(i,n)$ yielding a function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ is defined as follows:

(Def. 4) For every point x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $(\mathrm{Proj}(i,n))(x) = \langle (\mathrm{proj}(i,n))(x) \rangle$.

Let i be an element of $\mathbb{N}$ and let x be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{reproj}(i,x)$ yielding a function is defined as follows:

(Def. 5) $\mathrm{dom}\,\mathrm{reproj}(i,x) = \mathbb{R}$ and for every element r of $\mathbb{R}$ holds $(\mathrm{reproj}(i,x))(r) = \mathrm{Replace}(x,i,r)$.

Let n, i be elements of $\mathbb{N}$ and let x be an element of $\mathcal{R}^n$. Then $\mathrm{reproj}(i,x)$ is a function from $\mathbb{R}$ into $\mathcal{R}^n$.

Let n, i be elements of $\mathbb{N}$ and let x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. The functor $\mathrm{reproj}(i,x)$ yielding a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined by the condition (Def. 6).

(Def. 6) Let r be an element of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Then there exists an element q of $\mathbb{R}$ and there exists an element y of $\mathcal{R}^n$ such that $r = \langle q \rangle$ and $y = x$ and $(\mathrm{reproj}(i,x))(r) = (\mathrm{reproj}(i,y))(q)$.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. We say that f is differentiable in x if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and g is differentiable in y.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Let us assume that f is differentiable in x. The functor $f'(x)$ yields a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ and is defined as follows:

(Def. 8) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and $f'(x) = g'(y)$.

We now state four propositions:

(3) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$. Suppose $I = (\mathrm{proj}(1,1))^{-1}$. Then

(i) for every vector x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element y of $\mathbb{R}$ such that $x = I(y)$ holds $\|x\| = |y|$,

(ii) for all vectors x, y of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $x = I(a)$ and $y = I(b)$ holds $x + y = I(a + b)$,

(iii) for every vector x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element y of $\mathbb{R}$ and for every real number a such that $x = I(y)$ holds $a \cdot x = I(a \cdot y)$,

(iv) for every vector x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element a of $\mathbb{R}$ such that $x = I(a)$ holds $-x = I(-a)$, and

(v) for all vectors x, y of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $x = I(a)$ and $y = I(b)$ holds $x - y = I(a - b)$.

(4) Let J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $J = \text{proj}(1, 1)$. Then

(i) for every vector x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element y of $\mathbb{R}$ such that $J(x) = y$ holds $\|x\| = |y|$,

(ii) for all vectors x, y of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $J(x) = a$ and $J(y) = b$ holds $J(x + y) = a + b$,

(iii) for every vector x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element y of $\mathbb{R}$ and for every real number a such that $J(x) = y$ holds $J(a \cdot x) = a \cdot y$,

(iv) for every vector x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element a of $\mathbb{R}$ such that $J(x) = a$ holds $J(-x) = -a$, and

(v) for all vectors x, y of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $J(x) = a$ and $J(y) = b$ holds $J(x - y) = a - b$.

(5) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$ and J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $I = (\text{proj}(1, 1))^{-1}$ and $J = \text{proj}(1, 1)$. Then

(i) for every rest R of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $\langle \mathcal{E}^1, \| \cdot \| \rangle$ holds $J \cdot R \cdot I$ is a rest, and

(ii) for every linear operator L from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ holds $J \cdot L \cdot I$ is a linear function.

(6) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$ and J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $I = (\text{proj}(1, 1))^{-1}$ and $J = \text{proj}(1, 1)$. Then

(i) for every rest R holds $I \cdot R \cdot J$ is a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $\langle \mathcal{E}^1, \| \cdot \| \rangle$, and

(ii) for every linear function L holds $I \cdot L \cdot J$ is a bounded linear operator from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$.

In the sequel f is a partial function from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to $\langle \mathcal{E}^1, \| \cdot \| \rangle$, g is a partial function from $\mathbb{R}$ to $\mathbb{R}$, x is a point of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, and y is an element of $\mathbb{R}$.

We now state four propositions:

(7) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and f is differentiable in x, then g is differentiable in y and $g'(y) = (\text{proj}(1, 1) \cdot f'(x) \cdot (\text{proj}(1, 1))^{-1})(1)$.

(8) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and g is differentiable in y, then f is differentiable in x and $f'(x)(\langle 1 \rangle) = \langle g'(y) \rangle$.

(9) If $f = \langle g \rangle$ and $x = \langle y \rangle$, then f is differentiable in x iff g is differentiable in y.

(10) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and f is differentiable in x, then $f'(x)(\langle 1 \rangle) = \langle g'(y) \rangle$.

2. Partial Differentiation

For simplicity, we adopt the following rules: m, n are non empty elements of $\mathbb{N}$, i, j are elements of $\mathbb{N}$, f is a partial function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, g is a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, x is a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and y is an element of $\mathcal{R}^n$.

Let n, m be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. We say that f is partially differentiable in x w.r.t. i if and only if:

(Def. 9) $f \cdot \mathrm{reproj}(i, x)$ is differentiable in $(\mathrm{Proj}(i, m))(x)$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. The functor $\mathrm{partdiff}(f, x, i)$ yielding a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined as follows:

(Def. 10) $\mathrm{partdiff}(f, x, i) = (f \cdot \mathrm{reproj}(i, x))'((\mathrm{Proj}(i, m))(x))$.

Let n be a non empty element of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. We say that f is partially differentiable in x w.r.t. i if and only if:

(Def. 11) $f \cdot \mathrm{reproj}(i, x)$ is differentiable in $(\mathrm{proj}(i, n))(x)$.

Let n be a non empty element of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. The functor $\mathrm{partdiff}(f, x, i)$ yields a real number and is defined by:

(Def. 12) $\mathrm{partdiff}(f, x, i) = (f \cdot \mathrm{reproj}(i, x))'((\mathrm{proj}(i, n))(x))$.

We now state several propositions:

(11) $\mathrm{Proj}(i, n) = (\mathrm{proj}(1, 1))^{-1} \cdot \mathrm{proj}(i, n)$.

(12) If $x = y$, then $\mathrm{reproj}(i, y) \cdot \mathrm{proj}(1, 1) = \mathrm{reproj}(i, x)$.

(13) If $f = \langle g \rangle$ and $x = y$, then $\langle g \cdot \mathrm{reproj}(i, y) \rangle = f \cdot \mathrm{reproj}(i, x)$.

(14) Suppose $f = \langle g \rangle$ and $x = y$. Then f is partially differentiable in x w.r.t. i if and only if g is partially differentiable in y w.r.t. i.

(15) If $f = \langle g \rangle$ and $x = y$ and f is partially differentiable in x w.r.t. i, then $(\mathrm{partdiff}(f, x, i))(\langle 1 \rangle) = \langle \mathrm{partdiff}(g, y, i) \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. We say that f is partially differentiable in x w.r.t. i if and only if the condition (Def. 13) is satisfied.

(Def. 13) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and g is partially differentiable in y w.r.t. i.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Let us assume that f is partially differentiable in x w.r.t. i. The functor $\mathrm{partdiff}(f, x, i)$ yielding an element of $\mathcal{R}^n$ is defined as follows:

(Def. 14) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and $\mathrm{partdiff}(f, x, i) = (\mathrm{partdiff}(g, y, i))(\langle 1 \rangle)$.

One can prove the following four propositions:

(16) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$. Then F is partially differentiable in x w.r.t. i if and only if G is partially differentiable in y w.r.t. i.

(17) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$ and F is partially differentiable in x w.r.t. i. Then $(\mathrm{partdiff}(F, x, i))(\langle 1 \rangle) = \mathrm{partdiff}(G, y, i)$.

(18) Let g_1 be a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$. Suppose $g_1 = \langle g \rangle$. Then g_1 is partially differentiable in y w.r.t. i if and only if g is partially differentiable in y w.r.t. i.

(19) Let g_1 be a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$. Suppose $g_1 = \langle g \rangle$ and g_1 is partially differentiable in y w.r.t. i. Then $\mathrm{partdiff}(g_1, y, i) = \langle \mathrm{partdiff}(g, y, i) \rangle$.

3. Linearity of Partial Differential Operator

For simplicity, we use the following convention: X is a set, r is a real number, f, f_1, f_2 are partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, g, g_1, g_2 are partial functions from $\mathcal{R}^n$ to $\mathbb{R}$, h is a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x is a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, y is an element of $\mathcal{R}^n$, and z is an element of $\mathcal{R}^m$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. We say that f is partially differentiable in x w.r.t. i and j if and only if:

(Def. 15) $\mathrm{Proj}(j, n) \cdot f \cdot \mathrm{reproj}(i, x)$ is differentiable in $(\mathrm{Proj}(i, m))(x)$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

The functor $\mathrm{partdiff}(f, x, i, j)$ yields a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and is defined by:

(Def. 16) $\mathrm{partdiff}(f, x, i, j) = (\mathrm{Proj}(j, n) \cdot f \cdot \mathrm{reproj}(i, x))'((\mathrm{Proj}(i, m))(x))$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let h be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let z be an element of $\mathcal{R}^m$. We say that h is partially differentiable in z w.r.t. i and j if and only if:

(Def. 17) $\mathrm{proj}(j, n) \cdot h \cdot \mathrm{reproj}(i, z)$ is differentiable in $(\mathrm{proj}(i, m))(z)$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let h be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let z be an element of $\mathcal{R}^m$. The functor $\mathrm{partdiff}(h, z, i, j)$ yielding a real number is defined as follows:

(Def. 18) $\mathrm{partdiff}(h, z, i, j) = (\mathrm{proj}(j, n) \cdot h \cdot \mathrm{reproj}(i, z))'((\mathrm{proj}(i, m))(z))$.

The following propositions are true:

(20) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$. Then F is differentiable in x if and only if G is differentiable in y.

(21) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. If $F = G$ and $x = y$ and F is differentiable in x, then $F'(x) = G'(y)$.

(22) If $f = h$ and $x = z$, then $\mathrm{Proj}(j, n) \cdot f \cdot \mathrm{reproj}(i, x) = \langle \mathrm{proj}(j, n) \cdot h \cdot \mathrm{reproj}(i, z) \rangle$.

(23) Suppose $f = h$ and $x = z$. Then f is partially differentiable in x w.r.t. i and j if and only if h is partially differentiable in z w.r.t. i and j.

(24) If $f = h$ and $x = z$ and f is partially differentiable in x w.r.t. i and j, then $(\mathrm{partdiff}(f, x, i, j))(\langle 1 \rangle) = \langle \mathrm{partdiff}(h, z, i, j) \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let X be a set. We say that f is partially differentiable on X w.r.t. i if and only if:

(Def. 19) $X \subseteq \mathrm{dom}\, f$ and for every point x of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $x \in X$ holds $f{\restriction}X$ is partially differentiable in x w.r.t. i.

We now state the proposition

(25) If f is partially differentiable on X w.r.t. i, then X is a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let us consider X. Let us assume that f is partially differentiable on X w.r.t. i. The functor $f{\restriction}^i X$ yielding a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined by:

(Def. 20) $\mathrm{dom}(f\restriction^i X) = X$ and for every point x of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $x \in X$ holds $(f\restriction^i X)_x = \mathrm{partdiff}(f, x, i)$.

The following propositions are true:

(26) $(f_1 + f_2) \cdot \mathrm{reproj}(i, x) = f_1 \cdot \mathrm{reproj}(i, x) + f_2 \cdot \mathrm{reproj}(i, x)$ and $(f_1 - f_2) \cdot \mathrm{reproj}(i, x) = f_1 \cdot \mathrm{reproj}(i, x) - f_2 \cdot \mathrm{reproj}(i, x)$.

(27) $r\,(f \cdot \mathrm{reproj}(i, x)) = (r\,f) \cdot \mathrm{reproj}(i, x)$.

(28) Suppose f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i. Then $f_1 + f_2$ is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f_1 + f_2, x, i) = \mathrm{partdiff}(f_1, x, i) + \mathrm{partdiff}(f_2, x, i)$.

(29) Suppose g_1 is partially differentiable in y w.r.t. i and g_2 is partially differentiable in y w.r.t. i. Then $g_1 + g_2$ is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(g_1 + g_2, y, i) = \mathrm{partdiff}(g_1, y, i) + \mathrm{partdiff}(g_2, y, i)$.

(30) Suppose f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i. Then $f_1 - f_2$ is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f_1 - f_2, x, i) = \mathrm{partdiff}(f_1, x, i) - \mathrm{partdiff}(f_2, x, i)$.

(31) Suppose g_1 is partially differentiable in y w.r.t. i and g_2 is partially differentiable in y w.r.t. i. Then $g_1 - g_2$ is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(g_1 - g_2, y, i) = \mathrm{partdiff}(g_1, y, i) - \mathrm{partdiff}(g_2, y, i)$.

(32) Suppose f is partially differentiable in x w.r.t. i. Then $r\,f$ is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(r\,f, x, i) = r \cdot \mathrm{partdiff}(f, x, i)$.

(33) Suppose g is partially differentiable in y w.r.t. i. Then $r\,g$ is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(r\,g, y, i) = r \cdot \mathrm{partdiff}(g, y, i)$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[8] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[10] Noboru Endou and Yasunari Shidama. Completeness of the real Euclidean space. *Formalized Mathematics*, 13(**4**):577–580, 2005.

[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[12] Hiroshi Imura, Morishige Kimura, and Yasunari Shidama. The differentiable functions on normed linear spaces. *Formalized Mathematics*, 12(**3**):321–327, 2004.

[13] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.

[14] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[16] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.

[17] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.

[18] Laurent Schwartz. *Cours d'analyse.* Hermann, 1981.

[19] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(**1**):39–48, 2004.

[20] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[22] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[23] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[25] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[26] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[27] Hiroshi Yamazaki, Yoshinori Fujisawa, and Yatsuka Nakamura. On replace function and swap function for finite sequences. *Formalized Mathematics*, 9(**3**):471–474, 2001.

[28] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(**2**):171–175, 1992.

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