Coproducts in Categories without Uniqueness of cod and dom

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Summary. The paper introduces coproducts in categories without uniqueness of cod and dom. It is proven that set-theoretical disjoint union is the coproduct in the category Ens [9].

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The notation and terminology used in this paper have been introduced in the following articles: [10], [7], [6], [1], [11], [2], [3], [8], [4], [12], [14], and [5].

From now on \( I \) denotes a set and \( E \) denotes a non empty set.

Let \( I \) be a non empty set, \( A \) be a many sorted set indexed by \( I \), and \( i \) be an element of \( I \). Let us observe that \( \text{coprod}(i, A) \) is relation-like and function-like.

Let \( C \) be a non empty category structure, \( o \) be an object of \( C \), \( I \) be a set, and \( f \) be an objects family of \( I \) and \( C \). A morphisms family of \( f \) and \( o \) is a many sorted set indexed by \( I \) and is defined by

(Def. 1) Let us consider an element \( i \). Suppose \( i \in I \). Then there exists an object \( o_1 \) of \( C \) such that

(i) \( o_1 = f(i) \), and

(ii) \( it(i) \) is a morphism from \( o_1 \) to \( o \).

Let \( I \) be a non empty set. Let us note that a morphisms family of \( f \) and \( o \) can equivalently be formulated as follows:

(Def. 2) Let us consider an element \( i \) of \( I \). Then \( it(i) \) is a morphism from \( f(i) \) to \( o \).
Let $M$ be a morphisms family of $f$ and $o$ and $i$ be an element of $I$. Note that the functor $M(i)$ yields a morphism from $f(i)$ to $o$. Let $C$ be a functional non empty category structure. Let $I$ be a set. Let us note that every morphisms family of $f$ and $o$ is function yielding.

Now we state the proposition:

(1) Let us consider a non empty category structure $C$, an object $o$ of $C$, and an objects family $f$ of $\emptyset$ and $C$. Then $\emptyset$ is a morphisms family of $f$ and $o$.

Let $C$ be a non empty category structure, $I$ be a set, $A$ be an objects family of $I$ and $C$, $B$ be an object of $C$, and $P$ be a morphisms family of $A$ and $B$. We say that $P$ is feasible if and only if

(Def. 3) Let us consider a set $i$. Suppose $i \in I$. Then there exists an object $o$ of $C$ such that
(i) $o = A(i)$, and
(ii) $P(i) \in \langle o, B \rangle$.

Let $I$ be a non empty set. Let us observe that $P$ is feasible if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let us consider an element $i$ of $I$. Then $P(i) \in \langle A(i), B \rangle$.

Let $C$ be a category and $I$ be a set. We say that $P$ is coprojection morphisms if and only if

(Def. 5) Let us consider an object $X$ of $C$ and a morphisms family $F$ of $A$ and $X$. Suppose $F$ is feasible. Then there exists a morphism $f$ from $B$ to $X$ such that
(i) $f \in \langle B, X \rangle$, and
(ii) for every set $i$ such that $i \in I$ there exists an object $s_i$ of $C$ and there exists a morphism $P_i$ from $s_i$ to $B$ such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$, and
(iii) for every morphism $f_1$ from $B$ to $X$ such that for every element $i$ of $I$ there exists an object $s_i$ of $C$ and there exists a morphism $P_i$ from $s_i$ to $B$ such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f_1 \cdot P_i$ holds $f = f_1$.

Let $I$ be a non empty set. Let us note that $P$ is coprojection morphisms if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let us consider an object $X$ of $C$ and a morphisms family $F$ of $A$ and $X$. Suppose $F$ is feasible. Then there exists a morphism $f$ from $B$ to $X$ such that
(i) $f \in \langle B, X \rangle$, and
(ii) for every element $i$ of $I$, $F(i) = f \cdot P(i)$, and
(iii) for every morphism $f_1$ from $B$ to $X$ such that for every element $i$ of $I$, $F(i) = f_1 \cdot P(i)$ holds $f = f_1$.
Let $A$ be an objects family of $\emptyset$ and $C$. Note that every morphisms family of $A$ and $B$ is feasible.

Now we state the propositions:

(2) Let us consider a category $C$, an objects family $A$ of $\emptyset$ and $C$, and an object $B$ of $C$. Suppose $B$ is initial. Then there exists a morphisms family $P$ of $A$ and $B$ such that $P$ is empty and coprojection morphisms. The theorem is a consequence of (1).

(3) Let us consider an objects family $A$ of $I$ and $\text{Ens}_{\{\emptyset\}}$ and an object $o$ of $\text{Ens}_{\{\emptyset\}}$. Then $I \mapsto \emptyset$ is a morphisms family of $A$ and $o$.

(4) Let us consider an objects family $A$ of $I$ and $\text{Ens}_{\{\emptyset\}}$, an object $o$ of $\text{Ens}_{\{\emptyset\}}$, and a morphisms family $P$ of $A$ and $o$. If $P = I \mapsto \emptyset$, then $P$ is feasible and coprojection morphisms. Proof: $P$ is feasible by [11, (7)].

Reconsider $f = \emptyset$ as a morphism from $o$ to $Y$. For every set $i$ such that $i \in I$ there exists an object $s_i$ of $C$ and there exists a morphism $P_i$ from $s_i$ to $o$ such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$ by [11, (7)]. □

Let $C$ be a category. We say that $C$ has coproducts if and only if

(Def. 7) Let us consider a set $I$ and an objects family $A$ of $I$ and $C$. Then there exists an object $B$ of $C$ and there exists a morphisms family $P$ of $A$ and $B$ such that $P$ is feasible and coprojection morphisms.

Note that $\text{Ens}_{\{\emptyset\}}$ has coproducts and there exists a category which is strict and has products and coproducts.

Let $C$ be a category, $I$ be a set, $A$ be an objects family of $I$ and $C$, and $B$ be an object of $C$. We say that $B$ is $A$-category coproduct-like if and only if

(Def. 8) There exists a morphisms family $P$ of $A$ and $B$ such that $P$ is feasible and coprojection morphisms.

Let $C$ be a category with coproducts. Let us observe that there exists an object of $C$ which is $A$-category coproduct-like.

Let $C$ be a category and $A$ be an objects family of $\emptyset$ and $C$. Note that every object of $C$ which is $A$-category coproduct-like is also initial.

Now we state the propositions:

(5) Let us consider a category $C$, an objects family $A$ of $\emptyset$ and $C$, and an object $B$ of $C$. If $B$ is initial, then $B$ is $A$-category coproduct-like. The theorem is a consequence of (2).

(6) Let us consider a category $C$, an objects family $A$ of $I$ and $C$, and objects $C_1$, $C_2$ of $C$. Suppose

(i) $C_1$ is $A$-category coproduct-like, and

(ii) $C_2$ is $A$-category coproduct-like.

Then $C_1, C_2$ are iso.

From now on $A$ denotes an objects family of $I$ and $\text{Ens}_{E}$.
Let us consider $I$, $E$, and $A$. Assume $\bigcup \text{coprod}(A) \in E$. The functor $\prod A$ yielding an object of $\text{Ens}_E$ is defined by the term

(Def. 9) $\bigcup \text{coprod}(A)$.

The functor $\text{Coprod}(A)$ yielding a many sorted set indexed by $I$ is defined by

(Def. 10) Let us consider an element $i$. Suppose $i \in I$. Then there exists a function $F$ from $A(i)$ into $\bigcup \text{coprod}(A)$ such that

(i) $i(i) = F$, and

(ii) for every element $x$ such that $x \in A(i)$ holds $F(x) = \langle x, i \rangle$.

Let us observe that $\text{Coprod}(A)$ is function yielding.

Assume $\bigcup \text{coprod}(A) \in E$. The functor $\prod A$ yielding a morphisms family of $A$ and $\prod A$ is defined by the term

(Def. 11) $\text{Coprod}(A)$.

Now we state the propositions:

(7) If $\bigcup \text{coprod}(A) = \emptyset$, then $\text{Coprod}(A)$ is empty yielding.

(8) If $\bigcup \text{coprod}(A) = \emptyset$, then $A$ is empty yielding.

(9) If $\bigcup \text{coprod}(A) \in E$ and $\bigcup \text{coprod}(A) = \emptyset$, then $\prod A = I \mapsto \emptyset$. The theorem is a consequence of (7).

(10) If $\bigcup \text{coprod}(A) \in E$, then $\prod A$ is feasible and coprojection morphisms.

The theorem is a consequence of (7) and (8).

(11) If $\bigcup \text{coprod}(A) \in E$, then $\prod A$ is $A$-category coproduct-like. The theorem is a consequence of (10).

(12) If for every $I$ and $A$, $\bigcup \text{coprod}(A) \in E$, then $\text{Ens}_E$ has coproducts. The theorem is a consequence of (10).

References


Coproducts in categories without uniqueness of cod 


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Formulation of Cell Petri Nets

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Summary. Based on the Petri net definitions and theorems already formalized in the Mizar article [10], in this article we were able to formalize the definition of Cell Petri nets. It is based on [?]. Colored Petri net is already have been defined in [9]. In addition the conditions of the firing-rule and ColoredSet to this definition, that defines the Cell Petri nets extended to CPNT.i further. Although it was synthesis of two Petri nets in [9], it is synthesis from the family of Colored Petri nets (?? Colored-PT-net-Family of I) of finite number of pieces. That is, extension to a CPNT family is performed by defining the output arc from the transition of a certain Colored Petri nets to Place of a certain another Colored Petri nets (definition of the neighborhood). Finally, activation of Colored Petri nets was formalized.

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1. Preliminaries

Let $I$ be a non empty set and $C_1$ be a many sorted set indexed by $I$. We say that $C_1$ is colored-pt-net-family-like if and only if

(Def. 1) Let us consider an element $i$ of $I$. Then $C_1(i)$ is a colored place/transition net.

Note that there exists a many sorted set indexed by $I$ which is colored-pt-net-family-like.

A colored place/transition net family of $I$ is a colored-pt-net-family-like many sorted set indexed by $I$. Let $C_1$ be a colored place/transition net family of $I$
and \( i \) be an element of \( I \). One can check that the functor \( C_1(i) \) yields a colored place/transition net. Let \( C_2 \) be a colored place/transition net family of \( I \). We say that \( C_2 \) is disjoint valued if and only if

(Def. 2) Let us consider elements \( i, j \) of \( I \). Suppose \( i \neq j \). Then

(i) the carrier of \( C_2(i) \) misses the carrier of \( C_2(j) \), and

(ii) the carrier’ of \( C_2(i) \) misses the carrier’ of \( C_2(j) \).

Now we state the propositions:

(1) Let us consider a set \( I \) and many sorted sets \( F, D, R \) indexed by \( I \).
Suppose

(i) for every element \( i \) such that \( i \in I \) there exists a function \( f \) such that
\( f = F(i) \) and \( \text{dom } f = D(i) \) and \( \text{rng } f = R(i) \), and

(ii) for every elements \( i, j \) and for every functions \( f, g \) such that \( i, j \in I \)
and \( i \neq j \) and \( f = F(i) \) and \( g = F(j) \) holds \( \text{dom } f \) misses \( \text{dom } g \).

Then there exists a function \( G \) such that

(iii) \( G = \bigcup \text{rng } F \), and

(iv) \( \text{dom } G = \bigcup \text{rng } D \), and

(v) \( \text{rng } G = \bigcup \text{rng } R \), and

(vi) for every elements \( i, x \) and for every function \( f \) such that \( i \in I \) and
\( f = F(i) \) and \( x \in \text{dom } f \) holds \( G(x) = f(x) \).

**Proof:** For every element \( z \) such that \( z \in \bigcup \text{rng } F \) there exist elements
\( x, y, i \) such that \( z = \langle x, y \rangle \) and \( z \in F(i) \) and \( i \in I \). For every element \( z \) such that \( z \in \bigcup \text{rng } F \) there exist elements \( x, y \) such that \( z = \langle x, y \rangle \). Reconsider \( G = \bigcup \text{rng } F \) as a binary relation. \( G \) is a function. For every element \( x, x \in \text{dom } G \) iff \( x \in \bigcup \text{rng } D \) by [4, (3)]. For every element \( x \), \( x \in \text{rng } G \) iff \( x \in \bigcup \text{rng } R \) by [4, (3)]. For every elements \( i, x \) and for every function \( f \) such that \( i \in I \) and \( f = F(i) \) and \( x \in \text{dom } f \) holds \( G(x) = f(x) \) by [4] (1), (3). \( \square \)

(2) Let us consider a set \( I \) and many sorted sets \( Y, Z \) indexed by \( I \). Suppose elements \( i, j \). If \( i, j \in I \) and \( i \neq j \), then \( Y(i) \cap Z(j) = \emptyset \). Then \( \bigcup(Y \setminus Z) = \bigcup Y \setminus \bigcup Z \). **Proof:** Set \( X = Y \setminus Z \). For every element \( x \), \( x \in \bigcup \text{rng } X \) iff \( x \in \bigcup \text{rng } Y \setminus \bigcup \text{rng } Z \) by [4] (3). \( \square \)

(3) Let us consider a set \( I \) and many sorted sets \( X, Y, Z \) indexed by \( I \).
Suppose

(i) \( X \subseteq Y \setminus Z \), and

(ii) for every elements \( i, j \) such that \( i, j \in I \) and \( i \neq j \) holds \( Y(i) \cap Z(j) = \emptyset \).

Then \( \bigcup X \subseteq \bigcup Y \setminus \bigcup Z \). The theorem is a consequence of (2).
2. Synthesis of CPNT and I

Let $I$ be a non trivial set. The functor $\text{XorDelta} I$ yielding a non empty set is defined by the term

(Def. 3) $\{(i, j), \text{where } i, j \text{ are elements of } I : i \neq j\}$.

Now we state the proposition:

(4) Let us consider a non trivial finite set $I$ and a colored place/transition net family $C_2$ of $I$. Then $\bigcup \{(\text{the carrier of } C_2(j))^{\text{Outbds}(C_2(i))}, \text{where } i, j \text{ are elements of } I : i \neq j\}$ is not empty.

Let $I$ be a non trivial finite set and $C_2$ be a colored place/transition net family of $I$. A connecting mapping of $C_2$ is a many sorted set indexed by $\text{XorDelta} I$ and is defined by

(Def. 4) (i) $\text{rng } it \subseteq \bigcup \{(\text{the carrier of } C_2(j))^{\text{Outbds}(C_2(i))}, \text{where } i, j \text{ are elements of } I : i \neq j\}$, and

(ii) for every elements $i, j$ of $I$ such that $i \neq j$ holds $it((i, j))$ is a function from $\text{Outbds}(C_2(i))$ into the carrier of $C_2(j)$.

Now we state the proposition:

(5) Let us consider colored place/transition nets $C_4, C_5$, a function $O_{12}$ from $\text{Outbds} C_4$ into the carrier of $C_5$, and a function $q_{12}$. Suppose

(i) $\text{dom } q_{12} = \text{Outbds} C_4$, and

(ii) for every transition $t_{01}$ of $C_4$ such that $t_{01}$ is outbound holds $q_{12}(t_{01})$ is a function from the thin cylinders of the colored set of $C_4$ and $\ast\{t_{01}\}$ into the thin cylinders of the colored set of $C_4$ and $O_{12}\ast t_{01}$.

Then $q_{12} \in (\bigcup \{(\text{the thin cylinders of the colored set of } C_4 \text{ and } O_{12}\ast t_{01})^\alpha, \text{where } t_{01} \text{ is a transition of } C_4 : t_{01} \text{ is outbound}\})^{\text{Outbds} C_4}$, where $\alpha$ is the thin cylinders of the colored set of $C_4$ and $\ast\{t_{01}\}$.

Let $I$ be a non trivial finite set, $C_2$ be a colored place/transition net family of $I$, and $O$ be a connecting mapping of $C_2$. A connecting firing rule of $O$ is a many sorted set indexed by $\text{XorDelta} I$ and is defined by

(Def. 5) Let us consider elements $i, j$ of $I$. Suppose $i \neq j$. Then there exists a function $O_6$ from $\text{Outbds}(C_2(i))$ into the carrier of $C_2(j)$ and there exists a function $q_8$ such that $q_8 = it((i, j))$ and $O_6 = O((i, j))$ and $\text{dom } q_8 = \text{Outbds}(C_2(i))$ and for every transition $t_{01}$ of $C_2(i)$ such that $t_{01}$ is outbound holds $q_8(t_{01})$ is a function from the thin cylinders of the colored set of $C_2(i)$ and $\ast\{t_{01}\}$ into the thin cylinders of the colored set of $C_2(i)$ and $O_6\ast t_{01}$. 
3. Extension to a Family of Colored Petri Nets

Let $I$ be a non trivial finite set, $C_2$ be a colored place/transition net family of $I$, $O$ be a connecting mapping of $C_2$, and $q$ be a connecting firing rule of $O$. Assume $C_2$ is disjoint valued and for every elements $i, j_1, j_2$ of $I$ such that $i \neq j_1$ and $i \neq j_2$ there exist elements $x, y_1, y_2$ such that $\langle x, y_1 \rangle \in q(\langle i, j_1 \rangle)$ and $\langle x, y_2 \rangle \in q(\langle i, j_2 \rangle)$ holds $j_1 = j_2$. The functor synthesis $q$ yielding a strict colored place/transition net is defined by

(Def. 6) There exist many sorted sets $P, T, S_9, T_8, C_3, F$ indexed by $I$ and there exist functions $U_9, U_8$ such that for every element $i$ of $I$, $P(i) = \text{the carrier of } C_2(i)$ and $T(i) = \text{the carrier' of } C_2(i)$ and $S_9(i) = \text{the S-T arcs of } C_2(i)$ and $T_8(i) = \text{the T-S arcs of } C_2(i)$ and $C_3(i) = \text{the colored set of } C_2(i)$ and $F(i) = \text{the firing-rule of } C_2(i)$ and $U_9 = \bigcup \text{rng } F$ and $U_8 = \bigcup \text{rng } q$ and the carrier of it = $\bigcup \text{rng } P$ and the carrier' of it = $\bigcup \text{rng } T$ and the S-T arcs of it = $\bigcup \text{rng } S_9$ and the T-S arcs of it = $\bigcup \text{rng } T_8 \cup \bigcup \text{rng } O$ and the colored set of it = $\bigcup \text{rng } C_3$ and the firing-rule of it = $U_9 + U_8$.

4. Definition of Cell Petri Nets

Let $I$ be a non empty finite set and $C_2$ be a colored place/transition net family of $I$. We say that $C_2$ is cell Petri nets if and only if

(Def. 7) There exists a function $N$ from $I$ into $2^{\text{rng } C_2}$ such that for every element $i$ of $I$, $N(i) = \{C_2(j) \mid j \in I \ \text{where } j \neq i\}$.

Let $N$ be a function from $I$ into $2^{\text{rng } C_2}$ and $O$ be a connecting mapping of $C_2$. We say that $(N, O)$ is cell Petri nets if and only if

(Def. 8) Let us consider an element $i$ of $I$. Then $N(i) = \{C_2(j) \mid j \in I \ \text{where } j \neq i \}$ and there exists a transition $t$ of $C_2(i)$ and there exists an element $s$ such that $\langle t, s \rangle \in O(\langle i, j \rangle)$.

Now we state the proposition:

(6) Let us consider a non trivial finite set $I$, a colored place/transition net family $C_2$ of $I$, a function $N$ from $I$ into $2^{\text{rng } C_2}$, and a connecting mapping $O$ of $C_2$. Suppose

(i) $C_2$ is one-to-one, and

(ii) $(N, O)$ is cell Petri nets.

Let us consider an element $i$ of $I$. Then $C_2(i) \notin N(i)$.
5. Activation of Petri Nets

Let \( C_6 \) be a colored place/transition net structure. We say that \( C_6 \) has nontrivial colored set if and only if

(Def. 9) The colored set of \( C_6 \) is not trivial.

One can verify that there exists a strict colored-PT-net-like colored Petri net which has nontrivial colored set.

Let \( C_2 \) be a colored place/transition net with nontrivial colored set. One can verify that the colored set of \( C_2 \) is non trivial.

Let \( C_6 \) be a colored place/transition net with nontrivial colored set, \( S \) be a subset of the carrier of \( C_6 \), and \( D \) be a thin cylinder of the colored set of \( C_6 \) and \( S \). A color threshold of \( D \) is a function from \( \text{loc} \ D \) into the colored set of \( C_6 \).

(Def. 10) the set of all \( e \) where \( e \) is a color count of \( C_6 \).

A colored state of \( C_6 \) is a function from \( C_6 \) into the colored states of \( C_6 \).

Now we state the proposition:

(7) Let us consider a thin cylinder \( D \) of the colored set of \( C_6 \) and \( \{t\} \), there exists a color threshold \( C_a \) of \( D \) such that \( t \) is firable on \( m \) and \( C_a \).

Now we state the proposition:

(7) Let us consider a thin cylinder \( D \) of the colored set of \( C_6 \) and \( \{t\} \). Then there exists a color threshold \( C_a \) of \( D \) such that \( t \) is firable on \( m \) and \( C_a \).

Let us consider \( C_6 \), \( m \) and \( t \). Let \( L \) be a thin cylinder of the colored set of \( C_6 \) and \( \{t\} \), \( C_a \) be a color threshold of \( D \), and \( p \) be an element of \( C_6 \). Assume \( t \) is firable on \( m \) and \( C_a \). The Petri subtraction \((C_a,m,p)\) yielding a function from the colored set of \( C_6 \) into \( \mathbb{N} \) is defined by

(Def. 13) Let us consider an element \( x \) of the colored set of \( C_6 \). Then
Let us consider an element $x$ of the colored set of $C_6$. Then

(i) if $p \in \text{loc } D$ and $x = C_a(p)$, then $it(x) = m(p)(x) - 1$, and

(ii) if it is not true that $p \in \text{loc } D$ and $x = C_a(p)$, then $it(x) = m(p)(x)$.

Let $D$ be a thin cylinder of the colored set of $C_6$ and $\{t\}$. The Petri addition($C_a,m,p$) yielding a function from the colored set of $C_6$ into $\mathbb{N}$ is defined by

(Def. 14) Let us consider an element $x$ of the colored set of $C_6$. Then

(i) if $p \in \text{loc } D$ and $x = C_a(p)$, then $it(x) = m(p)(x) + 1$, and

(ii) if it is not true that $p \in \text{loc } D$ and $x = C_a(p)$, then $it(x) = m(p)(x)$.

Let $D$ be a thin cylinder of the colored set of $C_6$ and $^*\{t\}$ and $E$ be a thin cylinder of the colored set of $C_6$ and $\{t\}$. Let $C_d$ be a color threshold of $E$. The firing result($C_a,C_d,m,p$) yielding a function from the colored set of $C_6$ into $\mathbb{N}$ is defined by the term

(Def. 15) \[
\begin{cases}
\text{the Petri subtraction}(C_a,m,p), & \text{if } t \text{ is firable on } m \text{ and } C_a \text{ and } p \in \text{loc } D \setminus \text{loc } E,
\text{the Petri addition}(C_d,m,p), & \text{if } t \text{ is firable on } m \text{ and } C_a \text{ and } p \in \text{loc } E \setminus \text{loc } D,
\text{otherwise}.
\end{cases}
\]

Let us consider a thin cylinder $D_0$ of the colored set of $C_6$ and $^*\{t\}$, a thin cylinder $D_1$ of the colored set of $C_6$ and $\{t\}$, a color threshold $C_b$ of $D_0$, a color threshold $C_c$ of $D_1$, an element $x$ of the colored set of $C_6$, and an element $p$ of $C_6$. Now we state the propositions:

(8) $m(p)(x) - 1 \leq (\text{the firing result}(C_b,C_c,m,p))(x) \leq m(p)(x) + 1$.

(9) If $t$ is outbound, then $m(p)(x) - 1 \leq (\text{the firing result}(C_b,C_c,m,p))(x) \leq m(p)(x)$.

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Formulation of cell Petri nets


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Isometric Differentiable Functions on Real Normed Space

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Summary. In this article, we formalize isometric differentiable functions on real normed space \( \mathbb{R}^n \), and their properties.

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The notation and terminology used in this paper have been introduced in the following articles: \[3, 2, 8, 4, 5, 17, 10, 11, 18, 14, 16, 1, 6, 9, 15, 22, 23, 20, 21, 13, 24, \] and \[7. \]

From now on, \( S, T, W \) denote real normed spaces, \( f, f_1, f_2 \) denote partial functions from \( S \) to \( T \), \( Z \) denotes a subset of \( S \), \( i, n \) denote natural numbers, and \( Y \) denotes a real normed space.

Let us consider a real norm space sequence \( G \), a real normed space \( F \), a set \( i \), partial functions \( f, g \) from \( \prod G \) to \( F \), and a subset \( X \) of \( \prod G \). Now we state the propositions:

(1) Suppose \( X \) is open and \( i \in \text{dom} G \) and \( f \) is partially differentiable on \( X \) w.r.t. \( i \) and \( g \) is partially differentiable on \( X \) w.r.t. \( i \). Then

(i) \( f + g \) is partially differentiable on \( X \) w.r.t. \( i \), and

(ii) \( (f + g)^i X = (f^i X) + (g^i X). \)

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(2) Suppose $X$ is open and $i \in \text{dom} \ G$ and $f$ is partially differentiable on $X$ w.r.t. $i$ and $g$ is partially differentiable on $X$ w.r.t. $i$. Then 

(i) $f - g$ is partially differentiable on $X$ w.r.t. $i$, and 
(ii) $(f - g)^{\mid i}X = (f^{\mid i}X) - (g^{\mid i}X)$.

Now we state the propositions:

(3) Let us consider a real norm space sequence $G$, a real normed space $F$, a set $i$, a partial function $f$ from $\prod G$ to $F$, a real number $r$, and a subset $X$ of $\prod G$. Suppose 

(i) $X$ is open, and 
(ii) $i \in \text{dom} \ G$, and 
(iii) $f$ is partially differentiable on $X$ w.r.t. $i$.

Then 

(iv) $r \cdot f$ is partially differentiable on $X$ w.r.t. $i$, and 
(v) $r \cdot f^{\mid i}X = r \cdot (f^{\mid i}X)$.

**Proof:** Set $h = r \cdot f$. For every point $x$ of $\prod G$ such that $x \in X$ holds $h$ is partially differentiable in $x$ w.r.t. $i$ and $\text{partdiff}(h, x, i) = r \cdot \text{partdiff}(f, x, i)$ by [18, (24), (30)]. Set $f_3 = f^{\mid i}X$. For every point $x$ of $\prod G$ such that $x \in X$ holds $(r \cdot f_3)_x = \text{partdiff}(h, x, i)$.

(4) Let us consider sets $X, Y, Z$, functions $I, f$, and a set $X$. Then $(f^{\mid X}) \cdot I = (f \cdot I)^{\mid I^{-1}(X)}$.

Let us consider $S$ and $T$. Let $f$ be a function from $S$ into $T$. We say that $f$ is isometric if and only if 

(Def. 1) Let us consider an element $x$ of $S$. Then $\|f(x)\| = \|x\|$.

Now we state the propositions:

(5) Let us consider a linear operator $I$ from $S$ into $T$. If $I$ is isometric, then for every point $x$ of $S$, $I$ is continuous in $x$.

(6) Let us consider a linear operator $I$ from $S$ into $T$ and a subset $Z$ of $S$. If $I$ is isometric, then $I$ is continuous on $Z$. The theorem is a consequence of (5).

(7) Let us consider a linear operator $I$ from $S$ into $T$. Suppose $I$ is one-to-one, onto, and isometric. Then there exists a linear operator $J$ from $T$ into $S$ such that 

(i) $J = I^{-1}$, and 
(ii) $J$ is one-to-one, onto, and isometric.

**Proof:** Reconsider $J = I^{-1}$ as a function from $T$ into $S$. For every points $v, w$ of $T$, $J(v + w) = J(v) + J(w)$ by [5, (113)], [4, (34)]. For every point $v$ of $T$ and for every real number $r$, $J(r \cdot v) = r \cdot J(v)$ by [5, (113)], [4, (34)]. For every point $v$ of $T$, $\|J(v)\| = \|v\|$ by [5, (113)], [4, (34)]. \□
Let us consider a linear operator $I$ from $S$ into $T$ and a sequence $s_1$ of $S$. Now we state the propositions:

(8) If $I$ is isometric and $s_1$ is convergent, then $I \cdot s_1$ is convergent and $\lim (I \cdot s_1) = I (\lim s_1)$.

(9) If $I$ is one-to-one, onto, and isometric, then $s_1$ is convergent iff $I \cdot s_1$ is convergent.

Let us consider a linear operator $I$ from $S$ into $T$ and a subset $Z$ of $S$. Now we state the propositions:

(10) If $I$ is one-to-one, onto, and isometric, then $Z$ is closed iff $I \circ Z$ is closed.

(11) If $I$ is one-to-one, onto, and isometric, then $Z$ is open iff $I \circ Z$ is open.

(12) If $I$ is one-to-one, onto, and isometric, then $Z$ is compact iff $I \circ Z$ is compact.

Now we state the propositions:

(13) Let us consider a partial function $f$ from $T$ to $W$, a function $g$ from $S$ into $T$, and a point $x$ of $S$. Suppose

(i) $x \in \text{dom } g$, and

(ii) $g(x) \in \text{dom } f$, and

(iii) $g$ is continuous in $x$, and

(iv) $f$ is continuous in $g(x)$.

Then $f \cdot g$ is continuous in $x$. \textbf{Proof:} Set $h = f \cdot g$. For every real number $r$ such that $0 < r$ there exists a real number $s$ such that $0 < s$ and for every point $x_1$ of $S$ such that $x_1 \in \text{dom } h$ and $\|x_1 - x\| < s$ holds $\|h(x_1) - h(x)\| < r$ by [14, (7)], [12, (3), (4)]. □

(14) Let us consider a partial function $f$ from $T$ to $W$ and a linear operator $I$ from $S$ into $T$. Suppose $I$ is one-to-one, onto, and isometric. Let us consider a point $x$ of $S$. Suppose $I(x) \in \text{dom } f$. Then $f \cdot I$ is continuous in $x$ if and only if $f$ is continuous in $I(x)$. The theorem is a consequence of (7), (5), and (13).

(15) Let us consider a partial function $f$ from $T$ to $W$, a linear operator $I$ from $S$ into $T$, and a set $X$. Suppose

(i) $X \subseteq \text{the carrier of } T$, and

(ii) $I$ is one-to-one, onto, and isometric.

Then $f$ is continuous on $X$ if and only if $f \cdot I$ is continuous on $I^{-1}(X)$. The theorem is a consequence of (14) and (4). \textbf{Proof:} For every point $y$ of $T$ such that $y \in X$ holds $f|X$ is continuous in $y$ by [5, (113)], [22, (57)]. □

Let $X, Y$ be real normed spaces. The functor $\text{IsoCPN rSP} (X, Y)$ yielding a linear operator from $X \times Y$ into $\prod \langle X, Y \rangle$ is defined by
(Def. 2) (i) it is one-to-one and onto, and
(ii) for every point $x$ of $X$ and for every point $y$ of $Y$, $it(x, y) = \langle x, y \rangle$

(iii) $0 \prod_{\langle X, Y \rangle} = it(0_{X \times Y})$, and
(iv) it is isometric.

The functor $\text{IsoPCNrSP}(X, Y)$ yielding a linear operator from $\prod_{\langle X, Y \rangle}$ into $X \times Y$ is defined by

(Def. 3) (i) $it = (\text{IsoCPNrSP}(X, Y))^{-1}$, and
(ii) it is one-to-one and onto, and
(iii) for every point $x$ of $X$ and for every point $y$ of $Y$, $it(\langle x, y \rangle) = \langle x, y \rangle$

(iv) $0_{X \times Y} = it(0 \prod_{\langle X, Y \rangle})$, and
(v) it is isometric.

Now we state the propositions:

(16) Let us consider real normed spaces $X$, $Y$ and a point $z$ of $X \times Y$. Then $\text{IsoCPNrSP}(X, Y)$ is continuous in $z$. The theorem is a consequence of (5).

(17) Let us consider real normed spaces $X$, $Y$ and a point $z$ of $\prod_{\langle X, Y \rangle}$. Then $\text{IsoPCNrSP}(X, Y)$ is continuous in $z$. The theorem is a consequence of (5).

(18) Let us consider real normed spaces $X$, $Y$ and a subset $Z$ of $X \times Y$. Then

(i) $\text{IsoCPNrSP}(X, Y)$ is continuous on $Z$, and
(ii) $Z$ is closed iff $(\text{IsoCPNrSP}(X, Y))^\circ Z$ is closed, and
(iii) $Z$ is open iff $(\text{IsoCPNrSP}(X, Y))^\circ Z$ is open, and
(iv) $Z$ is compact iff $(\text{IsoCPNrSP}(X, Y))^\circ Z$ is compact.

The theorem is a consequence of (6), (10), (11), and (12).

(19) Let us consider real normed spaces $X$, $Y$ and a subset $Z$ of $\prod_{\langle X, Y \rangle}$. Then

(i) $\text{IsoPCNrSP}(X, Y)$ is continuous on $Z$, and
(ii) $Z$ is closed iff $(\text{IsoPCNrSP}(X, Y))^\circ Z$ is closed, and
(iii) $Z$ is open iff $(\text{IsoPCNrSP}(X, Y))^\circ Z$ is open, and
(iv) $Z$ is compact iff $(\text{IsoPCNrSP}(X, Y))^\circ Z$ is compact.

The theorem is a consequence of (6), (10), (11), and (12).

(20) Let us consider real normed spaces $S$, $T$, $W$, a point $f$ of the real norm space of bounded linear operators from $S$ into $W$, a point $g$ of the real norm space of bounded linear operators from $T$ into $W$, and a linear operator $I$ from $S$ into $T$. Suppose

(i) $I$ is one-to-one, onto, and isometric, and
(ii) \( f = g \cdot I. \)

Then \( \|f\| = \|g\|. \) The theorem is a consequence of (7). \textbf{Proof:} Consider \( J \) being a linear operator from \( T \) into \( S \) such that \( J = I^{-1} \) and \( J \) is one-to-one, onto, and isometric. Reconsider \( g_0 = g \) as a Lipschitzian linear operator from \( T \) into \( W. \) Reconsider \( g_4 = g \cdot I \) as a Lipschitzian linear operator from \( S \) into \( W. \) For every element \( x, x \in \{\|g_0(t)\|, \text{ where } t \text{ is a vector of } T : \|t\| \leq 1 \text{ iff } x \in \{\|g_4(w)\|, \text{ where } w \text{ is a vector of } S : \|w\| \leq 1 \} \text{ by } [13], (35). \) □

(21) Let us consider real normed spaces \( X, Y, \) a partial function \( f \) from \( \prod\langle X, Y \rangle \) to \( W, \) and a point \( z \) of \( X \times Y. \) Suppose \( \text{(IsoCPNrSP}(X, Y))(z) \in \text{dom } f. \) Then \( f \cdot \text{IsoCPNrSP}(X, Y) \) is continuous in \( z \) if and only if \( f \) is continuous in \( \text{(IsoCPNrSP}(X, Y))(z). \) The theorem is a consequence of (14).

(22) Let us consider real normed spaces \( X, Y, \) a partial function \( f \) from \( X \times Y \) to \( W, \) and a point \( z \) of \( \prod\langle X, Y \rangle. \) Suppose \( \text{(IsoCPNrSP}(X, Y))(z) \in \text{dom } f. \) Then \( f \cdot \text{IsoCPNrSP}(X, Y) \) is continuous in \( z \) if and only if \( f \) is continuous in \( \text{(IsoCPNrSP}(X, Y))(z). \) The theorem is a consequence of (14).

(23) Let us consider real normed spaces \( X, Y, \) a partial function \( f \) from \( \prod\langle X, Y \rangle \) to \( W, \) and a set \( D. \) Suppose \( D \subseteq \text{carrier of } \prod\langle X, Y \rangle. \) Then \( f \cdot \text{IsoCPNrSP}(X, Y) \) is continuous on \( \text{(IsoCPNrSP}(X, Y))^{-1}(D) \) if and only if \( f \) is continuous on \( D. \) The theorem is a consequence of (15).

(24) Let us consider real normed spaces \( X, Y, \) a partial function \( f \) from \( X \times Y \) to \( W, \) and a set \( D. \) Suppose \( D \subseteq \text{carrier of } X \times Y. \) Then \( f \cdot \text{IsoCPNrSP}(X, Y) \) is continuous on \( \text{(IsoCPNrSP}(X, Y))^{-1}(D) \) if and only if \( f \) is continuous on \( D. \) The theorem is a consequence of (15).

(25) Let us consider a linear operator \( I \) from \( S \) into \( T. \) If \( I \) is isometric, then \( I \) is a Lipschitzian linear operator from \( S \) into \( T. \)

Let us consider real normed spaces \( X, Y. \) Now we state the propositions:

(26) \( \text{IsoCPNrSP}(X, Y) \) is a Lipschitzian linear operator from \( X \times Y \) into \( \prod\langle X, Y \rangle. \)

(27) \( \text{IsoPCNrSP}(X, Y) \) is a Lipschitzian linear operator from \( \prod\langle X, Y \rangle \) into \( X \times Y. \)

Let \( X, Y \) be real normed spaces. Note that the functor \( \text{IsoCPNrSP}(X, Y) \) yields a Lipschitzian linear operator from \( X \times Y \) into \( \prod\langle X, Y \rangle. \) Let us observe that the functor \( \text{IsoCPNrSP}(X, Y) \) yields a Lipschitzian linear operator from \( \prod\langle X, Y \rangle \) into \( X \times Y. \)

Let us consider real normed spaces \( X, Y, W, \) a point \( f \) of the real norm space of bounded linear operators from \( X \times Y \) into \( W, \) and a point \( g \) of the real norm space of bounded linear operators from \( \prod\langle X, Y \rangle \) into \( W. \) Now we state the propositions:

(28) \( f = g \cdot \text{IsoCPNrSP}(X, Y), \) then \( \|f\| = \|g\|. \)
(29) If \( g = f \cdot \text{IsoPCNrSP}(X, Y) \), then \( \|g\| = \|f\| \).

Now we state the propositions:

(30) Let us consider real normed spaces \( S, T \), a Lipschitzian linear operator \( L \) from \( S \) into \( T \), and a point \( x_0 \) of \( S \). Then
   
   (i) \( L \) is differentiable in \( x_0 \), and
   (ii) \( L'(x_0) = L \).

**Proof:** Reconsider \( L = L_0 \) as a point of the real norm space of bounded linear operators from \( S \) into \( T \). Reconsider \( R = (\text{the carrier of } S) \mapsto 0_T \) as a partial function from \( S \) to \( T \). Set \( N = \) the neighbourhood of \( x_0 \). For every point \( x \) of \( S \) such that \( x \in N \) holds
   
   \[ L_0x - L_0x_0 = L(x - x_0) + R_{x-x_0} \]
   
   by [19, (7)], [20, (4)]. □

(31) Let us consider real normed spaces \( X, Y \) and a point \( x_0 \) of \( X \times Y \). Then
   
   (i) \( \text{IsoPCNrSP}(X,Y) \) is differentiable in \( x_0 \), and
   (ii) \( (\text{IsoPCNrSP}(X,Y))'(x_0) = \text{IsoPCNrSP}(X,Y) \).

(32) Let us consider real normed spaces \( X, Y \) and a point \( x_0 \) of \( \prod (X,Y) \). Then
   
   (i) \( \text{IsoPCNrSP}(X,Y) \) is differentiable in \( x_0 \), and
   (ii) \( (\text{IsoPCNrSP}(X,Y))'(x_0) = \text{IsoPCNrSP}(X,Y) \).

(33) Let us consider a partial function \( f \) from \( T \) to \( W \), a Lipschitzian linear operator \( I \) from \( S \) into \( T \), and a point \( I_0 \) of the real norm space of bounded linear operators from \( S \) into \( T \). Suppose \( I_0 = I \). Let us consider a point \( x \) of \( S \). Suppose \( f \) is differentiable in \( I(x) \). Then
   
   (i) \( f \cdot I \) is differentiable in \( x \), and
   (ii) \( (f \cdot I)'(x) = f'(I(x)) \cdot I_0 \).

The theorem is a consequence of (30).

(34) Let us consider real normed spaces \( X, Y \), a partial function \( f \) from \( \prod (X, Y) \) to \( W \), and a point \( I \) of the real norm space of bounded linear operators from \( X \times Y \) into \( \prod (X,Y) \). Suppose \( I = \text{IsoPCNrSP}(X,Y) \). Let us consider a point \( z \) of \( X \times Y \). Suppose \( f \) is differentiable in \( (\text{IsoPCNrSP}(X,Y))(z) \). Then
   
   (i) \( f \cdot \text{IsoPCNrSP}(X,Y) \) is differentiable in \( z \), and
   (ii) \( (f \cdot \text{IsoPCNrSP}(X,Y))'(z) = f'((\text{IsoPCNrSP}(X,Y))(z)) \cdot I \).

(35) Let us consider real normed spaces \( X, Y \), a partial function \( f \) from \( X \times Y \) to \( W \), and a point \( I \) of the real norm space of bounded linear operators from \( \prod (X,Y) \) into \( X \times Y \). Suppose \( I = \text{IsoPCNrSP}(X,Y) \). Let us consider a point \( z \) of \( \prod (X,Y) \). Suppose \( f \) is differentiable in \( (\text{IsoPCNrSP}(X,Y))(z) \). Then
Let us consider a partial function \( f \) from \( T \) to \( W \) and a linear operator \( I \) from \( S \) into \( T \). Suppose \( I \) is one-to-one, onto, and isometric. Let us consider a point \( x \) of \( S \). Then \( f \cdot I \) is differentiable in \( x \) if and only if \( f \) is differentiable in \( I(x) \). The theorem is a consequence of (7), (25), (30), and (33).

Let us consider real normed spaces \( X, Y \), a partial function \( f \) from \( \prod \langle X, Y \rangle \) to \( W \), and a point \( z \) of \( X \times Y \). Then \( f \cdot \operatorname{IsoCPNrSP}(X, Y) \) is differentiable in \( z \) if and only if \( f \) is differentiable in \( \operatorname{IsoCPNrSP}(X, Y)(z) \). The theorem is a consequence of (36).

Let us consider a partial function \( f \) from \( T \) to \( W \), a linear operator \( I \) from \( S \) into \( T \), and a set \( X \). Suppose

(i) \( X \subseteq \) the carrier of \( T \), and

(ii) \( I \) is one-to-one, onto, and isometric.

Then \( f \) is differentiable on \( X \) if and only if \( f \cdot I \) is differentiable on \( I^{-1}(X) \). The theorem is a consequence of (36) and (4). \( \square \)

Let us consider real normed spaces \( X, Y \), a partial function \( f \) from \( X \times Y \) to \( W \), and a point \( z \) of \( \prod \langle X, Y \rangle \). Then \( f \cdot \operatorname{IsoCPNrSP}(X, Y) \) is differentiable in \( z \) if and only if \( f \) is differentiable in \( \operatorname{IsoCPNrSP}(X, Y)(z) \). The theorem is a consequence of (36).

Let us consider real normed spaces \( X, Y \), a partial function \( f \) from \( \prod \langle X, Y \rangle \) to \( W \), and a set \( D \). Suppose \( D \subseteq \) the carrier of \( \prod \langle X, Y \rangle \). Then \( f \cdot \operatorname{IsoCPNrSP}(X, Y) \) is differentiable on \( \operatorname{IsoCPNrSP}(X, Y)^{-1}(D) \) if and only if \( f \) is differentiable on \( D \). The theorem is a consequence of (38).

Let us consider real normed spaces \( X, Y \), a partial function \( f \) from \( X \times Y \) to \( W \), and a set \( D \). Suppose \( D \subseteq \) the carrier of \( X \times Y \). Then \( f \cdot \operatorname{IsoCPNrSP}(X, Y) \) is differentiable on \( \operatorname{IsoCPNrSP}(X, Y)^{-1}(D) \) if and only if \( f \) is differentiable on \( D \). The theorem is a consequence of (38).

Let us consider real normed spaces \( X, Y \), a partial function \( f \) from \( \prod \langle X, Y \rangle \) to \( W \), and a subset \( D \) of \( \prod \langle X, Y \rangle \). Suppose \( f \) is differentiable on \( D \). Let us consider a point \( z \) of \( \prod \langle X, Y \rangle \). Suppose \( z \in \operatorname{dom} f_{\mid D} \). Then \( f_{\mid D}(z) = \left((f \cdot \operatorname{IsoCPNrSP}(X, Y))_{\mid \operatorname{IsoCPNrSP}(X, Y)^{-1}(D)} \cdot \operatorname{IsoCPNrSP}(X, Y)^{-1}\right)(z) \). The theorem is a consequence of (40) and (33). \( \square \)

Let us consider real normed spaces \( X, Y \), a partial function \( f \) from \( X \times Y \) to \( W \), and a subset \( D \) of \( X \times Y \). Suppose \( f \) is differentiable on
Let us consider a point $z$ of $X \times Y$. Suppose $z \in \dom f'_{(D)}$. Then $f'_{(D)}(z) = ((f \cdot \text{IsoPCNrSP}(X, Y))'_{(\text{IsoPCNrSP}(X, Y)^{-1}(D))})(\text{IsoPCNrSP}(X, Y))^{-1}(z) \cdot (\text{IsoPCNrSP}(X, Y))^{-1}$. The theorem is a consequence of (41) and (33).

**Proof:** Set $I = \text{IsoPCNrSP}(X, Y)$. Set $J = \text{IsoPCNrSP}(X, Y)$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point $z$ of $X \times Y$ such that $z \in \dom f'_{(D)}$ holds $f'_{(D)}(z) = (g'_{(E)})(I(z)) \cdot I^{-1}$ by [10 (31)], [5 (113)], [22 (36)]. □

Let $X, Y$ be real normed spaces and $x$ be an element of $X \times Y$. The functor $\text{reproj1} \, x$ yielding a function from $X$ into $X \times Y$ is defined by

(Def. 4) Let us consider an element $r$ of $X$. Then $it(r) = \langle r, x_2 \rangle$.

The functor $\text{reproj2} \, x$ yielding a function from $Y$ into $X \times Y$ is defined by

(Def. 5) Let us consider an element $r$ of $Y$. Then $it(r) = \langle x_1, r \rangle$.

Now we state the proposition:

(44) Let us consider real normed spaces $X, Y$ and a point $z$ of $X \times Y$. Then

(i) $\text{reproj1} \, z = \text{IsoPCNrSP}(X, Y) \cdot \text{reproj}(1(\in \dom(\langle X, Y \rangle)), (\text{IsoPCNrSP}(X, Y))(z))$, and

(ii) $\text{reproj2} \, z = \text{IsoPCNrSP}(X, Y) \cdot \text{reproj}(2(\in \dom(\langle X, Y \rangle)), (\text{IsoPCNrSP}(X, Y))(z))$.

Let $X, Y$ be real normed spaces and $z$ be a point of $X \times Y$. Observe that the functor $z_1$ yields a point of $X$. Let us note that the functor $z_2$ yields a point of $Y$. Let $X, Y, W$ be real normed spaces. Let $f$ be a partial function from $X \times Y$ to $W$. We say that $f$ is partial differentiable in'1 $z$ if and only if

(Def. 6) $f \cdot \text{reproj1} \, z$ is differentiable in $z_1$.

We say that $f$ is partial differentiable in'2 $z$ if and only if

(Def. 7) $f \cdot \text{reproj2} \, z$ is differentiable in $z_2$.

Now we state the propositions:

(45) Let us consider real normed spaces $X, Y$ and a point $z$ of $X \times Y$. Then

(i) $z_1 = \text{the projection onto } 1(\in \dom(\langle X, Y \rangle))(\text{IsoPCNrSP}(X, Y))(z))$, and

(ii) $z_2 = \text{the projection onto } 2(\in \dom(\langle X, Y \rangle))(\text{IsoPCNrSP}(X, Y))(z))$.

(46) Let us consider real normed spaces $X, Y, W$, a point $z$ of $X \times Y$, and a partial function $f$ from $X \times Y$ to $W$. Then

(i) $f$ is partial differentiable in'1 $z$ iff $f \cdot \text{IsoPCNrSP}(X, Y)$ is partially differentiable in $(\text{IsoPCNrSP}(X, Y))(z)$ w.r.t. 1, and

(ii) $f$ is partial differentiable in'2 $z$ iff $f \cdot \text{IsoPCNrSP}(X, Y)$ is partially differentiable in $(\text{IsoPCNrSP}(X, Y))(z)$ w.r.t. 2.

The theorem is a consequence of (44) and (45).

Let $X, Y, W$ be real normed spaces, $z$ be a point of $X \times Y$, and $f$ be a partial function from $X \times Y$ to $W$. The functor $\text{partdiff1}(f, z)$ yielding a point of the
real norm space of bounded linear operators from $X$ into $W$ is defined by the term

**(Def. 8)** $(f \cdot \text{reproj}_1 z)'(z_1)$.

The functor partdiff$^2(f, z)$ yielding a point of the real norm space of bounded linear operators from $Y$ into $W$ is defined by the term

**(Def. 9)** $(f \cdot \text{reproj}_2 z)'(z_2)$.

Now we state the propositions:

(47) Let us consider real normed spaces $X, Y, W$, a point $z$ of $X \times Y$, and a partial function $f$ from $X \times Y$ to $W$. Then

(i) $\text{partdiff}^1(f, z) = \text{partdiff}(f \cdot \text{IsoPCNrSP}(X, Y), (\text{IsoPCNrSP}(X, Y))(z), 1)$, and

(ii) $\text{partdiff}^2(f, z) = \text{partdiff}(f \cdot \text{IsoPCNrSP}(X, Y), (\text{IsoPCNrSP}(X, Y))(z), 2)$.

The theorem is a consequence of (44) and (45).

(48) Let us consider real normed spaces $X, Y, W$, a function $I$ from $X$ into $Y$, and partial functions $f_1, f_2$ from $Y$ to $W$. Then

(i) $(f_1 + f_2) \cdot I = f_1 \cdot I + f_2 \cdot I$, and

(ii) $(f_1 - f_2) \cdot I = f_1 \cdot I - f_2 \cdot I$. 

**Proof:** Set $D_1 = \text{the carrier of } X$. For every element $s$ of $D_1$, $s \in \text{dom}((f_1 + f_2) \cdot I)$ iff $s \in \text{dom}(f_1 \cdot I + f_2 \cdot I)$ by [4] (11). For every element $z$ of $D_1$ such that $z \in \text{dom}((f_1 + f_2) \cdot I)$ holds $((f_1 + f_2) \cdot I)(z) = (f_1 \cdot I + f_2 \cdot I)(z)$ by [4] (11), (12)]. For every element $s$ of $D_1$, $s \in \text{dom}((f_1 - f_2) \cdot I)$ iff $s \in \text{dom}(f_1 \cdot I - f_2 \cdot I)$ by [4] (11)]. For every element $z$ of $D_1$ such that $z \in \text{dom}((f_1 - f_2) \cdot I)$ holds $((f_1 - f_2) \cdot I)(z) = (f_1 \cdot I - f_2 \cdot I)(z)$ by [4] (11), (12)]. □

(49) Let us consider real normed spaces $X, Y, W$, a function $I$ from $X$ into $Y$, a partial function $f$ from $Y$ to $W$, and a real number $r$. Then $r \cdot (f \cdot I) = (r \cdot f) \cdot I$. **Proof:** Set $D_1 = \text{the carrier of } X$. For every element $s$ of $D_1$, $s \in \text{dom}(r \cdot (f \cdot I))$ iff $s \in \text{dom}(f \cdot I)$ by [4] (11)]. For every element $s$ of $D_1$, $s \in \text{dom}(r \cdot (f \cdot I))$ iff $I(s) \in \text{dom}(r \cdot f)$ by [4] (11)]. For every element $z$ of $D_1$ such that $z \in \text{dom}(r \cdot (f \cdot I))$ holds $(r \cdot (f \cdot I))(z) = (r \cdot f)(z)$ by [4] (12)]. □

Let us consider real normed spaces $X, Y, W$, a point $z$ of $X \times Y$, and partial functions $f_1, f_2$ from $X \times Y$ to $W$. Now we state the propositions:

(50) Suppose $f_1$ is partial differentiable in'1 $z$ and $f_2$ is partial differentiable in'1 $z$. Then

(i) $f_1 + f_2$ is partial differentiable in'1 $z$, and

(ii) $\text{partdiff}^1((f_1 + f_2), z) = \text{partdiff}^1(f_1, z) + \text{partdiff}^1(f_2, z)$, and

(iii) $f_1 - f_2$ is partial differentiable in'1 $z$, and
(iv) \( \text{partdiff}^1((f_1 - f_2), z) = \text{partdiff}^1(f_1, z) - \text{partdiff}^1(f_2, z) \).

(51) Suppose \( f_1 \) is partial differentiable in \('2 z\) and \( f_2 \) is partial differentiable in \('2 z\). Then

(i) \( f_1 + f_2 \) is partial differentiable in \('2 z\), and

(ii) \( \text{partdiff}^2((f_1 + f_2), z) = \text{partdiff}^2(f_1, z) + \text{partdiff}^2(f_2, z) \), and

(iii) \( f_1 - f_2 \) is partial differentiable in \('2 z\), and

(iv) \( \text{partdiff}^2((f_1 - f_2), z) = \text{partdiff}^2(f_1, z) - \text{partdiff}^2(f_2, z) \).

Let us consider real normed spaces \( X, Y, W \), a point \( z \) of \( X \times Y \), a real number \( r \), and a partial function \( f \) from \( X \times Y \) to \( W \). Now we state the propositions:

(52) If \( f \) is partial differentiable in \('1 z\), then \( r \cdot f \) is partial differentiable in \('1 z\) and \( \text{partdiff}^1((r \cdot f), z) = r \cdot \text{partdiff}^1(f, z) \).

(53) If \( f \) is partial differentiable in \('2 z\), then \( r \cdot f \) is partial differentiable in \('2 z\) and \( \text{partdiff}^2((r \cdot f), z) = r \cdot \text{partdiff}^2(f, z) \).

Let \( X, Y, W \) be real normed spaces, \( Z \) be a set, and \( f \) be a partial function from \( X \times Y \) to \( W \). We say that \( f \) is partial differentiable on \('1 Z\) if and only if

(Def. 10) (i) \( Z \subseteq \text{dom} f \), and

(ii) for every point \( z \) of \( X \times Y \) such that \( z \in Z \) holds \( f|Z \) is partially differentiable in \('1 z\).

We say that \( f \) is partial differentiable on \('2 Z\) if and only if

(Def. 11) (i) \( Z \subseteq \text{dom} f \), and

(ii) for every point \( z \) of \( X \times Y \) such that \( z \in Z \) holds \( f|Z \) is partially differentiable in \('2 z\).

Now we state the proposition:

(54) Let us consider real normed spaces \( X, Y, W \), a subset \( Z \) of \( X \times Y \), and a partial function \( f \) from \( X \times Y \) to \( W \). Then

(i) \( f \) is partial differentiable on \('1 Z\) iff \( f \cdot \text{IsoPCNrSP}(X, Y) \) is partially differentiable on \((\text{IsoPCNrSP}(X, Y))^{-1}(Z)\) w.r.t. 1, and

(ii) \( f \) is partial differentiable on \('2 Z\) iff \( f \cdot \text{IsoPCNrSP}(X, Y) \) is partially differentiable on \((\text{IsoPCNrSP}(X, Y))^{-1}(Z)\) w.r.t. 2.

The theorem is a consequence of (46) and (4). PROOF: Set \( I = \text{IsoPCNrSP}(X, Y) \).

Set \( g = f \cdot I \). Set \( E = I^{-1}(Z) \). \( f \) is partial differentiable on \('1 Z\) iff \( g \) is partially differentiable on \( E \) w.r.t. 1 by [5] (113), [4] (34), [5] (38)]. \( f \) is partial differentiable on \('2 Z\) iff \( g \) is partially differentiable on \( E \) w.r.t. 2 by [5] (113), [4] (34), [5] (38)]. □

Let \( X, Y, W \) be real normed spaces, \( Z \) be a set, and \( f \) be a partial function from \( X \times Y \) to \( W \). Assume \( f \) is partial differentiable on \('1 Z\). The functor \( f \cdot \text{partial'}1|Z \) yielding a partial function from \( X \times Y \) to the real norm space of bounded linear operators from \( X \) into \( W \) is defined by
(Def. 12) (i) \( \text{dom } it = Z \), and
   (ii) for every point \( z \) of \( X \times Y \) such that \( z \in Z \) holds \( it_z = \text{partdiff}'1(f, z) \).

Assume \( f \) is partial differentiable on'2 \( Z \). The functor \( f \text{’partial’2} | Z \) yielding a partial function from \( X \times Y \) to the real norm space of bounded linear operators from \( Y \) into \( W \) is defined by

(Def. 13) (i) \( \text{dom } it = Z \), and
   (ii) for every point \( z \) of \( X \times Y \) such that \( z \in Z \) holds \( it_z = \text{partdiff}'2(f, z) \).

Let us consider real normed spaces \( X, Y, W \), a subset \( Z \) of \( X \times Y \), and a partial function \( f \) from \( X \times Y \) to \( W \). Now we state the propositions:

(55) Suppose \( f \) is partial differentiable on'1 \( Z \). Then \( f \text{’partial’1} | Z = (f \cdot \text{IsoPCNrSP}(X, Y)^1(\text{IsoPCNrSP}(X, Y))^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y) \).

(56) Suppose \( f \) is partial differentiable on'2 \( Z \). Then \( f \text{’partial’2} | Z = (f \cdot \text{IsoPCNrSP}(X, Y)^2(\text{IsoPCNrSP}(X, Y))^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y) \).

(57) Suppose \( Z \) is open. Then \( f \) is partial differentiable on'1 \( Z \) if and only if \( Z \subseteq \text{dom } f \) and for every point \( x \) of \( X \times Y \) such that \( x \in Z \) holds \( f \) is partial differentiable in'1 \( x \).

(58) Suppose \( Z \) is open. Then \( f \) is partial differentiable on'2 \( Z \) if and only if \( Z \subseteq \text{dom } f \) and for every point \( x \) of \( X \times Y \) such that \( x \in Z \) holds \( f \) is partial differentiable in'2 \( x \).

Let us consider real normed spaces \( X, Y, W \), a subset \( Z \) of \( X \times Y \), and partial functions \( f, g \) from \( X \times Y \) to \( W \). Now we state the propositions:

(59) Suppose \( Z \) is open and \( f \) is partial differentiable on'1 \( Z \) and \( g \) is partial differentiable on'1 \( Z \). Then
   (i) \( f + g \) is partial differentiable on'1 \( Z \), and
   (ii) \( (f + g) \text{’partial’1} | Z = (f \text{’partial’1} | Z) + (g \text{’partial’1} | Z) \).

(60) Suppose \( Z \) is open and \( f \) is partial differentiable on'1 \( Z \) and \( g \) is partial differentiable on'1 \( Z \). Then
   (i) \( f - g \) is partial differentiable on'1 \( Z \), and
   (ii) \( (f - g) \text{’partial’1} | Z = (f \text{’partial’1} | Z) - (g \text{’partial’1} | Z) \).

(61) Suppose \( Z \) is open and \( f \) is partial differentiable on'2 \( Z \) and \( g \) is partial differentiable on'2 \( Z \). Then
   (i) \( f + g \) is partial differentiable on'2 \( Z \), and
   (ii) \( (f + g) \text{’partial’2} | Z = (f \text{’partial’2} | Z) + (g \text{’partial’2} | Z) \).

(62) Suppose \( Z \) is open and \( f \) is partial differentiable on'2 \( Z \) and \( g \) is partial differentiable on'2 \( Z \). Then
   (i) \( f - g \) is partial differentiable on'2 \( Z \), and
   (ii) \( (f - g) \text{’partial’2} | Z = (f \text{’partial’2} | Z) - (g \text{’partial’2} | Z) \).
Let us consider real normed spaces $X$, $Y$, $W$, a subset $Z$ of $X \times Y$, a real number $r$, and a partial function $f$ from $X \times Y$ to $W$. Now we state the propositions:

(63) Suppose $Z$ is open and $f$ is partial differentiable on $1Z$. Then

(i) $r \cdot f$ is partial differentiable on $1Z$, and

(ii) $r \cdot f \mid_{1Z} = r \cdot (f \mid_{1Z})$.

(64) Suppose $Z$ is open and $f$ is partial differentiable on $2Z$. Then

(i) $r \cdot f$ is partial differentiable on $2Z$, and

(ii) $r \cdot f \mid_{2Z} = r \cdot (f \mid_{2Z})$.

Let us consider real normed spaces $X$, $Y$, $W$, a subset $Z$ of $X \times Y$, and a partial function $f$ from $X \times Y$ to $W$. Now we state the propositions:

(65) Suppose $f$ is differentiable on $Z$. Then $f' \mid_Z$ is continuous on $Z$ if and only if $(f \mid_{\text{IsoPCNrSP}(X,Y)})' \mid_{(\text{IsoPCNrSP}(X,Y))^{-1}(Z)}$ is continuous on $(\text{IsoPCNrSP}(X,Y))^{-1}(Z)$.

(66) Suppose $Z$ is open. Then $f$ is partial differentiable on $1Z$ and $f$ is partial differentiable on $2Z$ and $f \mid_{1Z}$ is continuous on $Z$ and $f \mid_{2Z}$ is continuous on $Z$ if and only if $f$ is differentiable on $Z$ and $f' \mid_Z$ is continuous on $Z$.

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Differential Equations on Functions from $\mathbb{R}$ into Real Banach Space

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Summary. In this article, we described the differential equations on functions from $\mathbb{R}$ into real Banach space. The descriptions were based on the article [20]. As preliminary to prove these theorems, we proved some properties of differentiable functions on real normed space. For the proof we referred to descriptions and theorems in the article [21] and the article [31]. And applying the theorems of Riemann integral introduced in the article [22], we proved the ordinary differential equations on real Banach space. We referred to the methods of proof in [?].

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The notation and terminology used in this paper have been introduced in the following articles: [29], [5], [11], [3], [6], [7], [19], [13], [33], [30], [32], [11], [15], [25], [31], [18], [24], [23], [26], [27], [20], [2], [8], [14], [16], [28], [12], [36], [37], [9], [34], [35], [17], and [10].

1. Some Properties of Differentiable Functions on Real Normed Space

From now on $Y$ denotes a real normed space.

Now we state the propositions:

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(1) Let us consider a real normed space $Y$, a function $J$ from $\langle E^1, \| \cdot \| \rangle$ into $\mathbb{R}$, a point $x_0$ of $\langle E^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle E^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $J = \text{proj}(1, 1)$, and
(ii) $x_0 \in \text{dom } f$, and
(iii) $y_0 \in \text{dom } g$, and
(iv) $x_0 = \langle y_0 \rangle$, and
(v) $f = g \cdot J$.

Then $f$ is continuous in $x_0$ if and only if $g$ is continuous in $y_0$. Proof: If $f$ is continuous in $x_0$, then $g$ is continuous in $y_0$ by [14, (2)], [6, (39)], [36, (36)]. □

(2) Let us consider a real normed space $Y$, a function $I$ from $\mathbb{R}$ into $\langle E^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle E^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle E^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and
(ii) $x_0 \in \text{dom } f$, and
(iii) $y_0 \in \text{dom } g$, and
(iv) $x_0 = \langle y_0 \rangle$, and
(v) $f \cdot I = g$.

Then $f$ is continuous in $x_0$ if and only if $g$ is continuous in $y_0$. Proof: If $f$ is continuous in $x_0$, then $g$ is continuous in $y_0$ by [14, (1)], [21, (33)], [26, (15)]. □

(3) Let us consider a function $I$ from $\mathbb{R}$ into $\langle E^1, \| \cdot \| \rangle$. Suppose $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$. Then

(i) for every rest $R$ of $\langle E^1, \| \cdot \| \rangle$, $Y$, $R \cdot I$ is a rest of $\langle E^1, \| \cdot \| \rangle$, $Y$, and
(ii) for every linear operator $L$ from $\langle E^1, \| \cdot \| \rangle$ into $Y$, $L \cdot I$ is a linear of $Y$.

Proof: For every rest $R$ of $\langle E^1, \| \cdot \| \rangle$, $Y$, $R \cdot I$ is a rest of $Y$ by [15, (23)], [5, (47)], [14, (3)]. Reconsider $L_0 = L$ as a function from $\mathcal{R}^1$ into $Y$. Reconsider $L_1 = L_0 \cdot I$ as a partial function from $\mathbb{R}$ to $Y$. Reconsider $j_0 = 1$ as an element of $\mathbb{R}$. Reconsider $r = L_1(j_0)$ as a point of $Y$. For every real number $p$, $L_{1p} = p \cdot r$ by [6, (13)], [14, (3)], [6, (12)]. □

(4) Let us consider a function $J$ from $\langle E^1, \| \cdot \| \rangle$ into $\mathbb{R}$. Suppose $J = \text{proj}(1, 1)$. Then

(i) for every rest $R$ of $Y$, $R \cdot J$ is a rest of $\langle E^1, \| \cdot \| \rangle$, $Y$, and
(ii) for every linear $L$ of $Y$, $L \cdot J$ is a Lipschitzian linear operator from $\langle E^1, \| \cdot \| \rangle$ into $Y$. 

Proof: For every rest $R$ of $Y$, $R \cdot J$ is a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $Y$ by [14] (4), [15] (6), [3] (47). Consider $r$ being a point of $Y$ such that for every real number $p$, $L_p = p \cdot r$. □

(5) Let us consider a function $I$ from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and
(ii) $x_0 \in \text{dom } f$, and
(iii) $y_0 \in \text{dom } g$, and
(iv) $x_0 = \langle y_0 \rangle$, and
(v) $f \cdot I = g$, and
(vi) $f$ is differentiable in $x_0$.

Then

(vii) $g$ is differentiable in $y_0$, and
(viii) $g'(y_0) = f'(x_0)(\langle 1 \rangle)$, and
(ix) for every element $r$ of $\mathbb{R}$, $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$.

The theorem is a consequence of (3). Proof: Consider $N_1$ being a neighbourhood of $x_0$ such that $N_1 \subseteq \text{dom } f$ and there exists a point $L$ of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $Y$ and there exists a rest $R$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $Y$ such that for every point $x$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x \in N_1$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x - x_0}$. Consider $e$ being a real number such that $0 < e$ and $\{ z \in \langle \mathcal{E}^1, \| \cdot \| \rangle : \| z - x_0 \| < e \} \subseteq N_1$. Consider $L$ being a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $Y, R$ being a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $Y$ such that for every point $x_3$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x_3 \in N_1$ holds $f_{x_3} - f_{x_0} = L(x_3 - x_0) + R_{x_3 - x_0}$. Reconsider $R_0 = R \cdot I$ as a rest of $Y$. Reconsider $L_0 = L \cdot I$ as a linear of $Y$. Set $N = \{ z \in \langle \mathcal{E}^1, \| \cdot \| \rangle : \| z - x_0 \| < e \}$. $N \subseteq \text{carrier of } \langle \mathcal{E}^1, \| \cdot \| \rangle$. Set $N_0 = \{ z \in \langle \mathcal{E}^1, \| \cdot \| \rangle : \| z - y_0 \| < e \}$. $N_0 \subseteq \text{carrier of } \langle \mathcal{E}^1, \| \cdot \| \rangle$. Set $\mathbb{R}_0 = \{ z \in \langle \mathcal{E}^1, \| \cdot \| \rangle : \| z - y_0 \| < e \}$. $\mathbb{R}_0 \subseteq \text{carrier of } \langle \mathcal{E}^1, \| \cdot \| \rangle$. For every real number $y_1$ such that $y_1 \in N_0$ holds $(f \cdot I)_{y_1} - (f \cdot I)_{y_0} = L_{y_1 - y_0} + R_{y_1 - y_0}$. □

(6) Let us consider a function $I$ from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a real number $y_0$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and
(ii) $x_0 \in \text{dom } f$, and
(iii) $y_0 \in \text{dom } g$, and
(iv) \( x_0 = \langle y_0 \rangle \), and
(v) \( f \cdot I = g \).

Then \( f \) is differentiable in \( x_0 \) if and only if \( g \) is differentiable in \( y_0 \). The theorem is a consequence of (5) and (4). **Proof:** Reconsider \( J = \text{proj}(1, 1) \) as a function from \( \langle \mathcal{E}^1, \| \cdot \| \rangle \) into \( \mathbb{R} \). Consider \( N_0 \) being a neighbourhood of \( y_0 \) such that \( N_0 \subseteq \text{dom}(f \cdot I) \) and there exists a linear \( L \) of \( Y \) and there exists a rest \( R \) of \( Y \) such that for every real number \( y \) such that \( y \in N_0 \) holds \( (f \cdot I)y - (f \cdot I)y_0 = L_{y-y_0} + R_{y-y_0} \). Consider \( e_0 \) being a real number such that \( 0 < e_0 \) and \( N_0 = \|y_0 - e_0, y_0 + e_0\| \). Reconsider \( e = e_0 \) as an element of \( \mathbb{R} \). Set \( N = \{z, \text{ where } z \text{ is a point of } \langle \mathcal{E}^1, \| \cdot \| \rangle : \|z - x_0\| < e\} \). Consider \( L \) being a linear of \( Y \), \( R \) being a rest of \( Y \) such that for every real number \( y_1 \) such that \( y_1 \in N_0 \) holds \( (f \cdot I)y_1 - (f \cdot I)y_0 = L_{y_1-y_0} + R_{y_1-y_0} \). Reconsider \( R_0 = R \cdot J \) as a rest of \( \langle \mathcal{E}^1, \| \cdot \| \rangle \), \( Y \). Reconsider \( L_0 = L \cdot J \) as a Lipschitzian linear operator from \( \langle \mathcal{E}^1, \| \cdot \| \rangle \) into \( Y \). Then \( N \subseteq \text{the carrier of } \langle \mathcal{E}^1, \| \cdot \| \rangle \). For every point \( y \) of \( \langle \mathcal{E}^1, \| \cdot \| \rangle \) such that \( y \in N \) holds \( f_y - f_{x_0} = L_0(y - x_0) + R_{0y-x_0} \) by [6] (13), [7] (35), [14] (4). □

(7) Let us consider a function \( J \) from \( \langle \mathcal{E}^1, \| \cdot \| \rangle \) into \( \mathbb{R} \), a point \( x_0 \) of \( \langle \mathcal{E}^1, \| \cdot \| \rangle \), an element \( y_0 \) of \( \mathbb{R} \), a partial function \( g \) from \( \mathbb{R} \) to \( Y \), and a partial function \( f \) from \( \langle \mathcal{E}^1, \| \cdot \| \rangle \) to \( Y \). Suppose
(i) \( J = \text{proj}(1, 1) \), and
(ii) \( x_0 \in \text{dom } f \), and
(iii) \( y_0 \in \text{dom } g \), and
(iv) \( x_0 = \langle y_0 \rangle \), and
(v) \( f = g \cdot J \).

Then \( f \) is differentiable in \( x_0 \) if and only if \( g \) is differentiable in \( y_0 \). The theorem is a consequence of (6).

(8) Let us consider a function \( I \) from \( \mathbb{R} \) into \( \langle \mathcal{E}^1, \| \cdot \| \rangle \), a point \( x_0 \) of \( \langle \mathcal{E}^1, \| \cdot \| \rangle \), an element \( y_0 \) of \( \mathbb{R} \), a partial function \( g \) from \( \mathbb{R} \) to \( Y \), and a partial function \( f \) from \( \langle \mathcal{E}^1, \| \cdot \| \rangle \) to \( Y \). Suppose
(i) \( I = \text{proj}(1, 1) \) qua function\(^{-1} \), and
(ii) \( x_0 \in \text{dom } f \), and
(iii) \( y_0 \in \text{dom } g \), and
(iv) \( x_0 = \langle y_0 \rangle \), and
(v) \( f \cdot I = g \), and
(vi) \( f \) is differentiable in \( x_0 \).

Then \( \|g'(y_0)\| = \|f'(x_0)\| \). The theorem is a consequence of (5). **Proof:** Reconsider \( d_1 = f'(x_0) \) as a Lipschitzian linear operator from \( \langle \mathcal{E}^1, \| \cdot \| \rangle \) into \( Y \). Set \( A = \text{PreNorms}(d_1) \). For every real number \( r \) such that \( r \in A \) holds \( r \leq \|g(y_0)\| \) by [14] (1), (4). □
Let us consider real numbers $a, b, z$ and points $p, q, x$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$. Now we state the propositions:

(9) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then

(i) if $z \in ]a, b[,$ then $x \in ]p, q[,$ and

(ii) if $x \in ]p, q[,$ then $a \neq b$ and if $a < b$, then $z \in ]a, b[,$ and if $a > b$, then $z \in ]b, a[.$

(10) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then

(i) if $z \in [a, b]$, then $x \in [p, q]$, and

(ii) if $x \in [p, q]$, then if $a \leq b$, then $z \in [a, b]$ and if $a > b$, then $z \in [b, a]$.

Now we state the propositions:

(11) Let us consider real numbers $a, b$, points $p, q$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, and a function $I$ from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$. Suppose

(i) $p = \langle a \rangle$, and

(ii) $q = \langle b \rangle$, and

(iii) $I = (\text{proj}(1,1) \text{ qua function})^{-1}$.

Then

(iv) if $a \leq b$, then $I \circ [a, b] = [p, q]$, and

(v) if $a < b$, then $I \circ [a, b[ = ]p, q[.$

The theorem is a consequence of (10) and (9).

(12) Let us consider a real normed space $Y$, a partial function $g$ from $\mathbb{R}$ to the carrier of $Y$, and real numbers $a, b, M$. Suppose

(i) $a \leq b$, and

(ii) $[a, b] \subseteq \text{dom} \, g$, and

(iii) for every real number $x$ such that $x \in [a, b]$ holds $g$ is continuous in $x$, and

(iv) for every real number $x$ such that $x \in ]a, b[,$ holds $g$ is differentiable in $x$, and

(v) for every real number $x$ such that $x \in ]a, b[,$ holds $\| g'(x) \| \leq M$.

Then $\| g_b - g_a \| \leq M \cdot |b - a|$. The theorem is a consequence of (11), (10), (1), (9), (7), and (8).
2. Differential Equations

In the sequel $X, Y$ denote real Banach spaces, $Z$ denotes an open subset of $\mathbb{R}$, $a, b, c, d, e, r, x_0$ denote real numbers, $y_0$ denotes a vector of $X$, and $G$ denotes a function from $X$ into $X$.

Now we state the propositions:

(13) Let us consider a real Banach space $X$, a partial function $F$ from $\mathbb{R}$ to the carrier of $X$, and a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $[a, b] \subseteq \text{dom } f$, and

(ii) $]a, b[ \subseteq \text{dom } F$, and

(iii) for every real number $x$ such that $x \in ]a, b[$ holds $F_x = \int_a^x f(x)dx$, and

(iv) $x_0 \in ]a, b[$, and

(v) $f$ is continuous in $x_0$.

Then

(iii) $F$ is differentiable in $x_0$, and

(vii) $F'(x_0) = f_{x_0}$.

(14) Let us consider a partial function $F$ from $\mathbb{R}$ to the carrier of $X$ and a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $\text{dom } f = [a, b]$, and

(ii) $\text{dom } F = [a, b]$, and

(iii) for every real number $t$ such that $t \in [a, b]$ holds $F_t = \int_a^t f(x)dx$.

Let us consider a real number $x$. If $x \in [a, b]$, then $F$ is continuous in $x$.

(15) Let us consider a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$. If $a \in \text{dom } f$, then $\int_a^a f(x)dx = 0_X$.

Let us consider a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$ and a partial function $g$ from $\mathbb{R}$ to the carrier of $X$. Now we state the propositions:

(16) Suppose $a \leq b$ and $\text{dom } f = [a, b]$ and for every real number $t$ such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t f(x)dx$. Then $g_a = y_0$. 
Suppose \( \text{dom } f = [a, b] \) and \( \text{dom } g = [a, b] \) and \( Z = ]a, b[ \) and for every real number \( t \) such that \( t \in [a, b] \) holds \( g_t = y_0 + \int_a^t f(x)dx \). Then

(i) \( g \) is continuous and differentiable on \( Z \), and
(ii) for every real number \( t \) such that \( t \in Z \) holds \( g'(t) = f_t \).

Let us consider a partial function \( f \) from \( \mathbb{R} \) to the carrier of \( X \). Now we state the propositions:

(18) Suppose \( a < b \) and \( [a, b] \subset \text{dom } f \) and for every real number \( x \) such that \( x \in [a, b] \) holds \( f \) is continuous in \( x \) and \( f \) is differentiable on \( ]a, b[ \) and for every real number \( x \) such that \( x \in ]a, b[ \) holds \( f'(x) = 0_X \). Then \( f_b = f_a \).

(19) Suppose \( [a, b] \subset \text{dom } f \) and for every real number \( x \) such that \( x \in ]a, b[ \) holds \( f \) is continuous in \( x \) and \( f \) is differentiable on \( ]a, b[ \) and for every real number \( x \) such that \( x \in ]a, b[ \) holds \( f'(x) = 0_X \). Then \( f|]a, b[ \) is constant.

Now we state the propositions:

(20) Let us consider a continuous partial function \( f \) from \( \mathbb{R} \) to the carrier of \( X \). Suppose

(i) \( [a, b] = \text{dom } f \), and
(ii) \( f|]a, b[ \) is constant.

Let us consider a real number \( x \). If \( x \in [a, b] \), then \( f_x = f_a \).

(21) Let us consider continuous partial functions \( y, G_1 \) from \( \mathbb{R} \) to the carrier of \( X \) and a partial function \( g \) from \( \mathbb{R} \) to the carrier of \( X \). Suppose

(i) \( a \leq b \), and
(ii) \( Z = ]a, b[ \), and
(iii) \( \text{dom } y = ]a, b[ \), and
(iv) \( \text{dom } g = [a, b] \), and
(v) \( \text{dom } G_1 = [a, b] \), and
(vi) \( y \) is differentiable on \( Z \), and
(vii) \( y_a = y_0 \), and
(viii) for every real number \( t \) such that \( t \in Z \) holds \( y'(t) = G_{1t} \), and
(ix) for every real number \( t \) such that \( t \in ]a, b[ \) holds \( g_t = y_0 + \int_a^t G_1(x)dx \).

Then \( y = g \). The theorem is a consequence of (17), (16), (19), and (20).

Proof: Reconsider \( h = y - g \) as a continuous partial function from \( \mathbb{R} \) to the carrier of \( X \). For every real number \( x \) such that \( x \in \text{dom } h \) holds \( h_x = 0_X \) by [34, (15)]. For every element \( x \) of \( \mathbb{R} \) such that \( x \in \text{dom } y \) holds \( y(x) = g(x) \) by [34, (21)]. □
Let $X$ be a real Banach space, $y_0$ be a vector of $X$, $G$ be a function from $X$ into $X$, and $a$, $b$ be real numbers. Assume $a \leq b$ and $G$ is continuous on $\text{dom} G$. The functor $\text{Fredholm}(G, a, b, y_0)$ yielding a function from the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$ into the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$ is defined by

(Def. 1) Let us consider a vector $x$ of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$. Then there exist continuous partial functions $f, g, G_1$ from $\mathbb{R}$ to the carrier of $X$ such that

(i) $x = f$, and
(ii) $it(x) = g$, and
(iii) $\text{dom} f = [a, b]$, and
(iv) $\text{dom} g = [a, b]$, and
(v) $G_1 = G \cdot f$, and
(vi) for every real number $t$ such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t G_1(x)dx$.

Now we state the propositions:

(22) Suppose $a \leq b$ and $0 < r$ and for every vectors $y_1, y_2$ of $X$, $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$. Let us consider vectors $u, v$ of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$ and continuous partial functions $g, h$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $g = (\text{Fredholm}(G, a, b, y_0))(u)$, and
(ii) $h = (\text{Fredholm}(G, a, b, y_0))(v)$.

Let us consider a real number $t$. Suppose $t \in [a, b]$. Then $\|g_t - h_t\| \leq (r \cdot (t - a)) \cdot \|u - v\|$. PROOF: Set $F = \text{Fredholm}(G, a, b, y_0)$. Consider $f_1, g_1, G_3$ being continuous partial functions from $\mathbb{R}$ to the carrier of $X$ such that $u = f_1$ and $F(u) = g_1$ and dom $f_1 = [a, b]$ and dom $g_1 = [a, b]$ and $G_3 = G \cdot f_1$ and for every real number $t$ such that $t \in [a, b]$ holds $g_{1t} = y_0 + \int_a^t G_3(x)dx$. Consider $f_2, g_2, G_5$ being continuous partial functions from $\mathbb{R}$ to the carrier of $X$ such that $v = f_2$ and $F(v) = g_2$ and dom $f_2 = [a, b]$ and dom $g_2 = [a, b]$ and $G_5 = G \cdot f_2$ and for every real number $t$ such that $t \in [a, b]$ holds $g_{2t} = y_0 + \int_a^t G_5(x)dx$. Set $G_4 = G_3 - G_5$.

For every real number $t$ such that $t \in [a, b]$ holds $g_{4t} = y_0 + \int_a^t G_4(x)dx$. Set $G_4 = G_3 - G_5$.

For every real number $x$ such that $x \in [a, t]$ holds $\|G_{4x}\| \leq r \cdot \|u - v\|$ by [20, (26)], [12, (12)]. □

(23) Suppose $a \leq b$ and $0 < r$ and for every vectors $y_1, y_2$ of $X$, $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$. Let us consider vectors $u, v$ of the $\mathbb{R}$-norm space of
Differential equations on functions from $\mathbb{R}$ to $X$, an element $m$ of $\mathbb{N}$, and continuous partial functions $g$, $h$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $g = (\text{Fredholm}(G, a, b, y_0))^{m+1}(u)$, and

(ii) $h = (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)$.

Let us consider a real number $t$. Suppose $t \in [a, b]$. Then $\|g_t - h_t\| \leq \frac{(r - (t - a))^{m+1}}{(m+1)!} \cdot \|u - v\|$. The theorem is a consequence of (22).

Proof: Set $F = \text{Fredholm}(G, a, b, y_0)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every continuous partial functions $g$, $h$ from $\mathbb{R}$ to the carrier of $X$ such that $g = F^{s_1+1}(u_1)$ and $h = F^{s_1+1}(v_1)$ for every real number $t$ such that $t \in [a, b]$ holds $\|g_t - h_t\| \leq \frac{(r - (t - a))^{s_1+1}}{(s_1+1)!} \cdot \|u_1 - v_1\|$. $\mathcal{P}[0]$ by [4, (70)], [18, (5), (13)]. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (71)], [6, (13)], [36, (27)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(24) Let us consider a natural number $m$. Suppose

(i) $a \leq b$, and

(ii) $0 < r$, and

(iii) for every vectors $y_1, y_2$ of $X$, $\|Gy_1 - Gy_2\| \leq r \cdot \|y_1 - y_2\|$.

Let us consider vectors $u, v$ of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$. Then $\|(\text{Fredholm}(G, a, b, y_0))^{m+1}(u) - (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)\| \leq \frac{(r - (b - a))^{m+1}}{(m+1)!} \cdot \|u - v\|$. The theorem is a consequence of (23).

(25) If $a < b$ and $G$ is Lipschitzian on the carrier of $X$, then there exists a natural number $m$ such that $(\text{Fredholm}(G, a, b, y_0))^{m+1}$ is contraction. The theorem is a consequence of (24).

(26) If $a < b$ and $G$ is Lipschitzian on the carrier of $X$, then Fredholm($G, a, b, y_0$) has unique fixpoint. The theorem is a consequence of (25).

(27) Let us consider continuous partial functions $f$, $g$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $\text{dom } f = [a, b]$, and

(ii) $\text{dom } g = [a, b]$, and

(iii) $Z = ]a, b[$, and

(iv) $a < b$, and

(v) $G$ is Lipschitzian on the carrier of $X$, and

(vi) $g = (\text{Fredholm}(G, a, b, y_0))(f)$.

Then

(vii) $g_a = y_0$, and

(viii) $g$ is differentiable on $Z$, and

(ix) for every real number $t$ such that $t \in Z$ holds $g'(t) = (G \cdot f)_t$. 

continuous functions of $[a, b]$ and $X$, an element $m$ of $\mathbb{N}$, and continuous partial functions $g$, $h$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $g = (\text{Fredholm}(G, a, b, y_0))^{m+1}(u)$, and

(ii) $h = (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)$. 

The theorem is a consequence of (17) and (16).

(28) Let us consider a continuous partial function \( y \) from \( \mathbb{R} \) to the carrier of \( X \). Suppose

(i) \( a < b \), and

(ii) \( Z = ]a, b[ \), and

(iii) \( G \) is Lipschitzian on the carrier of \( X \), and

(iv) \( \text{dom} \, y = [a, b] \), and

(v) \( y \) is differentiable on \( Z \), and

(vi) \( y_a = y_0 \), and

(vii) for every real number \( t \) such that \( t \in Z \) holds \( y'(t) = G(y_t) \).

Then \( y \) is a fixpoint of Fredholm\((G, a, b, y_0)\). The theorem is a consequence of (21). \textbf{Proof}: Consider \( f, g, G_1 \) being continuous partial functions from \( \mathbb{R} \) to the carrier of \( X \) such that \( y = f \) and \( (\text{Fredholm}(G, a, b, y_0))(y) = g \) and \( \text{dom} \, f = [a, b] \) and \( \text{dom} \, g = [a, b] \) and \( G_1 = G \cdot f \) and for every real number \( t \) such that \( t \in [a, b] \) holds \( g_t = y_0 + \int_a^t G_1(x)dx \). For every real number \( t \) such that \( t \in Z \) holds \( y'(t) = G_1(\alpha) \). \( \square \)

(29) Let us consider continuous partial functions \( y_1, y_2 \) from \( \mathbb{R} \) to the carrier of \( X \). Suppose

(i) \( a < b \), and

(ii) \( Z = ]a, b[ \), and

(iii) \( G \) is Lipschitzian on the carrier of \( X \), and

(iv) \( \text{dom} \, y_1 = [a, b] \), and

(v) \( y_1 \) is differentiable on \( Z \), and

(vi) \( y_{1a} = y_0 \), and

(vii) for every real number \( t \) such that \( t \in Z \) holds \( y'_1(t) = G(y_{1t}) \), and

(viii) \( \text{dom} \, y_2 = [a, b] \), and

(ix) \( y_2 \) is differentiable on \( Z \), and

(x) \( y_{2a} = y_0 \), and

(xi) for every real number \( t \) such that \( t \in Z \) holds \( y'_2(t) = G(y_{2t}) \).

Then \( y_1 = y_2 \). The theorem is a consequence of (26) and (28).

(30) Suppose \( a < b \) and \( Z = ]a, b[ \) and \( G \) is Lipschitzian on the carrier of \( X \). Then there exists a continuous partial function \( y \) from \( \mathbb{R} \) to the carrier of \( X \) such that

(i) \( \text{dom} \, y = [a, b] \), and
Differential equations on functions from \( \mathbb{R} \) …

(ii) \( y \) is differentiable on \( Z \), and

(iii) \( y_a = y_0 \), and

(iv) for every real number \( t \) such that \( t \in Z \) holds \( y'(t) = G(y_t) \).

The theorem is a consequence of (26) and (27).

**References**


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Submodule of free $\mathbb{Z}$-module

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Summary. In this article, we formalize a free $\mathbb{Z}$-module and its property. Specially, we formalize a vector space of rational field corresponding to a free $\mathbb{Z}$-module and prove formally that submodules of a free $\mathbb{Z}$-module are free. $\mathbb{Z}$-module is necessary for lattice problems, LLL (Lenstra, Lenstra and Lovász) base reduction algorithm and cryptographic systems with lattice [20]. Some theorems in this article are described by translating theorems in [11] into theorems of $\mathbb{Z}$-module, however their proofs are different.

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1. Vector Space of Rational Field Generated by a Free $\mathbb{Z}$-module

From now on $V$ denotes a $\mathbb{Z}$-module and $W, W_1, W_2$ denote submodules of $V$.

Let us consider a $\mathbb{Z}$-module $V$, submodules $W_1, W_2$ of $V$, and submodules $W_5, W_6$ of $W_1 + W_2$. Now we state the propositions:

(1) If $W_5 = W_1$ and $W_6 = W_2$, then $W_1 + W_2 = W_5 + W_6$.

(2) If $W_5 = W_1$ and $W_6 = W_2$, then $W_1 \cap W_2 = W_5 \cap W_6$.

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Let $V$ be a $\mathbb{Z}$-module. Note that $(\text{the carrier of } V) \times (\mathbb{Z} \setminus \{0\})$ is non empty. Assume $V$ is cancelable on multiplication. The functor $\text{EQRZM}_V$ yielding an equivalence relation of $(\text{the carrier of } V) \times (\mathbb{Z} \setminus \{0\})$ is defined by

(Def. 1) Let us consider elements $S, T$. Then $(S, T) \in \text{it}$ if and only if $S, T \in (\text{the carrier of } V) \times (\mathbb{Z} \setminus \{0\})$ and there exist elements $z_1, z_2$ of $V$ and there exist integers $i_1, i_2$ such that $S = \langle\langle z_1, i_1 \rangle\rangle$ and $T = \langle\langle z_2, i_2 \rangle\rangle$ and $i_1 \neq 0$ and $i_2 \neq 0$ and $i_2 \cdot z_1 = i_1 \cdot z_2$.

Now we state the proposition:

(3) Let us consider a $\mathbb{Z}$-module $V$, elements $z_1, z_2$ of $V$, and integers $i_1, i_2$. Suppose $V$ is cancelable on multiplication. Then $\langle\langle \langle z_1, i_1 \rangle, \langle z_2, i_2 \rangle \rangle \rangle \in \text{EQRZM}_V$ if and only if $i_1 \neq 0$ and $i_2 \neq 0$ and $i_2 \cdot z_1 = i_1 \cdot z_2$.

Let $V$ be a $\mathbb{Z}$-module. Assume $V$ is cancelable on multiplication. The functor $\text{addCoset}_V$ yielding a binary operation on $\text{Classes EQRZM}_V$ is defined by

(Def. 2) Let us consider elements $A, B$. Suppose $A, B \in \text{Classes EQRZM}_V$. Let us consider elements $z_1, z_2$ of $V$ and integers $i_1, i_2$. Suppose

(i) $i_1 \neq 0$, and
(ii) $i_2 \neq 0$, and
(iii) $A = \langle\langle z_1, i_1 \rangle\rangle_{\text{EQRZM}_V}$, and
(iv) $B = \langle\langle z_2, i_2 \rangle\rangle_{\text{EQRZM}_V}$.

Then $\text{it}(A, B) = \langle\langle i_2 \cdot z_1 + i_1 \cdot z_2, i_1 \cdot i_2 \rangle\rangle_{\text{EQRZM}_V}$.

Assume $V$ is cancelable on multiplication. The functor $\text{zeroCoset}_V$ yielding an element of $\text{Classes EQRZM}_V$ is defined by

(Def. 3) Let us consider an integer $i$. Suppose $i \neq 0$. Then $it = \langle\langle 0_V, i \rangle\rangle_{\text{EQRZM}_V}$.

Assume $V$ is cancelable on multiplication. The functor $\text{lmultCoset}_V$ yielding a function from $(\text{the carrier of } \text{FRat}) \times \text{Classes EQRZM}_V$ into $\text{Classes EQRZM}_V$ is defined by

(Def. 4) Let us consider an element $q$ and an element $A$. Suppose

(i) $q \in \mathbb{Q}$, and
(ii) $A \in \text{Classes EQRZM}_V$.

Let us consider integers $m, n, i$ and an element $z$ of $V$. Suppose

(iii) $n \neq 0$, and
(iv) $q = \frac{m}{n}$, and
(v) $i \neq 0$, and
(vi) $A = \langle\langle z, i \rangle\rangle_{\text{EQRZM}_V}$.

Then $\text{it}(q, A) = \langle\langle m \cdot z, n \cdot i \rangle\rangle_{\text{EQRZM}_V}$.

Now we state the propositions:
(4) Let us consider a \( \mathbb{Z} \)-module \( V \), an element \( z \) of \( V \), and integers \( i, n \). Suppose
(i) \( i \neq 0 \), and
(ii) \( n \neq 0 \), and
(iii) \( V \) is cancelable on multiplication.
Then \( [(z, i)]_{EQRZM_V} = [(n \cdot z, n \cdot i)]_{EQRZM_V} \). The theorem is a consequence of (3).

(5) Let us consider a \( \mathbb{Z} \)-module \( V \) and an element \( v \) of \( V \). Suppose \( V \) is cancelable on multiplication. Then there exists an integer \( i \) and there exists an element \( z \) of \( V \) such that \( i \neq 0 \) and \( v = [(z, i)]_{EQRZM_V} \).

Let \( V \) be a \( \mathbb{Z} \)-module. Assume \( V \) is cancelable on multiplication. The functor \( \text{ZMQVectSp}_V \) yielding a vector space over \( \text{FRat} \) is defined by the term
(Def. 5) \( \langle \text{Classes EQRZM}_V, \text{addCoset}_V, \text{zeroCoset}_V, \text{lmultCoset}_V \rangle \).

Now we state the propositions:

(6) Let us consider a \( \mathbb{Z} \)-module \( V \). Suppose \( V \) is cancelable on multiplication. Let us consider a finite sequence \( s \) of elements of \( V \) and a finite sequence \( t \) of elements of \( \text{ZMQVectSp}_V \). Suppose
(i) \( \text{len} \ s = \text{len} \ t \), and
(ii) for every element \( i \) of \( \mathbb{N} \) such that \( i \in \text{dom} \ s \) there exists a vector \( s_1 \) of \( V \) such that \( s_1 = s(i) \) and \( t(i) = (\text{MorphsZQ}_V)(s_1) \).
Then \( \sum t = (\text{MorphsZQ}_V)(\sum s) \).
Proof: Define \( \mathcal{P}[\text{natural number}] \equiv \) for every finite sequence \( s \) of elements of \( V \) for every finite sequence \( t \) of elements of \( \text{ZMQVectSp}_V \) such that \( \text{len} \ s = \mathcal{P}[0] \) and \( \text{len} \ s = \text{len} \ t \) and for every element \( i \) of \( \mathbb{N} \) such that \( i \in \text{dom} \ s \) there exists a vector \( s_1 \) of \( V \) such that \( s_1 = s(i) \) and \( t(i) = (\text{MorphsZQ}_V)(s_1) \) holds \( \sum t = (\text{MorphsZQ}_V)(\sum s) \).
\( \mathcal{P}[0] \) by \([27] \ (43)\). For every natural number \( k \) such that \( \mathcal{P}[k] \) holds \( \mathcal{P}[k+1] \) by \([3] \ (59)\), \([3] \ (11)\), \([5] \ (4)\). For every natural number \( k \), \( \mathcal{P}[k] \) from \([3] \ Sch. 2\). □

(7) Let us consider a \( \mathbb{Z} \)-module \( V \), a subset \( I \) of \( V \), a subset \( I_6 \) of \( \text{ZMQVectSp}_V \), a \( z \) linear combination \( l \) of \( I \), and a linear combination \( l_5 \) of \( I_6 \). Suppose
(i) $V$ is cancelable on multiplication, and
(ii) $I_6 = (\text{MorphsZQ} V)^o I$, and
(iii) $l = l_5 \cdot \text{MorphsZQ} V$.

Then $\sum l_5 = (\text{MorphsZQ} V)(\sum l)$. The theorem is a consequence of (6).

(8) Let us consider a $\mathbb{Z}$-module $V$, a subset $I_6$ of $\text{ZMQVectSp} V$, and a linear combination $l_5$ of $I_6$. Then there exists an integer $m$ and there exists an element $a$ of $\text{FRat}$ such that $m \neq 0$ and $m = a$ and $\text{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. PROOF: Define $\mathcal{P}[$natural number$] \equiv$ for every linear combination $l_5$ of $I_6$ such that the support of $l_5 = \emptyset$ there exists an integer $m$ and there exists an element $a$ of $\text{FRat}$ such that $m \neq 0$ and $m = a$ and $\text{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. $\mathcal{P}[0]$ by [28, (28)], [8, (113)], [28, (3)]. For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [2, (44)], [10, (31)], [2, (42)]. For every natural number $n$, $\mathcal{P}[n]$ from [3, Sch. 2]. $\square$

(9) Let us consider a $\mathbb{Z}$-module $V$, a subset $I$ of $V$, a subset $I_6$ of $\text{ZMQVectSp} V$, and a linear combination $l_5$ of $I_6$. Suppose
(i) $V$ is cancelable on multiplication, and
(ii) $I_6 = (\text{MorphsZQ} V)^o I$.

Then there exists an integer $m$ and there exists an element $a$ of $\text{FRat}$ and there exists a $z$ linear combination $l$ of $I$ such that

$$l = (a \cdot l_5) \cdot \text{MorphsZQ} V \quad \text{and} \quad (\text{MorphsZQ} V)^{-1}(\text{the support of } l_5) = \text{the support of } l.$$ The theorem is a consequence of (8). PROOF: Consider $m$ being an integer, $a$ being an element of $\text{FRat}$ such that $m \neq 0$ and $m = a$ and $\text{rng}(a \cdot l_5) \subseteq \mathbb{Z}$. Reconsider $l = (a \cdot l_5) \cdot \text{MorphsZQ} V$ as an element of $\mathbb{Z}$ the carrier of $V$. Set $T = \{v, \text{ where } v \text{ is an element of } V : l(v) \neq 0\}$. Set $F = \text{MorphsZQ} V$. $T \subseteq F^{-1}(\text{the support of } l_5)$ by [71, (13)], [8, (38)]. $F^{-1}(\text{the support of } l_5) \subseteq T$ by [8, (38)], [71, (13)]. $\square$

(10) Let us consider a $\mathbb{Z}$-module $V$, a subset $I$ of $V$, a subset $I_6$ of $\text{ZMQVectSp} V$, a linear combination $l_5$ of $I_6$, an integer $m$, an element $a$ of $\text{FRat}$, and a $z$ linear combination $l$ of $I$. Suppose

(i) $V$ is cancelable on multiplication, and
(ii) $I_6 = (\text{MorphsZQ} V)^o I$, and
(iii) $m \neq 0$, and
(iv) $m = a$, and
(v) $l = (a \cdot l_5) \cdot \text{MorphsZQ} V$.

Then $a \cdot \sum l_5 = (\text{MorphsZQ} V)(\sum l)$. The theorem is a consequence of (7).

(11) Let us consider a $\mathbb{Z}$-module $V$, a subset $I$ of $V$, and a subset $I_6$ of $\text{ZMQVectSp} V$. Suppose

(i) $V$ is cancelable on multiplication, and
(ii) $I_6 = (\text{MorphsZQ} V)^o I$, and

(iii) $I$ is linearly independent.

Then $I_6$ is linearly independent. The theorem is a consequence of (9) and (10).

(12) Let us consider a $\mathbb{Z}$-module $V$, a subset $I$ of $V$, a $\mathbb{Z}$ linear combination $l$ of $I$, and a subset $I_6$ of $\text{ZMQVectSp} V$. Suppose

(i) $V$ is cancelable on multiplication, and

(ii) $I_6 = (\text{MorphsZQ} V)^o I$.

Then there exists a linear combination $l_5$ of $I_6$ such that

(iii) $l = l_5 \cdot \text{MorphsZQ} V$, and

(iv) the support of $l_5 = (\text{MorphsZQ} V)^o$ the support of $l$.

**Proof:** Reconsider $I_0$ = the support of $l$ as a finite subset of $V$. Reconsider $I_7 = (\text{MorphsZQ} V)^o I_0$ as a finite subset of $\text{ZMQVectSp} V$. Define $P[\text{element, element}] \equiv \$ \in I_7$ and there exists an element $v$ of $V$ such that $v \in I_0$ and $\$ = (\text{MorphsZQ} V)(v)$ and $\$ = l(v)$ or $\$ \in I_7$ and $\$ = 0_{\text{FRat}}$.

For every element $x$ such that $x \in$ the carrier of $\text{ZMQVectSp} V$ there exists an element $y$ such that $y \in \mathbb{Q}$ and $P[x, y]$ by [8, (64)], [25, (14)].

Consider $l_5$ being a function from the carrier of $\text{ZMQVectSp} V$ into $\mathbb{Q}$ such that for every element $x$ such that $x \in$ the carrier of $\text{ZMQVectSp} V$ holds $P[x, l_5(x)]$ from [3 Sch. 1]. The support of $l_5 \subseteq I_7$. For every element $x$ such that $x \in \text{dom} l$ holds $l(x) = (l_5 \cdot \text{MorphsZQ} V)(x)$ by [8] (35), (19)], [7] (12]], $I_7 \subseteq$ the support of $l_5$ by [8] (64)], [7] (12)], [14] (8)]. □

(13) Let us consider a free $\mathbb{Z}$-module $V$, a subset $I$ of $V$, a subset $I_6$ of $\text{ZMQVectSp} V$, a $\mathbb{Z}$ linear combination $l$ of $I$, and an integer $i$. Suppose

(i) $i \neq 0$, and

(ii) $I_6 = (\text{MorphsZQ} V)^o I$.

Then $\left\langle \sum l, i \right\rangle_{\text{EQRZM} V} \in \text{Lin}(I_6)$. The theorem is a consequence of (12) and (7).

Let us consider a free $\mathbb{Z}$-module $V$, a subset $I$ of $V$, and a subset $I_6$ of $\text{ZMQVectSp} V$. Now we state the propositions:

(14) If $I_6 = (\text{MorphsZQ} V)^o I$, then $\overline{I} = \overline{I_6}$.

(15) If $I_6 = (\text{MorphsZQ} V)^o I$ and $I$ is a basis of $V$, then $I_6$ is a basis of $\text{ZMQVectSp} V$.

Let $V$ be a finite-rank free $\mathbb{Z}$-module. Note that $\text{ZMQVectSp} V$ is finite dimensional.

Now we state the propositions:

(16) Let us consider a finite-rank free $\mathbb{Z}$-module $V$. Then $\text{rank } V = \dim(\text{ZMQVectSp} V)$.

The theorem is a consequence of (15) and (14).
(17) Let us consider a free $\mathbb{Z}$-module $V$ and finite subsets $I, A$ of $V$. Suppose
   (i) $I$ is a basis of $V$, and
   (ii) $\overline{I} + 1 = \overline{A}$.

   Then $A$ is linearly dependent. The theorem is a consequence of (15), (11), and (14).

(18) Let us consider a free $\mathbb{Z}$-module $V$ and subsets $A, B$ of $V$. If $A$ is linearly
   dependent and $A \subseteq B$, then $B$ is linearly dependent.

(19) Let us consider a free $\mathbb{Z}$-module $V$ and subsets $D, A$ of $V$. Suppose
   (i) $D$ is basis of $V$ and finite, and
   (ii) $\overline{D} \subset \overline{A}$.

   Then $A$ is linearly dependent. The theorem is a consequence of (17) and (18).

(20) Let us consider a free $\mathbb{Z}$-module $V$ and subsets $I, A$ of $V$. Suppose
   (i) $I$ is basis of $V$ and finite, and
   (ii) $A$ is linearly independent.

   Then $\overline{A} \subseteq \overline{I}$.

2. Submodule of Free $\mathbb{Z}$-module

Now we state the proposition:

(21) Let us consider a $\mathbb{Z}$-module $V$. If $\Omega_V$ is free, then $V$ is free.

Let us consider a $\mathbb{Z}$-module $V$, submodules $W_1, W_2$ of $V$, and strict submo-
   dules $W_3, W_4$ of $V$. Now we state the propositions:

(22) If $W_3 = \Omega_{W_1}$ and $W_4 = \Omega_{W_2}$, then $W_3 + W_4 = W_1 + W_2$.

(23) If $W_3 = \Omega_{W_1}$ and $W_4 = \Omega_{W_2}$, then $W_3 \cap W_4 = W_1 \cap W_2$.

Now we state the propositions:

(24) Let us consider a $\mathbb{Z}$-module $V$ and a strict submodule $W$ of $V$. Suppose
   $W \neq 0_V$. Then there exists a vector $v$ of $V$ such that
   (i) $v \in W$, and
   (ii) $v \neq 0_V$.

(25) Let us consider a subset $A$ of $V$ and $z$ linear combinations $l_1, l_2$ of $A$.
   Suppose (the support of $l_1) \cap (the support of $l_2) = \emptyset$. Then the support
   of $l_1 + l_2 = (the support of $l_1) \cup (the support of $l_2)$. Proof: (The support
   of $l_1) \cup (the support of $l_2) \subseteq the support of $l_1 + l_2 by [14, (8)]. □
Let us consider subsets $A_1$, $A_2$ of $V$ and a z linear combination $l$ of $A_1 \cup A_2$. Suppose $A_1 \cap A_2 = \emptyset$. Then there exists a z linear combination $l_1$ of $A_1$ and there exists a z linear combination $l_2$ of $A_2$ such that $l = l_1 + l_2$.

**Proof:** Define $P[\text{element, element}] \equiv$ if $S_1$ is a vector of $V$, then $S_1 \in A_1$ and $S_2 = l(S_1)$ or $S_1 \notin A_1$ and $S_2 = 0$. For every element $x$ such that $x \in$ the carrier of $V$ there exists an element $y$ such that $y \in \mathbb{Z}$ and $P[x, y]$. There exists a function $l_1$ from the carrier of $V$ into $\mathbb{Z}$ such that for every element $x$ such that $x \in$ the carrier of $V$ holds $P[x, l_1(x)]$ from [8, Sch. 1]. Consider $l_1$ being a function from the carrier of $V$ into $\mathbb{Z}$ such that for every element $x$ such that $x \in$ the carrier of $V$ holds $P[x, l_1(x)]$. For every element $x$ such that $x \in$ the support of $l_1$ holds $\mathbb{Z} \equiv l_1 \in A_1$ by [14, (8)]. Define $Q[\text{element, element}] \equiv$ if $S_1$ is a vector of $V$, then $S_1 \in A_2$ and $S_2 = l(S_1)$ or $S_1 \notin A_2$ and $S_2 = 0$. For every element $x$ such that $x \in$ the carrier of $V$ there exists an element $y$ such that $y \in \mathbb{Z}$ and $Q[x, y]$. There exists a function $l_2$ from the carrier of $V$ into $\mathbb{Z}$ such that for every element $x$ such that $x \in$ the carrier of $V$ holds $Q[x, l_2(x)]$ from [8, Sch. 1]. Consider $l_2$ being a function from the carrier of $V$ into $\mathbb{Z}$ such that for every element $x$ such that $x \in$ the carrier of $V$ holds $Q[x, l_2(x)]$. For every element $x$ such that $x \in$ the support of $l_2$ holds $l(x) \in A_2$ by [13, (8)]. For every vector $v$ of $V$, $l(v) = (l_1 + l_2)(v)$. □

Let us consider a $\mathbb{Z}$-module $V$, free submodules $W_1$, $W_2$ of $V$, a basis $I_1$ of $W_1$, and a basis $I_2$ of $W_2$. If $V$ is the direct sum of $W_1$ and $W_2$, then $I_1 \cap I_2 = \emptyset$.

Let us consider a $\mathbb{Z}$-module $V$, free submodules $W_1$, $W_2$ of $V$, a basis $I_1$ of $W_1$, a basis $I_2$ of $W_2$, and a subset $I$ of $V$. Now we state the propositions:

(28) If $V$ is the direct sum of $W_1$ and $W_2$ and $I = I_1 \cup I_2$, then $\text{Lin}(I) =$ the $\mathbb{Z}$-module structure of $V$.

(29) If $V$ is the direct sum of $W_1$ and $W_2$ and $I = I_1 \cup I_2$, then $I$ is linearly independent.

Let us consider a $\mathbb{Z}$-module $V$ and free submodules $W_1$, $W_2$ of $V$. Now we state the propositions:

(30) If $V$ is the direct sum of $W_1$ and $W_2$, then $V$ is free.

(31) If $W_1 \cap W_2 = 0_V$, then $W_1 + W_2$ is free.

Let us consider a free $\mathbb{Z}$-module $V$, a basis $I$ of $V$, and a vector $v$ of $V$. Now we state the propositions:

(32) If $v \in I$, then $\text{Lin}(I \setminus \{v\})$ is free and $\text{Lin}(\{v\})$ is free.

(33) If $v \in I$, then $V$ is the direct sum of $\text{Lin}(I \setminus \{v\})$ and $\text{Lin}(\{v\})$.

Let $V$ be a finite-rank free $\mathbb{Z}$-module. One can verify that every submodule of $V$ is free.

Now we state the propositions:
(34) Let us consider a \( \mathbb{Z} \)-module \( V \), a submodule \( W \) of \( V \), and free submodules \( W_1, W_2 \) of \( V \). Suppose

(i) \( W_1 \cap W_2 = 0_V \), and

(ii) the \( \mathbb{Z} \)-module structure of \( W = W_1 + W_2 \).

Then \( W \) is free. The theorem is a consequence of (31).

(35) Let us consider a prime number \( p \) and a free \( \mathbb{Z} \)-module \( V \). If \( \mathbb{Z} \text{MtoMQV}(V, p) \) is finite dimensional, then \( V \) is finite-rank.

(36) Let us consider a prime number \( p \), a \( \mathbb{Z} \)-module \( V \), an element \( s \) of \( V \), an integer \( a \), and an element \( b \) of \( \text{GF}(p) \). Suppose \( b = a \mod p \). Then \( b \cdot \mathbb{Z} \text{MtoMQV}(V, p, s) = \mathbb{Z} \text{MtoMQV}(V, p, a \cdot s) \).

(37) Let us consider a prime number \( p \), a free \( \mathbb{Z} \)-module \( V \), a subset \( I \) of \( V \), a subset \( I_6 \) of \( \mathbb{Z} \text{MtoMQV}(V, p) \), and a \( \mathbb{Z} \)-linear combination \( l \) of \( I \). Suppose \( I_6 = \{ \mathbb{Z} \text{MtoMQV}(V, p, u), \text{ where } u \text{ is a vector of } V : u \in I \} \). Then \( \mathbb{Z} \text{MtoMQV}(V, p, \sum l) \in \text{Lin}(I_6) \).

(38) Let us consider a prime number \( p \), a free \( \mathbb{Z} \)-module \( V \), a subset \( I \) of \( V \), and a subset \( I_6 \) of \( \mathbb{Z} \text{MtoMQV}(V, p) \). Suppose

(i) \( \text{Lin}(I) \) is the \( \mathbb{Z} \)-module structure of \( V \), and

(ii) \( I_6 = \{ \mathbb{Z} \text{MtoMQV}(V, p, u), \text{ where } u \text{ is a vector of } V : u \in I \} \).

Then \( \text{Lin}(I_6) \) is the vector space structure of \( \mathbb{Z} \text{MtoMQV}(V, p) \). The theorem is a consequence of (37). PROOF: For every element \( v_3 \) of \( \mathbb{Z} \text{MtoMQV}(V, p) \), \( v_3 \in \text{Lin}(I_6) \) by [15 (22)], [14 (64)]. □

(39) Let us consider a finitely-generated free \( \mathbb{Z} \)-module \( V \). Then there exists a finite subset \( A \) of \( V \) such that \( A \) is a basis of \( V \). The theorem is a consequence of (38). PROOF: Set \( p = \) the prime number. Consider \( B \) being a finite subset of \( V \) such that \( \text{Lin}(B) \) is the \( \mathbb{Z} \)-module structure of \( V \). Set \( B_1 = \{ \mathbb{Z} \text{MtoMQV}(V, p, u), \text{ where } u \text{ is a vector of } V : u \in B \} \). Define \( \mathcal{F}(\text{element of } V) = \mathbb{Z} \text{MtoMQV}(V, p, x) \). Consider \( f \) being a function from the carrier of \( V \) into \( \mathbb{Z} \text{MtoMQV}(V, p) \) such that for every element \( x \) of \( V \), \( f(x) = \mathcal{F}(x) \) from [8 Sch. 4]. For every element \( y \) such that \( y \in B_1 \) there exists an element \( x \) such that \( x \in \text{dom}(f|B) \) and \( y = (f|B)(x) \) by [31 (62)], [7 (47)]. Consider \( I_6 \) being a basis of \( \mathbb{Z} \text{MtoMQV}(V, p) \) such that \( I_6 \subseteq B_1 \). □

One can verify that every finitely-generated free \( \mathbb{Z} \)-module is finite-rank and every finite-rank free \( \mathbb{Z} \)-module is finitely-generated.

Now we state the proposition:

(40) Let us consider a finite-rank free \( \mathbb{Z} \)-module \( V \) and a subset \( A \) of \( V \). If \( A \) is linearly independent, then \( A \) is finite. The theorem is a consequence of (19).
Let $V$ be a $\mathbb{Z}$-module and $W_1, W_2$ be finite-rank free submodules of $V$. One can check that $W_1 \cap W_2$ is free. Note that $W_1 \cap W_2$ is finite-rank.

Let $V$ be a finite-rank free $\mathbb{Z}$-module. Note that every submodule of $V$ is finite-rank.

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