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PAPERS ON LOGIC AND RATIONALITY

Festschrift in Honour of Andrzej Grzegorczyk

Edited by
Kazimierz Trzęsicki, Stanisław Krajewski,
Jan Woleński

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Refereed by Stanisław Krajewski, Marek Kuś, Wiktor Marek, Mieczysław Omyła, Kazimierz Trzęsicki, and Jan Woleński

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Edited by Halina Święczkowska
University of Bialystok, Faculty of Law, Section of Semiotics

in collaboration with Kazimierz Trzęsicki
University of Bialystok
Chair of Logic, Informatics and Philosophy of Science – logika@uwb.edu.pl

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WYDAWNICTWO UNIWERSYTETU W BIAŁYMSTOKU
15-097 Białystok, ul. Marii Skłodowskiej-Curie 14, tel. 857457120
http://wydawnictwo.uwb.edu.pl, e-mail: ac-dw@uwb.edu.pl

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INTRODUCTION

This volume is dedicated to Professor Andrzej Grzegorczyk on the occasion of his 90th birthday. It is rare to be able to pay respects to someone who has reached this impressive age and is still actively involved in creative research. To wish "one hundred years," as Polish tradition has it, seems much too modest. We are happy that we can present to him this volume, and we hope that Professor Grzegorczyk and ourselves will be able to participate in a future project in which scholars who have been influenced by him can present their research and reflections related to his fields of activities.

Only a very brief outline of Grzegorczyk's life and achievements is given below. More detailed accounts can be found in the following two papers: S. Krajewski and J. Woleński, "Andrzej Grzegorczyk: Logic and Philosophy", *Fundamenta Informaticae* 81, 1–3 (2007), pp. 1–10; S. Krajewski, "Andrzej Grzegorczyk" (in Polish), *Edukacja Filozoficzna* 37 (2004), pp. 185–204 (also published in *Polska filozofia powojenna III* [*Polish Postwar Philosophy III*], ed. by W. Mackiewicz, Agencja Wydawnicza Witmark, Warszawa 2005, pp. 99–118).

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Andrzej Grzegorczyk was born in Warsaw on August 22, 1922. World War II interrupted his high school education, but he continued his studies in a clandestine school system organized in occupied Poland. Then he studied in a chemical college and attended clandestine classes in philosophy (given by Władysław Tatarkiewicz) and logic (by Fr Jan Salamucha and Henryk Hiż).

Grzegorczyk took part in the 1944 Warsaw uprising, and after the war went to study at the Jagiellonian University in Cracow. He graduated in philosophy, having written his master's thesis, *The Ontology of Properties*, and he then returned to Warsaw in 1946 where he became Tatarkiewicz's

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assistant and the secretary of *Przegląd Filozoficzny* (Philosophical Review). He began research in the field of logic and foundations of mathematics, and obtained his PhD at the University of Warsaw in 1950. His dissertation, *On Topological Spaces in Topologies without Points*, was written under the supervision of Andrzej Mostowski. Then he worked at the Institute of Mathematics of the Polish Academy of Sciences, where he became a docent in 1953 (the paper *Some Classes of Recursive Functions* served as a *de facto* Habilitation dissertation), an associate professor in 1961 and a full professor in 1972. He also lectured at the University of Warsaw, and in 1974 moved to the Institute of Philosophy of the Polish Academy of Sciences where he became the head of the Ethics Group in 1982; he retired in 1990. Married to Renata Majewska, a professor at the University of Warsaw, Grzegorczyk has two children and six grandchildren.

Active in organizing scholarly activities, Grzegorczyk headed the Logical Semester at the International Mathematical Center (the Banach Center) of the Polish Academy of Sciences in 1973; he led a special project, "One Hundred Years of the Lvov-Warsaw School", in 1995–1997; he worked as an assessor on the Executive Committee of the International Union of History and Philosophy of Science, the Division of Logic, Methodology and Philosophy of Science; and from 1999 to 2003 he served as the President of the Committee of Philosophy of the Polish Academy of Sciences.

Grzegorczyk published popular books on logic and computability as well as a widely used textbook: *An Outline of Mathematical Logic*, *Fundamental Results and Notions Explained in All Details*. These all played an important role in logical education in Poland – and also outside its borders, as his popular books, the first presentations of the theory of computability for a general public, were translated into Czech and Russian.

Grzegorczyk's best known achievement, the so-called Grzegorczyk's hierarchy, was introduced in 1953. He described and investigated classes of recursive functions obtainable by superposition, restricted recursion and the operation of restricted minimum from some initial functions containing addition, multiplication and, in addition, for each class the appropriate, more complicated, primitive recursive function. The resulting subrecursive hierarchy fills the class of primitive recursive functions. Grzegorczyk also co-authored (with Mostowski and Ryll-Nardzewski) a fundamental paper about second-order arithmetic and the infinitary omega-rule.

During his career, Grzegorczyk studied computable real numbers, axiomatic geometry based on the concept of solid, and the theory of Boolean algebras. He showed how to interpret Lesniewski's ontology as Boolean algebra without zero and demonstrated the undecidability of the theory of

Boolean algebras with the operation of closure. He investigated intuitionistic logic, and a modal interpretation of Grzegorczyk's semantics for intuitionism leads to the system known in the literature as $S4.Grz$, defined as $S4$ plus the formula $\Box(\Box(A \Rightarrow \Box A) \Rightarrow A) \Rightarrow A$, called Grzegorczyk's axiom.

Grzegorczyk's recent contribution, the undecidability of the theory of concatenation, has a philosophical motivation: studying formal systems should be performed by operating on objects which are visually comprehensible. The most natural is the notion of text.

Grzegorczyk has always believed that logic is the morality of speech and thought, something that is also applicable to moral discussions. Logic conceived broadly, including the methodology of science, forms a basic component of the intellectual attitude identified by Grzegorczyk as European rationalism. It is rationalism open to the realm of values that makes it possible to acquire reliable knowledge and advocate ethics in social relations. Logic appears from this point of view as a human affair. Interestingly, Grzegorczyk opts for psychologism in logic: semantic relations are always relations for someone and are mediated by language. As a result, for example, paradoxes should not be interpreted as showing that our language is inconsistent, but rather that our concepts and theoretical systems are limited.

Grzegorczyk is a devout Roman Catholic who feels an affinity to Russian Orthodox Christianity. He has always been highly independent in his views and has expressed critical opinions about various policies of the Church. His religious reflection is focused on the moral dimension of Christianity as well as on its links with the European cultural tradition. According to him, Christianity is deeply involved in the same values as European rationalism. The history of Christianity (including its Biblical roots) can be considered the history of how a sense and understanding of the world can be deepened by contemplating the sacred and transcendent. In particular, Jesus provides a moral pattern because he demanded and demonstrated coherent individual testimony.

Grzegorczyk has applied his ethical views to the field of conflict resolution, attaching special importance to the methods of non-violence, such as those advocated by Mahatma Gandhi or Martin Luther King. He cooperated with leaders of non-violent movements. He was also one of the first figures visible in Polish public life who took seriously ecological issues. Before it was widely understood in Poland, he popularized warnings made by the Club of Rome that the resources of our planet are scarce and, therefore, the idea of permanent growth is dangerous.

Andrzej Grzegorczyk's philosophical and axiological views have not become as influential or even as known as their author expected. Still, his achievements in logic, such as the Grzegorczyk hierarchy, the geometry of solids, results about undecidability, results about second-order arithmetic, the S4Grz system and semantics for intuitionistic logic, have secured his place in the history of this field. Moreover, his results in concatenation theory and, most recently, regarding propositional calculus with the descriptive equivalence connective, provide an important addition to his signal achievements.

Kazimierz Trzęsicki, Stanisław Krajewski, Jan Woleński

Zofia Adamowicz

A GENERALIZATION OF THE CONSISTENCY PREDICATE

Abstract. We modify the usual arithmetical consistency predicate. We study the behavior of our predicate in fragments of arithmetic. The definition of our predicate depends on a formula J defining an initial segment and on a set Γ of arithmetical formulas. We formulate conditions on J and on Γ under which our predicate has the properties usually required from a consistency predicate. As a result we obtain a well behaving consistency predicate. Suitably choosing J and Γ we obtain a well behaving consistency predicate whose arithmetical complexity is unusual, namely is Σ_1.

1. Motivation

Our motivation is a deeper understanding of the independence phenomenon in arithmetic. To this end we generalize the usual consistency predicate. Since consistency is dual to provability, as well we may deal with provability. On one hand we restrict some usual consistency predicate, like the Hilbert or the Gentzen or the Herbrand consistency to some definable initial segment; on the other hand we include into the theory whose consistency is considered some set of true sentences. We study those properties of the initial segment involved which guarantee that our consistency predicate behaves regularly, including the Gödel phenomenon. We axiomatize those properties. Finally we illustrate our considerations by showing a consistency predicate which behaves regularly although it is Σ_1 definable, not as usually Π_1.

Restricting the consistency predicate to some definable initial segment was already considered in the literature for instance by P. Pudlak [1985]. He proved that as far as the Hilbert consistency predicate is concerned many interesting restrictions to a cut still satisfy the Gödel independence phenomenon; however as far as the Herbrand or Gentzen predicate is concerned for many natural cuts this is not the case. A similar approach for Herbrand consistency was studied carefully in [Adamowicz and Zdanowski, 2011]. Also S. Feferman [1960] studied consistency predicates for which the second Gödel

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independence theorem failed. We go in a different direction than P. Pudlak and S. Feferman.

The main theorem is formulated and proved in section 15. The last section is devoted to an illustration of the main theorem by a special case of it. The remaining sections are introductory.

2. Fragments of Arithmetic

Peano Arithmetic is the theory of the non negative part of a discretely ordered ring together with the induction scheme:

$$\forall x(\phi(x) \Rightarrow \phi(x+1)) \Rightarrow \forall x \phi(x),$$

where ϕ runs over all formulas of the language.

This scheme is equivalent to its parameter version:

$$\forall a\Big(\phi(a,0) \& \forall x(\phi(a,x) \Rightarrow \phi(a,x+1)) \Rightarrow \forall x \phi(a,x)\Big).$$

When the range of the formula ϕ is restricted to Σ_n formulas, we get the fragment $I\Sigma_n$.

For $n \geq 1$ this is considered as an strong fragment, for $n = 0$ this is the theory $I\Sigma_0$, which is considered as Weak Arithmetic. The distinction between strong and weak refers to the provability of the totality of the exponential function:

$$\forall x \exists y \; y = 2^x,$$

where the notation $y = 2^x$ is an abbreviation of an arithmetical formula defining the graph of the function 2^x.

Let the above sentence be denoted by *exp*. We have

$$I\Sigma_1 \vdash exp,$$

and

$$I\Sigma_0 \nvdash exp\,.$$

Note that a Σ_0 formula, which is also denoted by Δ_0, is a formula whose quantifiers are all bounded.

Thus, the theory $I\Sigma_0$ is more often denoted by $I\Delta_0$.

One may also consider the theory $I\Delta_0 + exp$, or intermediate theories $I\Delta_0 + \Omega_n$.

We define the following functions:

$$\omega_0(x) \quad = x^2,$$

$$\begin{aligned}\omega_1(x) &= 2^{(log\,x)^2},\\ \omega_2(x) &= 2^{2^{(log\,log\,x)^2}},\\ \omega_{i+1}(x) &= 2^{\omega_i(log\,x)}.\end{aligned}$$

Note that $\omega_1(x) = 2^{(log\,x)^2} = (2^{log\,x})^{log\,x} = x^{log\,x}$ and has an intermediate growth between polynomials x^n, for a fixed n, and the exponential function 2^x.

By Ω_i we mean the sentence stating the totality of the ω_i function:

$$\Omega_i: \ \forall x \exists y \ y = \omega_i(x),$$

where the notation $y = \omega_i(x)$ is an abbreviation of an arithmetical formula defining the graph of the function $\omega_i(x)$.

We have

$$I\Delta_0 + \Omega_i \nvdash exp\,.$$

Hence the theories $I\Delta_0 + \Omega_n$ are weak fragments of artithmetic, while $I\Delta_0 + exp$ is a strong fragment.

In a model M of $I\Delta_0 + \Omega_i$ exponentiation may not be total, hence $log(M) = \{x \in M: \ M \models \exists 2^x\}$ may be a proper initial segment. Similarly the segment $log_n(M) = \{x \in M: \ M \models \exists\, exp_n(x)\}$, where log_n denotes the n times iterated logarithm (where $log_0(x) = x$) and exp_n denotes the n times iterated exponentiation, may be a proper initial segment.

For a model M of $I\Delta_0$ we have $M \models \Omega_i$ iff $log_i(M)$ is closed under multiplication. Consequently, in a model of $I\Delta_0 + \Omega_i$, $log_{i+1}(M)$ is closed under addition, $log_{i+2}(M)$ is closed under successor and $log_{i-k}(M)$ is closed under ω_k.

Note that $log_k(M)$ is a Σ_1 definable initial segment of M.

Assume $\langle a, b\rangle$ denotes the pair-number $\frac{(a+b)(a+b+1)}{2}$.

3. Π_2 axiomatizable fragments of arithmetic

The theory $I\Delta_0$ may be axiomatized by Π_1 sentences, namely

$$\forall a, b\Big(\phi(a,0) \& \forall x < b(\phi(a,x) \Rightarrow \phi(a,x+1)) \Rightarrow \forall x \leq b\phi(a,x)\Big),$$

where ϕ runs over Δ_0 formulas.

The theories $I\Delta_0 + \Omega_n$ additionally require the axiom $\forall x \exists y \ y = \omega_n(x)$, which is Π_2. Similarly the axiom exp is Π_2: $\forall x \exists y \ y = 2^x$. Hence $I\Delta_0 + \Omega_n$, $I\Delta_0 + exp$ are Π_2 axiomatizable. Other examples of Π_2 axiomatizable fragments of arithmetic are theories stronger than $I\Delta_0 + exp$, e.g. $I\Delta_0 + \sup exp$, where sup exp is an axiom stating the totality of the super exponential function, or $I\Delta_0 + \forall x \exists y\ y = F_n(x)$, where F_n is the nth function in the Grzegorczyk hierarchy, [Grzegorczyk, 1953], or $I\Delta_0 + \forall x \exists y\ y = F_\alpha(x)$, where F_α is the α's function in the Grzegorczyk-Wainer hierarchy [Cichon and Wainer, 1983].

Assume T is a Π_2 axiomatizable theory. Let $M \models T$.

Definition 1

An element $x \in M$ is syntactically Σ_1 definable in M if there is a Δ_0 formula θ, such that

$$M \models \exists y \Big(\theta(x,y) \& \forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta(x', y')) \Big).$$

An element $x \in M$ is Σ_1 definable in M if there is a Σ_1 formula η such that $M \models \eta(x) \& \forall x' (x' \neq x \Rightarrow \neg \eta(x'))$.

Remark 2

An element $x \in M$ is syntactically Σ_1 definable in M iff it is Σ_1 definable in M.

Proof.

Assume that x is syntactically Σ_1 definable. Let $\eta(x)$ be the formula

$$\exists y \Big(\theta(x,y) \& \forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta(x', y')) \Big).$$

Then η is as required.

Assume conversely, that η is a Σ_1 definition of x. Assume η is of the form $\exists y \eta'(x,y)$, where η' is Δ_0. Let $\theta(x,y)$ be $\eta'(x,y) \& \forall y' < y \neg \eta'(x,y')$. Suppose for some $x', y' \in M$, we have $\langle x', y' \rangle < \langle x, y \rangle$ and $\theta(x', y')$. Then either $x' \neq x$ or $x' = x$ and $y' < y$. The second case contradicts the definition of θ. In the first case $M \models \exists y \eta'(x', y)$, whence $M \models \eta(x')$, contradicting $M \models \forall x' (x' \neq x \Rightarrow \neg \eta(x'))$. Hence $\forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta(x', y'))$, whence θ is as required in 1. ∎

Remark 3

If x is Σ_1 definable in M, then w.l.o.g. we may assume that it is definable by the formula $\exists y \theta(x,y)$, where θ is of the form $\theta'(x,y) \& \forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta'(x', y'))$. Hence θ provably can have at most one witness.

Proof.

In 1 instead of θ we may take $\theta'(x,y) \& \forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta'(x', y'))$. ∎

Definition 4

Let $\mathcal{K}(M)$ denote the set of those x in M which are Σ_1 definable in M. In $\mathcal{K}(M)$ we consider addition and multiplication inherited from M.

Theorem 5

$$\mathcal{K}(M) \prec_{\Sigma_1} M.$$

Proof.

Recall the Tarski Vaught criterium:

For $M_1 \subseteq M_2$ we have $M_1 \prec_{\Sigma_1} M_2$ iff for any $a_1, \ldots, a_n \in M_1$ and $\eta \in \Sigma_1$, whenever there is an $x \in M_2$ such that $M_2 \models \eta(x, a_1, \ldots, a_n)$, then there is an $x \in M_1$ such that $M_2 \models \eta(x, a_1, \ldots, a_n)$.

We apply this criterium.

So, assume $a_1, \ldots, a_n \in \mathcal{K}(M)$, $\eta \in \Sigma_1$ and there is an $x \in M$ such that $M \models \eta(x, a_1, \ldots, a_n)$. Assume η is of the form $\exists y \eta'(y, \ldots)$, where η' is Δ_0. Hence $M \models \exists x, y \eta'(x, y, a_1, \ldots, a_n)$. Let $\eta_1, \ldots, \eta_n$ define $a_1, \ldots, a_n$, respectively and assume η_i is $\exists y \eta'_i$. Assume that $\eta', \eta'_1, \ldots, \eta'_n$ provably can have at most one witness.

Thus,

$$M \models \exists x, y, y_1, \ldots, y_n (\eta'(x, y, a_1, \ldots, a_n) \& \eta'_1(y_1, a_1) \& \ldots \& \eta'_n(y_n, a_n)).$$

We have

$$M \models \exists u \exists \tilde{a}_1, \ldots, \tilde{a}_n, \tilde{x}, \tilde{y}, \tilde{y}_1, \ldots, \tilde{y}_n \leq u \\ (\eta'(\tilde{x}, \tilde{y}, \tilde{a}_1, \ldots, \tilde{a}_n) \& \eta'_1(\tilde{y}_1, \tilde{a}_1) \& \ldots \& \eta'_n(\tilde{y}_n, \tilde{a}_n)).$$

Hence, for some $\tilde{x} \in M$,

$$M \models \exists u \exists \tilde{a}_1, \ldots, \tilde{a}_n, \tilde{y}, \tilde{y}_1, \ldots, \tilde{y}_n \leq u \\ (\eta'(\tilde{x}, \tilde{y}, \tilde{a}_1, \ldots, \tilde{a}_n) \& \eta'_1(\tilde{y}_1, \tilde{a}_1) \& \ldots \& \eta'_n(\tilde{y}_n, \tilde{a}_n)).$$

By the fact that η'_i and η have at most one witness in M, we have $\tilde{a}_1 = a_1, \ldots, \tilde{a}_n = a_n$ and $\tilde{x}$ is Σ_1 definable in M by the formula

$$\exists u \exists \tilde{a}_1, \ldots, \tilde{a}_n, \tilde{y}, \tilde{y}_1, \ldots, \tilde{y}_n \leq u \\ (\eta'(\tilde{x}, \tilde{y}, \tilde{a}_1, \ldots, \tilde{a}_n) \& \eta'_1(\tilde{y}_1, \tilde{a}_1) \& \ldots \& \eta'_n(\tilde{y}_n, \tilde{a}_n)).$$

Hence $\tilde{x} \in \mathcal{K}(M)$ and $M \models \eta(\tilde{x}, a_1, \ldots, a_n)$. Hence for some $x \in \mathcal{K}(M)$ and $M \models \eta(x, a_1, \ldots, a_n)$ and the Tarski Vaught criterium is fulfilled. ∎

Corollary 6

Every element of $\mathcal{K}(M)$ is Σ_1 definable in $\mathcal{K}(M)$.

Proof.

Let $x \in \mathcal{K}(M)$ and let $\eta \in \Sigma_1$ define syntactically x. Then $M \models \eta(x)$. By Σ_1 elementariness, $\mathcal{K}(M) \models \eta(x)$. ■

Corollary 7

$\mathcal{K}(M) \models T$.

Proof.

Let $\forall x \exists y \eta(x,y)$ be an axiom of T, where $\eta \in \Delta_0$. Let $x \in \mathcal{K}(M)$. Then, $M \models \exists y \eta(x,y)$. By Σ_1 elementariness, $\mathcal{K}(M) \models \exists y \eta(x,y)$. ■

Corollary 8

A Π_2 axiomatizable theory T has a model which is pointwise Σ_1 definable, i.e. whose every element is Σ_1 definable in it.

4. Coding of truth

Definition 9

Let $x, t \in M$, $t \in \{0,1\}^x$. We say that t codes the Σ_1 truth of M if $x > \mathbb{N}$ and for every sentence $\phi \in \Sigma_1$ we have

$$t(\phi) = 1 \text{ iff } M \models \phi.$$

5. Existence of models whose Σ_1 truth is not coded

Theorem 10

If $M \models T$ is pointwise Σ_1 definable, then $\Sigma_1(M)$ is not coded in M.

Proof.

Suppose the converse. Let $x \in M$ be a code for $\Sigma_1(M)$. Let η be the Σ_1 definition of x. Then we have for ϕ running over Σ_1 sentences:

$$\phi \text{ iff } \forall x(\eta(x) \Rightarrow \phi \in x).$$

This gives a Π_1 definition of the Σ_1 truth, a contradiction with the Tarski theorem. ■

Theorem 11 [Wilkie and Paris, 1978]

Every model for $I\Delta_0 + B\Sigma_1$ has a Σ_1 elementary submodel satisfying $I\Delta_0 + B\Sigma_1$ whose Σ_1 truth is not coded.

6. Maximal theories

Definition 12

A set of Σ_1 sentences $T^{\#}$ is maximal w.r.t. T if it is maximal consistent with T.

Remark 13

A Π_2 axiomatizable theory T has a model which is pointwise Σ_1 definable and satisfies a maximal theory $T^{\#}$.

Proof.

Let $M \models T + T^{\#}$ and take $\mathcal{K}(M)$. ■

7. Initial segments

I is an initial segment of M if $I \subseteq M$ and for every $x \in I$, $y \leq x$ we have $y \in I$. I is definable if there is a formula η such that $I = \{x \in M : M \models \eta(x)\}$.

If I is definable we identify I with its definition.

Note that $log_k(M)$ is a Σ_1 definable initial segment of M.

If $a \in M$ is definable, then $\{x \in M : x \leq a\}$ is a definable initial segment. If $M \models PA$ then every definable initial segment of M is of this form.

I is a cut if I is definable and is, provably in T, an initial segment and I is provably closed under successor.

Note that log_k is not a cut.

In the case where $T = PA$, there are no proper definable cuts.

In a model of $I\Delta_0 + \Omega_n$ or of $I\Delta_0 + exp$, $\mathbb{N}$ may be a definable proper initial segment.

Example 14

Let $\theta(x)$ be the formula expressing "x is the least proof of the inconsistency of $I\Delta_0 + \Omega_n$". Let $M \models I\Delta_0 + \Omega_n + \exists x\theta(x)$ and let a satisfy θ in M. Assume that elements of the form $\omega_1^n(a)$, for $n \in \mathbb{N}$, are cofinal in M. Then $\mathbb{N}$ is definable in M by the formula $\eta(u)$:

$$\exists x \exists y(\theta(x) \& y = \omega_1^u(x)).$$

It is more difficult to show a model of $I\Delta_0 + \Omega_n$, where $\mathbb{N}$ is Π_1 definable.

8. Defining $\mathbb{N}$

We assume that J_T is a Σ_1 or Π_1 formula. We shall identify J_T with the set definable by the formula J_T.

We assume the following:

Key properties

1) J_T is an initial segment provably in T,
2) $\mathbb{N} \subseteq J_T$ provably in T
3) J_T is $\mathbb{N}$ in some models of T.

9. What J_T can be

Assume $T \supseteq I\Delta_0 + exp$.

Consider the following formula $\mathbb{N}_{T,\Sigma_1}(x)$ expressing the meaning that there is a set (i.e. a characteristic function of a set) of size x consisting of Σ_1 sentences containing all true Σ_1 sentences and x-consistent with T:

We may call $\mathbb{N}_{T,\Sigma_1}(x)$, the amount of the codability of the Σ_1 truth.

$$\exists t \in \{0,1\}^x \Big(\forall \varphi < x(Sat_{\Sigma_1}(\varphi) \Rightarrow t(\varphi) = 1)$$

$$\& \text{ the theory } \{\varphi < x : \ t(\varphi) = 1\} \text{ is } x\text{-consistent with } T\Big)$$

We can refine $\mathbb{N}_{T,\Sigma_1}(x)$ so that it will be Π_1 definable. Assume T contains a Σ_1 sentence η which is a Σ_1 definition and is false in $\mathbb{N}$. Assume η is of the form $\exists y \eta'(y)$. Consider the formula $\mathbb{N}_{T,\Sigma_1}(x)$:

$$\forall y \forall w \Big(\eta'(w) \Rightarrow \exists t \in \{0,1\}^x \Big(t \leq w \& \forall \varphi < x(Sat_{\Sigma_1}(\varphi^y) \Rightarrow t(\varphi) = 1)$$

$$\& \text{ the theory } \{\varphi < x : \ t(\varphi) = 1\} \text{ is } x\text{-consistent with } T\Big)\Big)$$

This formula is Π_1.

Dual

Consider the following formula $\mathbb{N}_{T,\Pi_1}(x)$ expressing the meaning that there is a set (i.e. a characteristic function of a set) of size x consisting of Π_1 sentences containing all true Π_1 sentences and x-consistent with T:

We may call $\mathbb{N}_{T,\Pi_1}(x)$, the amount of the codability of the Π_1 truth.

$$\exists t \in \{0,1\}^x \Big(\forall \varphi < x(Sat_{\Pi_1}(\varphi) \Rightarrow t(\varphi) = 1)$$

$$\& \text{ the theory } \{\varphi < x : \ t(\varphi) = 1\} \text{ is } x\text{-consistent with } T\Big)$$

This can be made Σ_1.

10. For what T, do $\mathbb{N}_{T,\Pi_1}$, $\mathbb{N}_{T,\Sigma_1}$ have key properties?

Let T denote a Π_2 axiomatizable consistent recursive theory containing $I\Delta_0 + exp$.

E.g. $I\Delta_0 + exp$, $I\Delta_0 + \Omega_1$. We may deal with the language containing a constant $\underline{a}$ and we include into our theories the sentence $\zeta(\underline{a})$, where ζ is a fixed Δ_0 formula. In this case T usually is not a true theory, i.e. every model of T is non standard.

- T has pointwise Σ_1 definable models. Every model of T has a Σ_1 elementary submodel pointwise Σ_1 definable.
- T has models in which witnesses for true Σ_1 sentences are cofinal.
- T has models in which the set $\Sigma_1(M)$ of true Σ_1 sentences is not coded.

11. The key properties of $\mathbb{N}_{T,\Pi_1}$, $\mathbb{N}_{T,\Sigma_1}$

Lemma 15

For every $n \in \mathbb{N}$ and every model M of T, $M \models \mathbb{N}_{T,\Pi_1}(n)$, $M \models \mathbb{N}_{T,\Sigma_1}(n)$.

Lemma 16

For every theory $T^{\#} \subseteq \Sigma_1$ which is maximal consistent w.r.t. T and every model M of $T + T^{\#}$ having the property that $T^{\#}$ is not coded in M, $\mathbb{N}_{T,\Sigma_1}$ defines $\mathbb{N}$ in M.

For every theory $T^{\#} \subseteq \Pi_1$ which is maximal consistent w.r.t. T and every model M of $T + T^{\#}$ having the property that $T^{\#}$ is not coded in M, $\mathbb{N}_{T,\Pi_1}$ defines $\mathbb{N}$ in M.

Proof.

Let M satisfy the requirements of the first part of the lemma.

We shall show that $\mathbb{N}_{T,\Sigma_1}$ defines $\mathbb{N}$ in M.

For, assume $x \in \mathbb{N}$. Let $t \in \{0,1\}^x$ be such that

$$t(\varphi) = 1 \text{ iff } M \models Sat_{\Sigma_1}(\varphi).$$

Then t is as required in $\mathbb{N}_{T,\Sigma_1}$.

Assume now $\mathbb{N}_{T,\Sigma_1}(x)$ and suppose $x > \mathbb{N}$. Take the $t \in M$ existing by $\mathbb{N}_{T,\Sigma_1}$. Then the theory

$$\{\varphi : \ M \models t(\varphi) = 1\}$$

is consistent with T since

$$M \models \text{the theory } \{\varphi < x : \ t(\varphi) = 1\} \text{ is } x\text{-consistent with } T.$$

On the other hand this theory contains $T^{\#}$, since whenever φ is true i.e. $M \models Sat_{\Sigma_1}(\varphi)$, then $t(\varphi) = 1$.

So, by the maximality of $T^{\#}$, the theory

$$\{\varphi : \ M \models t(\varphi) = 1\}$$

equals $T^{\#}$. But so, t is its code on M, a contradiction.

For $\mathbb{N}_{T,\Pi_1}$ the proof is similar. ■

12. Provability and consistency

Let $Pr_T(\phi)$ express the meaning "there is a proof of ϕ in the theory T". The most known predicates Pr are "there is a Hilbert proof", "there is a Gentzen proof", "there is a Herbrand proof", there is a "Tableau proof". All these predicates are definable by Σ_1 formulas of the form $\exists x\, Pr_T^x(\phi)$, where $Pr_T^x(\phi)$ expresses the meaning "x is a proof of ϕ in the theory T".

Dual to $Pr_T(\phi)$ is the predicate $Cons(T+\phi)$, defined as

$$Cons(T+\phi) \text{ iff } \neg Pr_T(\neg\phi).$$

Consequently, $Cons(T+\phi)$ is Π_1 in each of the above cases.

If J is an initial segment of a model M of $I\Delta_0 + \Omega_n$, then let $Pr_T^J(\phi)$ express the meaning "there is a proof belonging to J of ϕ in the theory T". Consequently, $Cons^J(T+\phi)$ expresses the meaning "there is no proof belonging to J of $\neg\phi$ in the theory T".

If J is definable by a formula $J(x)$, then $Pr_T^J(\phi)$ can be defined as $\exists x(J(x) \& Pr_T^x(\phi)$. Similarly, $Cons^J(T+\phi)$ can be defined as $\forall x(J(x) \Rightarrow \neg Pr_T^x(\neg\phi))$.

Our focus will be on $Cons^J(\cdot)$ (consistency relativized to J), for some definable initial segment J.

Assume T is recursive, consistent and contains $I\Delta_0$.

Usually a predicate $Cons(\cdot)$ is considered as expressing consistency if

$$T \text{ is consistent iff } \mathbb{N} \models Cons(T).$$

Let $Pr_T(\cdot)$ be defined as $\neg Cons(T + \neg\cdot)$.

Some other properties are usually expected, e.g. the Hilbert Bernays derivability conditions:

- $T \vdash \phi$ implies $T \vdash Pr_T(\phi)$
- $T \vdash (Pr_T(\phi) \Rightarrow Pr_T(Pr_T(\phi)))$
- $T \vdash \Big((Pr_T(\phi) \& Pr_T(\phi \Rightarrow \psi)) \Rightarrow Pr_T(\psi)\Big)$

Note two other useful properties:

- $Cons(T) \& Pr_T(\phi)$ implies $Cons(T+\phi)$.
- If $Cons^J(\cdot)$ denotes $Cons$ relativized to a definable initial segment J, then
$$Cons^{2J}(T) \& Pr^J_T(\phi) \text{ implies } Cons^J(T+\phi),$$
where $2J = \{2x : x \in J\}$.

We shall call the above properties **basic**.

Later we shall consider some unusual consistency predicates $Cons^J(\cdot)$, for some initial segments J.

13. Usual properties of consistency

Here we formulate the most important properties of the usual consistency predicates like the Hilbert, Gentzen or Herbrand ones.

- $Cons(\cdot)$ is Π_1

Σ_1 **completeness**:

- $T \vdash (\eta \Rightarrow Pr_T(\eta))$ for $\eta \in \Sigma_1$

Gödel:

- $T \nvdash Cons(T)$;
- If T is true then $T \nvdash \neg Cons(T)$ (note that $T + \neg Cons(T) \vdash \neg Cons(T + \neg Cons(T))$
- If $\theta \Leftrightarrow Cons(T + \neg\theta)$ provably in T, then $\theta \Leftrightarrow Cons(T)$ provably in T

14. Consistency over J_T

Let $Cons(\cdot)$ denote the Hilbert or the Herbrand consistency predicate. Assume $T \supseteq I\Delta_0 + exp$. Let Γ be a class of formulas, for instance, Γ can be Σ_1 or Π_1. Assume that we are given a formula $Sat_\Gamma(\cdot)$ universal for sentences in Γ which itself is in Γ. Let $Cons(\cdot + \Gamma)$ mean the sentence stating the following: for every sentence η in Γ, if $Sat_\Gamma(\eta)$ holds, then $Cons(T + \eta)$ holds.

We consider consistency $Cons^J(T + \Gamma)$ over an initial segment $J = J_T$ depending on T. The definition of T is built into the definition of J_T.

By $Cons^{J_T}(T + \Gamma)$ we shall mean the sentence stating the following: for every $x \in J_T$ and every sentence η in Γ such that $\eta \in J_T$ if $Sat_\Gamma(\eta)$ holds, then $Cons^x(T + \eta)$ holds.

For a sentence ϕ, by $Cons^{J_T}(T + \Gamma + \phi)$ we shall mean a sentence stating the following: for every $x \in J_T$ and every sentence η in Γ such that

$\eta \in J_T$ if $Sat_\Gamma(\eta)$ holds, then $Cons^x(T+\eta)$ holds and if $\eta\&\phi \in J_T$, then $Cons^x(T+\eta+\phi)$ holds.

We assume that J_T is Γ definable and has the key properties. Below, we see that the usual properties of consistency from section 13 generalize to this case.

15. Consistency with true Γ sentences

Theorem 17

Assume T is a recursive consistent theory containing $I\Delta_0$. Let Γ be a recursive set of formulas. Let J_T be a formula in Γ having the key properties, i.e.

1. J_T is an initial segment provably in T,
2. $\mathbb{N} \subseteq J_T$ provably in T
3. J_T is $\mathbb{N}$ in some models of T.

Assume that there is a universal formula $Sat_\Gamma(\cdot)$ available in T, which is itself in Γ and is universal for sentences in Γ. We assume that Γ is closed under conjunction, double negation and has enough closure properties so that $Cons^{J_T}(T+\Gamma+\cdot)$, defined in section 14, and the Gödel sentence θ equivalent in T to $Cons^{J_T}(T+\Gamma+\neg\theta)$ are in $\neg\Gamma$.

Then $Cons^{J_T}(T+\Gamma+\cdot)$ has the following properties:

- $T \vdash \phi$ implies $T \vdash Pr^{J_T}_{T,\Gamma}(\phi)$

The Γ completeness:

- $T \vdash (\eta \Rightarrow Pr^{J_T}_{T+\Gamma}(\eta))$ for $\eta \in \Gamma$

The Gödel properties:

- $T \nvdash Cons^{J_T}(T+\Gamma)$
- $T \nvdash \neg Cons^{J_T}(T+\Gamma)$
- If $\theta \Leftrightarrow Cons^{J_T}(T+\Gamma+\neg\theta)$ provably in T, then $\theta \Leftrightarrow Cons^{J_T}(T+\Gamma)$ provably in T

Proof.

The Γ completeness is immediate. Let us focus on the Gödel properties.

Lemma 18

Let θ be the diagonal sentence such that

$$T \vdash (\theta \Leftrightarrow Cons^{J_T}(T+\Gamma+\neg\theta)).$$

Call θ the Gödel sentence.

Then

$$T \vdash (\theta \Leftrightarrow Cons^{J_T}(T+\Gamma)).$$

Proof.

Work in T. Assume θ. Then $Cons^{J_T}(T+\Gamma+\neg\theta)$, whence, in particular, $Cons^{J_T}(T+\Gamma)$. Assume $Cons^{J_T}(T+\Gamma)$. Suppose $\neg\theta$. Since $\neg\theta$ is Γ we infer $Cons^{J_T}(T+\Gamma+\neg\theta)$, whence θ. ■

Corollary 19

$T \nvdash Cons^{J_T}(T+\Gamma)$.

Proof.

We shall prove that $T \nvdash Cons^{J_T}(T+\Gamma)$. Suppose the converse. Let θ Gödel sentence. Then, by 18, $T \vdash \theta$. Let M be a model of T. Then $M \models \theta$. Thus, $M \models Cons^{J_T}(T+\Gamma+\neg\theta)$. Since ${J_T}^M \supseteq \mathbb{N}$, the theory $T+\neg\theta$ is consistent. But on the other hand $T \vdash \theta$. Contradiction. ■

Corollary 20

The sentence $Cons^{J_T}(T+\Gamma)$ is independent from T.

Proof.

To see that the theory $T + Cons^{J_T}(T+\Gamma)$ is consistent it suffices to observe that is is true in every model M of T in which ${J_T}^M = \mathbb{N}$. On the other hand, $T+\neg Cons^{J_T}(T+\Gamma)$ is consistent, by 19. ■

Thus, the theorem follows. ■

16. Consistency which is Σ_1

Here we illustrate our general considerations on the predicate $Cons^{J_T}(T+\Gamma+\cdot)$ by considering the case where T is $S+B\Sigma_1$, where S is a Π_2 axiomatizable fragment of arithmetic including $I\Delta_0 + exp + \zeta$, where ζ is a Σ_1 sentence false in $\mathbb{N}$, Γ is the class of Π_1 sentences and J_T is Π_1 definable, e.g. $J_T = \mathbb{N}_{T,\Sigma_1}$. We have the following properties:

- $T \vdash \phi$ implies $T \vdash Pr^{J_T}_{T,\Pi_1}(\phi)$
- $Cons^{J_T}(T+\Pi_1+\cdot)$ is Σ_1

Π_1 **completeness**

- $T \vdash (\eta \Rightarrow Pr^{J_T}_{T+\Pi_1}(\eta))$ for $\eta \in \Pi_1$

Gödel:

- $T \nvdash Cons^{J_T}(T+\Pi_1)$
- $T \nvdash \neg Cons^{J_T}(T+\Pi_1)$
- If $\theta \Leftrightarrow Cons^{J_T}(T+\Pi_1+\neg\theta)$ provably in T, then $\theta \Leftrightarrow Cons^{J_T}(T+\Pi_1)$ provably in T

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Zofia Adamowicz
Institute of Mathematics of the Polish Academy of Sciences
Z.Adamowicz@impan.pl

Tomasz Bigaj

ENTANGLEMENT OF N DISTINGUISHABLE PARTICLES

Abstract. In their 2002 article, Ghirardi, Marinatto and Weber proposed a formal analysis of the entanglement properties for a system consisting of N distinguishable particles. Their analysis leads to the differentiation of three possible situations that can arise in such systems: complete entanglement, complete non-entanglement, and the remaining cases. This categorization can be extended by adding one important possibility in which a system is completely entangled, and yet some of its subsystems are mutually non-entangled. As an example I present and discuss the state of a three-particle system which cannot be decomposed into two non-entangled systems, and yet particle number one is not entangled with particle number three. Consequently, I introduce a new notion of utter entanglement, and I argue that some systems may be completely but not utterly entangled.

Keywords: entanglement, composite systems.

1.

The notion of entanglement remains at the centre of the foundational analysis of quantum mechanics. To date, one of the most comprehensive studies of mathematical and conceptual features of quantum entanglement in various settings is the 2002 paper co-authored by G. Ghirardi, L. Marinatto and T. Weber 2002 (other, more recent surveys of quantum entanglement can be found in [Horodecki *et al.*, 2009; Amico *et al.*, 2008]. One section of this extensive article has been devoted to the analysis of the entanglement relations that can occur in a system containing N distinguishable particles. Because N particles can remain in different entanglement settings relative to one another, we need to distinguish various types of entanglement relations that may emerge in the entire composite system. Ghirardi, Marinatto and Weber (henceforth referred to as GMW) formulate precise mathematical definitions of such possible categories of entanglement, including cases of complete entanglement and complete non-entanglement. However, it turns out that their categorization is not exhaustive. This article contains an at-

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tempt to amend *GMW*'s analysis by adding one special case of entanglement of N distinguishable particles.

In the first section I briefly outline the original method of analysing possible correlations among N particles proposed by *GMW*. The second section sketches a proof, missing from *GMW*'s article, that their procedure is consistent. In the third section I present a case of a three-particle system prepared in a state such that although the system as a whole cannot be bipartitioned into two non-entangled subsystems (and hence qualifies as completely entangled), two particles within the system are arguably not entangled with one another. An interesting physical realisation of such a situation is provided by interpreting the states of the particles as consisting of spatial and internal (e.g. spin) degrees of freedom. In that case the mathematical form of the initial state implies that particles 1 and 2 have their spins entangled, while the entanglement of particles 2 and 3 affects only their positions. In the fourth section I argue that this new case cannot be classified with the help of another distinction introduced by *GMW* between partially and totally entangled systems. To categorize it, I introduce a new notion of utter entanglement, showing that complete entanglement does not have to be utter.

2.

Following *GMW*, our main goal will be to categorize all possible entanglement relations that may arise in a composite system consisting of N distinguishable particles. The starting assumption is that the system S is prepared in a pure state described by the vector $|\psi(1,\ldots,N)\rangle$ (thus, in this paper I will ignore the important problem of how to classify entangled mixed states of many particles) This state, in turn, determines the states of all subsystems of S, which are obtained by reducing $|\psi(1,\ldots,N)\rangle$. The general method of getting the reduced states is by applying the partial trace operation to $|\psi(1,\ldots,N)\rangle$. Thus, the subsystem $S_{(1\ldots M)}$ consisting of particles $1,2,\ldots,M$, where $M<N$, will be assigned the state represented by the following density operator

$$\rho^{(1\ldots M)} = Tr^{(M+1\ldots N)}(|\psi(1,\ldots,N)\rangle\langle\psi(1,\ldots,N)|)$$

where $Tr^{M+1\ldots N)}$ is the partial trace calculated over the spaces corresponding to the particles $M+1,\ldots,N$. It is worth noting that the state assigned to a given subsystem $S_{(1\ldots M)}$ is independent from what system $S_{(1\ldots M)}$ is considered to be a subsystem of. That is, if we decide first to calculate, using the above formula, the reduced density operator for a bigger subsystem consisting of particles $1, \ldots, M, M+1, \ldots, K$, and then we apply the same procedure to reduce the resulting state to the subsystem $S_{(1\ldots M)}$, the final state will be precisely the same as above. This follows directly from the fact that the application of two partial trace operations is equivalent to one partial trace operation over the sum of both systems associated with the separate trace operations.

The first question we have to ask with respect to the global entanglement of S is whether it is possible to decompose it into two subsystems such that they are not entangled with each other. There are several equivalent ways of presenting the condition of non-entanglement between two subsystems containing particles $1, \ldots, N$ and $N+1, \ldots, M$. The most popular definition of non-entanglement is based on the factorizability condition.

Definition 1

The subsystem $S_{(1\ldots M)}$ is non-entangled with the subsystem $S_{(M+1\ldots N)}$ iff there exist vectors $|\lambda(1, \ldots, M)\rangle$ and $|\phi(M+1, \ldots, N)\rangle$, representing possible states of $S_{(1\ldots M)}$ and $S_{(M+1\ldots N)}$ respectively, such that $|\psi(1, \ldots, N)\rangle = |\lambda(1, \ldots, M)\rangle \otimes |\phi(M+1, \ldots, N)\rangle$.

Another possible definition of non-entanglement uses the notion of the reduced state.

Definition 2

The subsystem $S_{(1\ldots M)}$ is non-entangled with the subsystem $S_{(M+1\ldots N)}$ iff the reduced density operator $\rho^{(1\ldots M)}$ is a projection operator onto a one-dimensional subspace of the space $H_1 \otimes H_2 \ldots \otimes H_M$. ($\rho^{(1\ldots M)}$ can be presented as $|\lambda(1, \ldots, M)\rangle\langle\lambda(1, \ldots, M)|$.

Other equivalent definitions of non-entanglement are possible too, but we won't write them down, referring the reader to literature 1 instead.[1]

The procedure used by *GMW* in order to analyze the entanglement of the composite system S consisting of N particles is quite straightforward. First, we have to check whether it is possible to split S into two

[1] The main definition of non-entanglement given in [Ghirardi *et al.*, 2002, p. 68] refers to the existence of a one-dimensional projection operator characterising the subsystem $S_{(1\ldots M)}$, whose expectation value in the initial state is 1. Definitions of non-entanglement based on the notion of the Schmidt number and von Neumann entropy are mentioned in [Ghirardi, 2004, p. 012109–4]. Another popular criterion of non-entanglement is that the trace operator of the square of the reduced density operator should equal one (cf [Barnett, 2009, p. 50]).

non-entangled subsystems S' and S''. If this can be done, then the procedure has to be repeated for each of the subsystems S' and S'' in order to bipartition them into even smaller subsystems not entangled with one another, if possible. That way we can arrive at the finest partitioning of S into several independent subsystems $S_1, S_2, \dots, S_k$ such that none of the subsystems S_i is further decomposable into non-entangled components. Now, two possibilities have to be considered. One is that the systems $S_1, S_2, \dots, S_k$ may turn out to be one-particle systems. This means that the initial system S is *completely unentangled*, and each particle constituting it has its own pure state. In other words, the state vector $|\psi(1, \dots, N)\rangle$ can be presented as the product of N vectors each belonging to a one-particle Hilbert space. But it is also possible that S does not have any non-entangled subsystems, i.e. there is only one system in the set of non-decomposable subsystems $S_1, S_2, \dots, S_k$, and this system is S itself. In this case S is said to be *completely entangled*.[2]

This distinction can be conveniently presented as follows. In accordance with the adopted notation let k be the number of mutually non-entangled subsystems of S which are not decomposable into further non-entangled parts. Then, if $k = N$, the system S is completely non-entangled, and if $k = 1$, S is completely entangled. If k falls between 1 and N, we have a case in which S is decomposable into non-entangled composite subsystems which themselves are completely entangled.

3.

It turns out, however, that the above analysis has to be amended in two respects. First, let us start with a relatively minor issue. We have to make sure that the procedure of identifying the smallest entangled components of a given system is consistent, i.e. that it leads to a unique outcome which is independent of the initial separation into two non-entangled subsystems. The uniqueness property can be argued for as follows. Suppose that it is possible to make two bipartitions of S into subsystems S_K and $S_{K'}$ and into subsystems S_L and $S_{L'}$ and that both pairs S_K, $S_{K'}$ and S_L, $S_{L'}$ are mutually non-entangled. To ensure the uniqueness of the procedure of separation into smallest non-entangled components of S, we have to prove that the subsystems $S_{KL} = S_K \cap S_L$, $S_{KL'} = S_K \cap S_{L'}$, $S_{K'L} = S_{K'} \cap S_L$,

[2] Completely entangled states are called "N-partite entangled" in [Horodecki *et al.*, 2009, p. 890].

$S_{K'L'} = S_{K'} \cap S_{L'}$ are also mutually non-entangled. That way we can argue that no matter which initial bipartition we start with, we will end up with the same decomposition into the smallest mutually non-entangled subsystems of the system S.

A proof of the above-mentioned fact can be sketched as follows. By assumption the initial state of the system S factorizes into the product of the components describing the states of $S_K, S_{K'}$ and $S_L, S_{L'}$ respectively:

$$|\psi(1,\ldots,N)\rangle = |\psi\rangle_K |\psi\rangle_{K'} = |\psi\rangle_L |\psi\rangle_{L'}$$

Now we can write down the Schmidt decompositions for the vectors $|\psi\rangle_K$ and $|\psi\rangle_{K'}$ in the bases of subsystems $S_{KL}, S_{KL'}$ and $S_{K'L}, S_{K'L'}$.

$$\begin{aligned} |\psi\rangle_K &= \sum_n a_n |\lambda_n\rangle_{KL} |\phi_n\rangle_{KL'} \\ |\psi\rangle_{K'} &= \sum_l b_l |\chi_l\rangle_{K'L} |\mu_l\rangle_{KL'} \end{aligned}$$

The state vector of the system S can be thus presented as follows:

$$\psi(1,\ldots,N)\rangle = \textstyle\sum_{nl} a_n b_l |\lambda_n\rangle_{KL} |\phi_n\rangle_{KL'} |\chi_l\rangle_{KL'} |\mu_l\rangle_{K'L'}$$

But we know that $|\psi(1,\ldots,N)\rangle$ factorizes into the direct product of vectors $|\psi\rangle_L$ and $|\psi\rangle_{L'}$. This is possible only when all coefficients a_n and b_l but one equal zero. But in this case clearly $|\psi(1,\ldots,N)\rangle$ decomposes into the product of four vectors, describing the states of the subsystems S_{KL}, $S_{KL'}$, $S_{K'L}$, and $S_{K'L'}$. Therefore these subsystems are not entangled.

4.

However, the analysis proposed by *GMW* can benefit from the following amendment. It turns out that even if the system S is not fully decomposable into two non-entangled subsystems, there may be some 'pockets' of mutually non-entangled subsystems within S left. This is possible, because when a given subsystem S' receives a reduced density operator ρ' as the representation of its state, ρ' may turn out to be the product of two density operators ρ'_1 and ρ'_2 each representing the state of one subsystem of S'. In such a case the subsystems are deemed non-entangled (cf. [Barnett, 2009, p. 52]).

It has to be noted, though, that *GMW* start their analysis with a slightly different concept of non-entanglement based on the notion of possessing a complete set of properties by the separate components of a system.

This concept cannot be directly applied to a composite system whose state is not pure, because in this case its components can never possess complete sets of properties. This may be one reason why GMW chose not to consider the above-mentioned case in which the impure state of a subsystem S' factorizes into the product of two density operators. However, at the end of their extensive paper they briefly consider the case of non-pure states [Ghirardi *et al.*, 2002, pp. 119–120], and they present a simple argument showing that if the state of a system of two particles is a statistical mixture of factorized states, then no violation of Bell's inequality can occur in this state. Because violation of Bell's inequality is taken as indicative of entanglement, I will continue to classify the cases in question as non-entanglement.

Below I will present and carefully examine a particular example of such a situation. This example involves three particles whose state spaces are four-dimensional Hilbert spaces spanned by orthonormal vectors $|0\rangle, |1\rangle, |2\rangle, |3\rangle$. The considered state of the system S is given as follows:

$$(\star) \quad |\psi(1,2,3)\rangle = \tfrac{1}{2}(|0\rangle_1|1\rangle_2|2\rangle_3 + |0\rangle_1|3\rangle_2|0\rangle_3 + |1\rangle_1|0\rangle_2|2\rangle_3 + |1\rangle_1|2\rangle_2|0\rangle_3)$$

We can now calculate the reduced density operators for particles 1, 2 and 3 separately.

$$\rho_1 = Tr^{(2,3)}(|\psi(1,2,3)\rangle\langle\psi(1,2,3)|) = \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$\rho_2 = Tr^{(1,3)}(|\psi(1,2,3)\rangle\langle\psi(1,2,3)|) = \tfrac{1}{4}(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|)$$

$$\rho_3 = Tr^{(1,2)}(|\psi(1,2,3)\rangle\langle\psi(1,2,3)|) = \tfrac{1}{2}(|0\rangle\langle 0| + |2\rangle\langle 2|)$$

Clearly, all reduced one-particle states are mixed rather than pure, and therefore the system S cannot be decomposed into non-entangled subsystems. However, let us now calculate the reduced density operator for the two-particle subsystem $S_{(1,3)}$:

$$\rho_{1,3} = Tr^{(2)}(|\psi(1,2,3)\rangle\langle\psi(1,2,3)|) =$$

$$\tfrac{1}{4}(|0\rangle_{11}\langle 0|\otimes|2\rangle_{33}\langle 2|+|0\rangle_{11}\langle 0|\otimes|0\rangle_{33}\langle 0|+|1\rangle_{11}\langle 1|\otimes|2\rangle_{33}\langle 2|+|1\rangle_{11}\langle 1|\otimes|0\rangle_{33}\langle 0|)=$$

$$\tfrac{1}{4}(|0\rangle_{11}\langle 0| + |1\rangle_{11}\langle 1|) \otimes (|2\rangle_{33}\langle 2| + |0\rangle_{33}\langle 0|) = \rho_1 \otimes \rho_3$$

Because the reduced state $\rho_{1,3}$ is the product of the states of particle 1 and 3, it has to be concluded that 1 is not entangled with 3. Thus we have an interesting case of entanglement here. Particle 1 is entangled with the subsystem containing particles 2 and 3, but this entanglement affects only the relation between 1 and 2, not 1 and 3. In particular, no non-local correlations can be detected between outcomes of measurements performed on particles 1 and 3. Similarly, the entanglement of particle 2 with the two-particle system $\{1,3\}$ arises entirely in virtue of the entanglement between 2 and 3. It can be verified by analogous calculations that particle 1 is entangled with 2, and 2 is entangled with 3, as neither reduced density operator $\rho_{1,2}$ nor $\rho_{2,3}$ factorizes. But clearly the relation of entanglement is not transitive, hence 1 and 3 may be, and actually are, non-entangled.

The state $|\psi(1,2,3)\rangle$ can be given a suggestive physical interpretation when we identify the vectors with states having both internal and spatial degrees of freedom. Let us assume that the particles can be characterized by their spin-half values up ($|\uparrow\rangle$) and down ($|\downarrow\rangle$), and by their two possible locations left ($|L\rangle$) and right ($|R\rangle$). In addition, let us make the following identifications:

$$|0\rangle = |R\rangle|\uparrow\rangle$$

$$|1\rangle = |R\rangle|\downarrow\rangle$$

$$|2\rangle = |L\rangle|\uparrow\rangle$$

$$|3\rangle = |L\rangle|\downarrow\rangle$$

Under this interpretation the initial state of the system ($\star$) can be rewritten in the form of the following vector:

$$(\star\star) \quad |\psi(1,2,3)\rangle = \tfrac{1}{2}|R\rangle_1(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2)(|R\rangle_2|L\rangle_3 + |L\rangle_2|R\rangle_3)|\uparrow\rangle_3$$

The mathematical form of the above vector already suggests the interpretation according to which the spins of particles 1 and 2 and positions of particles 2 and 3 are entangled, while particle 1 has a precise location and particle 3 has a precise spin. Calculation of reduced density matrices confirms this observation:

$$\rho_1 = |R\rangle\langle R|(\tfrac{1}{2}|\uparrow\rangle\langle\uparrow| + \tfrac{1}{2}|\downarrow\rangle\langle\downarrow|)$$

$$\rho_2 = \tfrac{1}{4}(|R\rangle\langle R| + |L\rangle\langle L|)(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$$

$$\rho_3 = (\tfrac{1}{2}|R\rangle\langle R| + \tfrac{1}{2}|L\rangle\langle L|)|\uparrow\rangle\langle\uparrow|$$

The reduced state for particle 1 is a mixture of spins but its location is precisely R, whereas particle 3 has the precise spin up, but its location is a mixture of R and L. Particle number 2 has neither spin nor position well-defined. Particle 2 is entangled both with 1 (via spins) and with 3 (via positions). But no direct entanglement between particles 1 and 3 is present. By looking at the formula ($\star\star$) we can immediately see that a measurement of spin on particle 1 changes non-locally the spin state of particle 2 (forcing it to admit one of the two definite values depending on the outcome), but doesn't affect the state of particle 3. On the other hand, a position measurement performed on particle 3 affects the position of particle 2 without influencing in any way the reduced state of particle 1.

5.

It may be observed that the entanglement between system S_1 and system $S_{(2,3)}$, as well as between S_3 and $S_{(1,2)}$, is of the type that GMW call *partial entanglement* (cf. [Ghirardi *et al.*, 2002, p. 69]). The general definition of partial entanglement is as follows.

Definition 3

The subsystem $S_{(1\dots M)}$ is partially entangled with the subsystem $S_{(M+1\dots N)}$ iff the range of the reduced density operator $\rho^{(1\dots M)}$ is a proper submanifold (whose dimensionality is greater than one) of the total state space $H_1 \otimes H_2 \otimes \dots \otimes H_M$.

If definition 3 is satisfied, the entangled systems can be ascribed some definite properties in the form of projection operators which are projecting onto a subspace which is more than one-dimensional, but does not coincide with the entire state space. In our case the range of the operator ρ_1 describing the state of the first particle is a proper subset of the entire state space, as it coincides with the product of the entire spin space and the one-dimensional ray spanned by vector $|R\rangle$. Analogously, the range of ρ_3 is the product of the whole two-dimensional position space and the one-dimensional ray spanned by $|\uparrow\rangle$. Consequently, particle 1 is only partially entangled with the remaining subsystem, and so is particle 3. In contrast with this, the density operator ρ_2 for particle number 2 has its range identical with the product of two entire spaces for spins and positions. As a result, no definite property can be associated with this system, and in GMW's terminology particle 2 is *totally* (i.e. not partially) entangled with the system consisting of particles 1 and 3.

However, it would be incorrect to claim that the special character of the entanglement of the state ($\star\star$) can be fully expressed by categorizing it as a case of complete but not total entanglement. It can be easily verified that there are completely and not totally entangled states which nevertheless lack the unique feature of the state ($\star\star$), i.e. the non-entanglement of some small subsystems within the entire completely entangled system. Consider, for instance, the following three-particle state:

$$|\psi(1,2,3)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\uparrow\rangle_2|\uparrow\rangle_3 + |\downarrow\rangle_1|\downarrow\rangle_2|\downarrow\rangle_3)|A\rangle_1|B\rangle_2|C\rangle_3$$

where A, B, C denote three distinct locations. It is clear that the three particles are not totally entangled, as their positions are well-defined, and yet each particle is entangled with any other particle (the spin measurement on any particle changes the state of the remaining two). In order to distinguish this case from the cases similar to ($\star\star$), we should introduce a new category of entanglement – let's call it *utter entanglement* – with the help of the following definition.

Definition 4

A composite system S is utterly entangled iff S is completely entangled and for every proper subsystem S' of S, its state ρ' cannot be written in the form $\rho_q \otimes \rho_b$, where ρ_a and ρ_b are states of the subsystems composing S'.

As we know from the above-mentioned example, there are states which are completely but not utterly entangled. The impossibility of dividing a system S into two non-entangled subsystems does not imply that every subsystem of S is entangled with every other subsystem.

In order to clarify better the physical meaning of the concept of utter entanglement as expressed in definition 4, let us focus our attention on its complement, i.e. the notion of complete but not utter entanglement. As I explained earlier, this type of entanglement arises in a multipartite system S when it is impossible to partition it into two subsystems each characterized by its own pure state, and yet there is a subsystem S' whose state (mixed, not pure) factorizes into a product of two density operators. This means that the subsystems S_a and S_b jointly composing the larger subsystem S' are effectively separated from one another, even though they are not separated from the remaining particles in system S. This separation can be best characterized in terms of the lack of non-classical correlations between measurements on one system and the physical state of the other system. The measurement on system S_a which projects its initial state onto any vector within the range of the operator ρ_a leaves the other system in the same initial state ρ_b.

This general observation can be illustrated with the help of the state defined in ($\star$). The state in which particle 1 will be found after a particular measurement can be written in its most general form as $a|0\rangle + b|1\rangle$, where $|a|^2 + |b|^2 = 1$. The resulting state of the remaining two particles can be shown to be the following (up to the normalization constant):

$$a^*|1\rangle_2|2\rangle_3 + a^*|3\rangle_2|0\rangle_3 + b^*|0\rangle_2|2\rangle_3 + b^*|2\rangle_2|0\rangle_3$$

It is now easy to observe that the reduced density operator for particle 3 calculated with the help of the above state is precisely the same as before the measurement. Hence no non-classical connection exists between particles 1 and 3. Due to the separation of particles 1 and 3, other non-classical phenomena, such as the violation of Bell's inequality or entanglement swapping, are also impossible to produce. On the other hand, particle 2 clearly assumes a new state after the measurement, which is now the equally weighted mixture of states $a^*|1\rangle + b^*|0\rangle$ and $a^*|3\rangle + b^*|2\rangle$. Thus a non-classical correlation between particles 1 and 2 is present.

It may be suggested that my definition of non-utter entanglement should be corrected in order to include cases in which the state of the subsystem S' is a mixture of products of density operators $\sum_{ij} P_{ij}\rho_a^i\rho_b^j$, where $\sum_{ij} P_{ij} = 1$. Such states are commonly referred to in the literature as "correlated but not entangled" (cf [Barnett, 2009, pp. 52–53]). However, in our case such a modification would lead to unacceptable conclusions. Consider, for instance, the well-known GHZ state:

$$\frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2|0\rangle_3 + |1\rangle_1|1\rangle_2|1\rangle_3)$$

In this state each pair of particles is assigned the following mixture as its reduced state:

$$\tfrac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

which would incorrectly imply that no two particles in the GHZ state are mutually entangled. But in fact the entanglement between any two particles is clearly present because a measurement on one of them can change the state of the remaining two. In my opinion the decision to categorize mixtures of density operators as non-entangled states is justified when we limit ourselves to proper mixtures, i.e. ensembles of particles prepared in different but unknown states. In this case the change of the state of one component of the system brought about by a measurement on another component can be interpreted as a mere change in our knowledge about the real state of the system. However, the mixed state assigned to a subsystem by taking the partial trace of the state of a larger system does not admit an ignorance interpretation. In this case it is better not to classify mixtures of products of density operators as non-entangled.

In conclusion, we can distinguish the following categories of entanglement which can occur in a system consisting of N distinguishable particles. To begin with, the system can be completely unentangled, which means that its state is a direct product of N states of separate particles. If the system can be split into k subsystems ($k < N$) which are mutually non-entangled but cannot be further divided into non-entangled components, this is a case of incomplete entanglement. A system which cannot be divided into two non-entangled subsystems is called completely entangled. Within the category of completely entangled systems we can distinguish systems which are not utterly entangled, i.e. such that they still contain two or more subsystems (which however do not jointly compose the entire system) which are not entangled with one another. The last category of entanglement is utter entanglement, which means that every subsystem is entangled with every other subsystem. Finally, it should be added that the concept of total entanglement as presented above is orthogonal to the introduced distinctions. That is, each of the above-mentioned cases of entanglement can be a case of total or partial entanglement.

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Tomasz Bigaj
Institute of Philosophy
University of Warsaw
Warsaw, Poland
`t.f.bigaj@uw.edu.pl`

Andrzej Biłat

A NEO-FREGEAN THEORY OF OBJECTS AND FUNCTIONS

Abstract. Aside from the most well-known semantic postulates underlying classical logic, the main postulate of Frege's philosophy of logic is the ontological principle that there are exactly two logical types of entities, functions and objects. The aim of this paper is to reconstruct a neo-Fregean theory which implements this principle in the simplest possible way and to examine the philosophical properties of this theory. Indicated and formalized here the so called NOF-theory has the following properties: (1) The only existential assumption of the logic underlying the NOF is the thesis of the existence of at least one object and at least one unary function. (2) The only non-tautological axiom of NOF is the thesis that two arbitrarily chosen objects are different from each other. It is also one of the axioms of Tarski-Grzegorczyk's theory of concatenation (TC). (3) A nominalistic interpretation of the NOF is acceptable, where all functions are determined in the field of linguistic expressions. (4) The concepts of class, of membership and equinumerosity are definable in the NOF. (5) The monadic second-order logic (MSO) is interpretable in the NOF. (6) In the NOF-formalization of Tarski-Grzegorczyk's theory – in contrast to the normal version of this theory (TC) – the concept of sequence is definable.

1. What is the neo-Fregean theory of objects and functions?

As we know from the history and philosophy of logic, the first clear formulation of the main semantic principles defining classical logic – the principles of bivalence, compositionality of extensions, and non-emptiness of names – are all derived from Gottlob Frege. Less known is the fact that the following ontological postulate occurs among the specific principles of Frege's philosophy of logic.

(O) There are exactly two logical types of entities (i.e. values of the quantified logical variables): functions ("unsaturated" entities) and objects (arguments of functions).

The first clear articulation of the idea of logicism also comes from Frege. According to the articulation, arithmetic based on natural numbers like finite cardinals is derivable from classical logic and some meaning postulates.

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Among these postulates, the most important role is played by the so called Hume's Principle (the term was introduced by George Boolos), which states that the powers of two classes are equal if and only if the classes are equinumerous. The methodological and philosophical status of the postulate has become a main theme of reflections and discussions in neo-Fregean philosophy of mathematics in recent decades.[1] These reflections and discussions alone – regardless of the evaluation of the results – clearly show that the concept of equinumerosity is a key component of Fregean foundations for mathematics.

These observations suggest that at the heart of the neo-Fregean philosophy of mathematics is a second order theory in which: a) there are exactly two types of quantified variables, object (individual) variables and *one-place* function variables, and b) the concept of equinumerosity is expressible. Frege did not assume (as far as I know) that the category of *many-place* functions were derivable from the category of one-place functions. However, this assumption provides the simplest way to formalize the postulate (O). It is also compatible with contemporary set-theoretic logicism, i.e. the widely accepted programme of reducing mathematics to standard set theory (since many-place functions are defined in the theory as a special kind of one-place function). The purpose of this paper is to simply reconstruct the suggested neo-Fregean theory of objects and functions, meaning: NOF, and to give a description of some of their philosophical properties.

2. Formalization of NOF-theory

NOF-language is the result of the reduction of "functional" second order logic on its extra-logical constants to a set of only two names, "**1**" and "**0**". Intuitively, these names denote two arbitrarily chosen objects. In more detail, the alphabet (of the NOF-language) consists of the following symbols.

1. Logical connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$.
2. Identity predicate: $=$.
3. Logical quantifiers: $\exists, \forall$.
4. Names: $\mathbf{0}, \mathbf{1}$.
5. Object variables: $x_1, x_2, \ldots$.
6. Function variables: $f_1, f_2, \ldots$.
7. Parentheses: $($, $)$.

[1] This trend is directly derived from Crispin Wright and Bob Hale, and indirectly from Boolos. See: [Hale and Wright, 2001; Boolos, 1998].

We define the set of terms and formulas (of the NOF-language) in the usual manner.

1. The object variables and the names are terms.
2. If f is a function variable and t is a term, then '$f(t)$' is a term.
3. If s, t are terms or function variables, then '$s = t$' is a formula.
4. If α is a formula, then '$\exists x_i \alpha$', '$\forall x_i \alpha$', '$\exists f_i \alpha$', '$\forall f_i \alpha$' are formulas.
5. If α, β are formulas, then '$\neg\alpha$', '$(\alpha \wedge \beta)$', '$(\alpha \vee \beta)$', '$(\alpha \Rightarrow \beta)$', '$(\alpha \Leftrightarrow \beta)$' are formulas.
6. No other sequence of symbols is a formula.

We will sometimes use the (metalogical) letters x, y, z as object variables, f, g, h – as function variables, α, β, γ – as variables ranging over formulas, s, t – as variables ranging over terms and function variables.

Axioms of the NOF-theory consist of logical axioms (1–5) and a specific axiom ($\mathbf{NOF^{01}}$).

1. Every instance of the tautology of classical propositional calculus.
2. Every instance of the axioms of classical logic, common to first and second order logic (i.e. schemes of two versions of axioms, objectual and functional, *dictum de omni* and existential introduction).
3. Every instance of the comprehension schema for functions (FCP):[2]

$$\forall x \exists! y \alpha(x, y) \Rightarrow \exists f \forall x \forall y (f(x) = y \Leftrightarrow \alpha(x, y)),$$

provided that f does not occur free in $\alpha(x, y)$.

4. Every instance of the axioms for identity:

$$\forall x x = x.$$

$\forall s \forall t (s = t \Rightarrow (\alpha(s) \Rightarrow \alpha(t)))$, provided that t is free for s in $\alpha(s)$.

5. The axiom of extensionality (for functions):

$$\forall f \forall g (\forall x (f(x) = g(x)) \Rightarrow f = g).$$

NOF^{01}. $\neg \mathbf{1} = \mathbf{0}$.

NOF-theory (in short: NOF) is determined by the axioms and standard rules of inferences: *modus ponens*, two versions of (objectual and functional) rules of generalization and two versions of rules for existential introduction. A thesis of the NOF-theory is a formula derivable from the axioms with the use of the rules. If α is a thesis of NOF, we will sometimes write: $\emptyset \vdash_{NOF} \alpha$.

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[2] We assume, as usual, that every expression of the form '$\exists! z \beta(z)$' is an abbreviation for the formula '$\exists z \forall y (\beta(z) \Leftrightarrow z = y)$'.

3. Reconstruction of the concepts of class and equinumerosity

We define (following Frege and John von Neumann) the concepts of class and membership relation:
Df1. $CL(f) =_{df} \forall x(f(x) = \mathbf{1} \vee f(x) = \mathbf{0})$.
Df2. $x \in f =_{df} CL(f) \wedge f(x) = \mathbf{1}$.

Then classes are (total) characteristic functions in NOF. Instead of function variables running over classes, we will sometimes use meta-variables X, Y, Z etc.[3]

We derive the principles of extensionality and comprehension for classes from definitions Df1, Df2 and axiom $\mathrm{NOF}^{\mathbf{01}}$ (and also from logical axioms).

Fact 1

$\emptyset \vdash_{NOF} \forall X \forall(\forall x(x \in X \Leftrightarrow x \in Y) \Rightarrow X = Y)$.

Fact 2

$\emptyset \vdash_{NOF} \exists f \forall x(x \in f \Leftrightarrow \alpha(x))$, provided that f is not free in α.

Sketch of the proof.

We acknowledge this fact by transforming the scheme obtained from the substitution of the formula:

$$\alpha(x) \wedge y = \mathbf{1} \vee \neg\alpha(x) \wedge y = \mathbf{0},$$

where y is not free in $\alpha(x)$, for $\alpha(x, y)$ in Axiom 3 (the comprehension scheme for functions). Since the antecedent of the obtained scheme is true, we can detach the consequent. Now we can substitute the constants $\mathbf{1}$ and $\mathbf{0}$ for y in this consequent and then use Df1, Df2 and $\mathrm{NOF}^{\mathbf{01}}$. By simple logical transforming of the result, we obtain the formula in question. ∎

We may, as usual, define – with the use of the obtained comprehension scheme – Boolean operations for classes.

Fact 3

Boolean algebra of classes is a fragment of NOF.

We define a translation function $\star$ from the set of formulae of the monadic second order logic (MSO) to the set of NOF-formulae:

[3] Formulas '$\forall X\alpha$', '$\exists X\alpha$' represent, respectively, '$\forall f(CL(f) \Rightarrow \alpha(X/f))$', '$\exists f(CL(f) \wedge \alpha(X/f))$' (provided that f is not free in α).

$$
\begin{aligned}
(x = y)^\star &= \text{`}x = y\text{'},\\
(X_i x)^\star &\equiv \text{`}\forall x(f_i(x) = \mathbf{1} \vee f_i(x) = \mathbf{0}) \wedge f(x) = \mathbf{1}\text{'},\\
(\neg\alpha)^\star &\equiv \text{`}\neg(\alpha)^\star\text{'},\\
(\alpha \wedge \beta)^\star &\equiv \text{`}(\alpha)^\star \wedge (\beta)^\star\text{'},\\
(\alpha \vee \beta)^\star &\equiv \text{`}(\alpha)^\star \vee (\beta)^\star\text{'},\\
(\alpha \Rightarrow \beta)^\star &\equiv \text{`}(\alpha)^\star \Rightarrow (\beta)^\star\text{'},\\
(\alpha \Leftrightarrow \beta)^\star &\equiv \text{`}(\alpha)^\star \Leftrightarrow (\beta)^\star\text{'},\\
(\exists x\alpha)^\star &\equiv \text{`}\exists x(\alpha)^\star\text{'},\\
(\forall x\alpha)^\star &\equiv \text{`}\forall x(\alpha)^\star\text{'},\\
(\exists X_i\alpha)^\star &\equiv \text{`}\exists f_i(CL(f_i) \wedge (\alpha)^\star)\text{'},\\
(\forall X_i\alpha)^\star &\equiv \text{`}\forall f_i(CL(f_i) \Rightarrow (\alpha)^\star)\text{'}.
\end{aligned}
$$

We can easily state that the determined function leads all MSO-theses to NOF-theses.

Fact 4

MSO-system is a fragment of (is interpretable in) NOF-theory.

We may also define the concept of mapping of sets in NOF-theory:

Df3. $f : X \to Y =_{df} \forall x(x \in X \Rightarrow f(x) \in Y) \wedge \forall x(\neg x \in X \Rightarrow f(x) = \mathbf{0})$,

and then, as usual, the concepts of bijection and equinumerosity.

4. Is neo-Fregean logic a set theory in disguise?

Let NF be a system obtained from NOF by deletion of the axiom $\mathrm{NOF}^{\mathbf{01}}$. NF is a system without extra-logical constants that forms the logical basis for NOF.

NF does not include – unlike the full version of second-order logic and MSO – any existential commitments to classes. Two facts are its sources (quite nice from the philosophical point of view). First, NF-language does not have separate types of variables ranging over classes. Second, if classes were definable in NF, then they would be characteristic functions; however, this would require extra-logical assumption about the existence of at least two different objects. Since the said assumption does not apply to this logic, no version of set theory is interpretable in NF.

Moreover, this logic has no strong existential commitments to functions. The source of this property is in turn the fact that the comprehension scheme for functions (FCP) is the conditional form. It is easy to verify (considering even the minimal model of NF, thus any singleton) that we can define exactly one function on the basis of FCP, namely the identity function (obtained by the substitution of the formula '$x = y$' for '$\alpha(x, y)$'). Based on this, we can state the fact:

Fact 5

The set of the existential commitments of NF consists of exactly two claims:

- there is at least one object,
- there is at least one function.

This conclusion may seem quite surprising from the philosophical point of view. Previous discussions concerning the issue of the assumptions underlying the consistent and interesting (for logicists) fragments of Frege's system seem to suggest that one of the greatest difficulties of neo-Fregeanism is the question of the ontological commitment of higher order logic. This difficulty is usually associated with Quine's thesis that second-order logic is set theory in disguise.[4] Indeed, if the thesis were correct, then neo-Fregeanism would not be essentially different from the usual set-theoretical logicism.

The previous discussions assumed – as far as I know – that MSO is contained in each adequate (for logicists) fragment of Frege's logic. Under this assumption, a defense against Quine's thesis was sometimes developed, replacing the objectual interpretation of second-order quantifiers by a substitutional one. However, such a solution is not compatible with the spirit of the neo-Fregean philosophy of logic, both because of the typical assignment of this philosophy, the objectual interpretation of the logic, and because of the common tasks of neologicism (contrasting with the limited power of expression of the substitutional quantification).[5]

The concept of foundations of mathematics, in which NOF plays a central role, provides a simple method to avoid Quine's objection. The method is to exclude, from the scope of mathematical logic, systems in which the predicate variables (including monadic variables) and many-placed functional variables are quantified. As a result, we get a system of second-

[4] See for example: [Hale and Wright, 2005, pp. 197–200].

[5] See ibid. In this article, other ways to avoid Quine's difficulty are also presented (for example the interpretation of MSO as "logic of plurality" in Boolos's style).

order logic with modest existential commitments to keep an ordinary, non-substitutional interpretation (for both types of quantification). At the same time, the system has sufficient strength of expression for introducing – through such theories as NOF – fundamental theories in mathematics.

5. Does NOF have ontological commitments?

Since the NOF is an extremely general theory of objects and functions, its extent does not exclude nominalistic interpretations in which some expressions of NOF-language are values of the object variables. A simple example of such an interpretation is the structure:

$$\mathrm{M}^{1,0} = \langle\{\mathbf{1}, \mathbf{0}\}, \mathrm{ID}\rangle,$$

where ID is the identity function (i.e. $\mathrm{ID}(\mathbf{1}) = \mathbf{1}, \mathrm{ID}(\mathbf{0}) = \mathbf{0}$). It is quite reasonable to postulate that ontological commitments of a theory are reduced to objects that are not linguistic expressions of the theory. In this sense, we can assume that NOF does not contain any ontological commitments in $\mathrm{M}^{1,0}$.

Fact 6

There are acceptable interpretations of NOF that are free of any ontological commitment.

In this context, it is quite interesting that $\mathrm{NOF}^{\mathbf{01}}$ is one of the axioms of Tarski-Grzegorczyk's theory of concatenation TC.[6] From the neo-Fregeanism perspective, as to mathematical basis, there is nothing in the way of formulating TC theory on the basis of NOF. In making such formalization, we get a "nominalistic" definition of the sequence:

$$f = (x_1, x_2, x_3 \ldots)$$

as a function defined on successive "powers" of names (the ˆ is here a symbol of the concatenation operation):

$$f(\mathbf{1}) = x_1, f(\mathbf{1}\hat{}\mathbf{1}) = x_2, f(\mathbf{1}\hat{}\mathbf{1}\hat{}\mathbf{1}) = x_3 \ldots.$$

Now the n-tuples can be represented as follows. All the arguments, different from $\mathbf{1}^k$, for $1 \leq k \leq n$, are assigned by the function f to the

[6] TC theory, derived from [Tarski, 1933], in recent years has been described and applied in examining the issue of decidability in [Grzegorczyk, 2005].

name **0** (which may be identified with the empty string).[7] With the finite strings, you can then pose the problem of neo-Fregean reconstruction of the concepts of relation and the many-placed function. This problem, like the question of the details of the project outlined above, we leave here as open.[8] Here we only note the fact.

Fact 7

Let $\mathrm{TC}^{\mathrm{NOF}}$ be the result of the extension of the NOF-theory by addition of TC-axioms. In the $\mathrm{TC}^{\mathrm{NOF}}$ theory, the concept of (finite or infinite) sequence is definable.

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Andrzej Biłat
Department of Social Sciences, College of Enterprise and Administration in Lublin, Lublin, Poland
`a.bilat@wspa.pl`

[7] The symbol "$\mathbf{1}^k$" ("power" of the letter **1**) means the product of k-times writing of the letter **1** to the empty string.

[8] This is not a very apparent problem: ordered pairs are not definable in TC (see: [Visser, 2009]).

Izabela Bondecka-Krzykowska

THE BEGINNINGS OF MECHANICAL COMPUTING IN POLAND*

Abstract. The paper presents the first computing devices which were constructed in Poland in the 18th and 19th centuries. Most of the attention has been devoted to the inventions of Abraham Stern, Chaim Słonimski and Izrael Staffel, especially to the construction and the rules of operating on their calculating machines. Presented inventors were Jewish artisans who, in spite of difficult conditions, succeeded in creating numerous interesting inventions (including calculating machines). This suggests the existence of a dynamic Jewish artisan community in Warsaw at that time.

Keywords: history of mechanical computation, calculating machines, inventors of calculating machines

1. Introduction

The history of mechanical computation is long and interesting, but very often researchers limit it to inventors from Western Europe such as Schickard, Pascal, Leibniz and their successors. Since the aim of this paper is to present calculating machines built in Poland (Eastern Europe), similar machines built at that time or earlier in Western Europe[1] have been completely omitted here. There are only a few papers concerned with this subject. The article has been based mainly on archival sources: contemporary newspapers, publications of scientific associations, materials from exhibitions and descriptions of the machines drawn by their constructors.

Probably the oldest mechanical calculating machine in the Polish territory (then belonging to Russia) was the invention of Gevna Jakobson who before 1770 built a machine for addition, subtraction and multiplication of

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[1] There are many publications on the history of calculating machines in Western Europe, in particular papers in IEE Annals of the History of Computing.

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10-digit numbers. It is extremely difficult to find in source materials descriptions of this machine and the way of operating it. The only available data say that Jakobson's machine was a brass case 34 cm × 21,8 cm × 3,4 cm size, built with gears serving to transfer digits from one row to another similarly to Schicard's construction. Jacobson's calculating machine is preserved in the Łomonosov Museum of Science in St. Petersburg, though very difficult to investigate.[2]

No mention of calculating machines built on Polish territory in the period between 1770 to the 19th century can be found either in the press or others source materials. The first information on the subject concerns three inventors: Abraham Stern, Chaim Słonimski and Izrael Staffel.[3] All of them were Jewish artisans who spent most of their lives in Warsaw (the present capital of Poland) which was part of the Russian Empire at that time. Their Jewish origin and social background affected not only their personal lives but also the history of their inventions (including calculating machines). Therefore it is worth describing briefly the political and economic situation in Warsaw of that time.

Warsaw was the capital of the Polish-Lithuanian Commonwealth until 1795, when it was annexed by the Kingdom of Prussia to become the capital of the province of South Prussia. Liberated by Napoleon's army in 1806, Warsaw was made the capital of the newly created Duchy of Warsaw. Following the Congress of Vienna of 1815, Warsaw became the centre of the Congress Poland (called also the Kingdom of Poland), a constitutional monarchy under a personal union with Imperial Russia. At that time Warsaw was the centre of Poland's national life; many Polish patriotic organizations had their seats there. In 1897 Warsaw was the third-largest city of the Russian Empire after St. Petersburg and Moscow. According to the Russian population census of 1897 the territory of the Kingdom was inhabited by six nations; the second nation (after the Poles) were the Jews, who constituted 13.8% of the whole population [Eberhardt, 2003, pp. 76–77].

The situation of the Jews in Warsaw in the 19th century had been changing together with the changes in the political and economic situation of the city. In 1791 tsarina Catherine the Great created a special region of Russia, in which permanent residency of the Jews was allowed (beyond

2 In spite of many attempts we did not succeed in getting permission from the museum's management to investigate Jacobson's machine.

3 We concentrate on Abraham Stern, Chaim Słonimski and Izrael Staffel because their inventions are sufficiently documented.

that region, Jewish permanent residency was generally prohibited) called the Settlement of Pale, which included lands formerly belonging to Poland. In the 19th century most of the Jews in the Pale (including Warsaw) were poor, living and working in very bad conditions. One of the reasons was the tremendous growth of the Jewish population in Warsaw; others were political and legal regulations. The Jews were obliged to live in special areas in most cities;[4] they faced restrictions on education, business activities, and occupation. Additional taxes were imposed on members of the Jewish community. Jewish boys were obliged to serve in the Russian army, where they were often forced to convert to Christianity. That situation combined with too many artisans in the same area resulted in the reduction of orders and a lack of work, which in consequence led to the pauperization of the Jewish artisan community. The legislation was changing along with tsars. Tsar Nicolas introduced special restrictions against the Jews (among others Cantonist Laws which kept the traditional double taxation on the Jews). Those restrictions were softened by tsar Alexander II (also known as Alexander the Liberator) and reintroduced by tsar Alexander III who tightened restrictions on where Jews could live in the Pale of Settlement and restricting the occupations that Jews could attain.[5]

The Jews at that time formed rather a hermetic community with a separate religion and system of education. Jewish children either were not educated at all or attended Jewish religious schools. Most Jews could read neither Polish nor Russian – this made impossible learning about the latest technical and scientific achievements. It is worth remarking here that Chaim Zelig Słonimski[6] (detailed presentation included below), first in history, began writing and publishing science books in Hebrew to enlighten the Jewish population and in 1862 launched the popular science magazine "Hazefirah", the first Hebrew journal with an emphasis on science, which continued after his death in 1931.

In contrast to the restrictions mentioned above, in the 19th century Haskalah, the Jewish Enlightenment, began on the Polish territory. Supporters of that movement stressed secular ideas and values and pressed for assimilation and integration into European society and for educational development in secular studies. In spite of the that fact, most Polish Jews were

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4 The special area with restriction on the permanent residency of Jewish was founded in Warsaw in 1809 [Dubnow, 1916–1920, p. 145].

5 Detailed information about the history of Jews on Polish and Russian territory can be found in [Dubnow, 1916–1920; Klier, 1986].

6 More information about Ch. Z. Słonimski can be found in section 3.

indifferent to Haskalah; they focused on a continuation of religious tradition as a base of their lives. Nevertheless the Jewish Enlightenment had prominent supporters mainly in Warsaw, including Abraham Stern and Chaim Zelig Słonimski.

Abraham Stern, Chaim Słonimski, and Izrael Staffel, like most Jewish artisans, lived in poverty – that made their living conditions hard. They often contended with financial problems which prevented them from developing their talents (including the construction of prototypes of their inventions). However, despite the fact that Jewish inventors lived in isolation, had difficult access to the knowledge of technical and scientific achievements, and faced financial problems, they were very talented inventors and their inventions were not limited to calculating machines.

Summing up, in spite of the difficulties that the Jews faced in 19$^{\text{th}}$ century Warsaw, Jewish artisans succeeded in creating numerous interesting inventions (including calculating machines). It can be supposed that there was a dynamic Jewish artisan community in Warsaw at that time.

2. The calculating machines of Abraham Stern

One of the members of that community was Abraham Stern. ABRAHAM STERN (1769–1842) was born in a poor Jewish family in Hrubieszów (Eastern Poland). Thanks to help from Polish nobleman and scientist Stanislaw Staszic, Stern moved to Warsaw, where he designed several inventions: among others, a mechanical harvester, a rangefinder, a "topographical cart" which allowed the drawing of maps of regions to scale (a cart was pulled by horses along the boundary of a region and at the same time the map of the region was drawn on paper), a thresher, a mechanical brake for droshky, a sawmill, and a series of calculating machines. Due to a shortage of money, Stern did not manage to build prototypes of most of his machines. Stern was not only an inventor, he wrote poems and was known as an expert in Hebrew writings. He was also engaged in political activity to support the Jews. There is the following mention about Abraham Stern in a book by S. Dubnow [1916–1920, pp. 248–249]:

> In 1825 the Polish Government appointed a special body to deal with Jewish affairs. It was called "Committee of Old Testament Believers," though composed in the main of Polish officials. It was supplemented by an advisory council consisting of five public-spirited Jews and their alternates. Among the members of the Committee, which included several prominent Jewish merchants of Warsaw, such as Jacob Bergson, M. Kavski, Solomon Posner and

> T. Teplitz, was also the well-known mathematician[7] Abraham Stern, one of the few cultured Jews of that period who remained a steadfast upholder of Jewish tradition.

At the end of his life he became the Rector of the School for Rabbis. He died in 1842 in Warsaw.

Stern gained the reputation of a splendid inventor. He presented his inventions a couple of times at the Society's meetings. His calculating machines Stern presented to the Royal Warsaw Society of the Friends of Science (predecessor of the Polish Academy of Science): his first machine for only four arithmetical operations in December 1812, a second machine for extracting square roots in January 1817 and finally a combined machine for four operations and square roots in April 1818.[8] The last machine was probably the first machine for five arithmetical operations in Europe [Trzesicki, 2006].

There are two pictures of Stern's machine: the first one is only a fragment of the machine visible in a portrait of Stern by Antoni Blank (1823, The National Museum in Poznań) and the other one (published in [Sawicka and Sawicki, 1956]) is a picture of a copy which was exhibited in the Museum of Industry in Kraków, between the wars (however, the copy has not been preserved to our times). Because of this, the description of the machine and its use is based on a presentation by Abraham Stern given at meetings of the Royal Warsaw Society of the Friends of Science (see [Stern, 1818]).

The machine was a cuboid with five rows of wheels. In the first row there were 13 wheels with discs, on which there were ordinary digits of numbers engraved. They were seen singly through the apertures. Each wheel corresponded to one position in number: from units, tens, hundreds, etc. The 13 wheels of the second row were only a part of the machine's mechanism and didn't have any engravings. The next two rows of wheels with engraved digits (visible by the windows) were on a carriage which was moving with the use of cylinders. There were 7 wheels in the first row on the carriage and 8 wheels in the second one. There were also seven small folding cranks attached to 7 wheels in the first row on the carriage (this row was called by the inventor "crank row") and one big removable crank on the cover of the machine. Above the carriage Stern placed the fifth row of

[7] There is no evidence that Abraham Stern was interested in mathematics, so it is a mistake to call him a mathematician [my remark – I. B.-K.].

[8] For his inventions he was admitted to the Royal Warsaw Society of the Friends of Science as a corresponding member in February 1817, then as a qualifying member (in February 1821), and finally as a full member (in January 1830).

seven wheels with engraved digits visible through apertures. Besides these five rows of wheels, there were two more rows of wheels on the cover of the machine: one above the first row of apertures and the other one above the last row of apertures (above the lowermost row). On the wheels in these two rows there were Roman numerals engraved which were visible singly through the apertures. These two rows were used to check the results of calculations.

Abraham Stern in his presentation in Warsaw (see above) described in detail the way of using the machine. To prepare the machine to carry out four basic arithmetic operations, the operator had to place the carriage using a handle in such a position that on a carriage on the left-hand side, the word Species showed through an aperture and all the numerical apertures of the second row were covered. Then the operator with two handles on the right and left-hand side of the machine moved the carriage up – if the operation to carry out was addition or multiplication – and down if the operation to carry out was subtraction or division. At the same time the words: *Addition – Multiplication or Subtraction – Division* (respectively) were seen on the machine through the aperture and the machine was ready to perform.

To add or subtract two numbers, the operator put one of them in the uppermost row, and the other one in the crank row on the carriage. Then he performed the operation by a single circular rotation of a big crank in the middle of the carriage. The machine had a brake, located on the left-hand side of the carriage which stopped further movement of the crank. The result of the operation appeared in the uppermost row of apertures (replacing the first number). Thanks to that the machine helped to add long rows of numbers, because the current sum was always in the first row, the added numbers were placed one after another in the crank row. Additionally, the machine had a counter which showed how many numbers had been added, which facilitated adding long rows or tables of numbers without mistakes like adding the same number twice or skipping some numbers.

To carry out multiplication the operator put one factor in the crank row in the carriage, and the other in the lowermost row. In the uppermost row there were only zeroes. Then the carriage was moved from the right to the left side, to the very end of the machine, by the handle placed on the left-hand side of the carriage. After releasing the handle, the carriage returned by itself, and stopped in an appropriate position. In this position the operator started the rotation of the main crank. During the rotation, the carriage moved by itself from one number to the other towards the right-hand side, back through the end of the machine. The ringing of a bell (built into the machine) informed about the operation's completion. At the same time the desired product appeared in the uppermost row. This method of multiplication enabled calculating the sum of any number of products. The operator put the factors of the first product in the machine and operated until the ring of the bell indicated to stop; then without putting zeros in the uppermost row he put the factor of the second product, third, and so on, and when after the last operation the ring of the bell indicated to stop the rotations, at that time the sum of all the products appeared in the uppermost row. Stern stated that [1818, p. 118]:

> [...] the Machine has a particular superiority over calculations in an ordinary manner, that from several given multiplications one can obtain a general product without performing an addition operation, that is, without combining individually calculated products together.

To divide numbers, the operator set the dividend in the uppermost row, the divisor in the crank row of the carriage and zeroes in the lowermost row (designated for the quotient). Then he moved the carriage towards the left-hand side, until the divisor stood straight under the dividend number. At that time the main crank was rotated as long as the dividend number became smaller than the divisor, at which point the operator pressed with a finger a flap situated on the right-hand side of the carriage. As a result, the carriage moved by itself towards the right-hand side and stopped at the appropriate place, where further operation continued in a similar manner till the divisor placed in the carriage "passed" the dividend. The quotient appeared in the lowermost row. If the quotient was a whole number then in the uppermost row there were only zeroes; if it was a fraction, then the numerator appeared in the uppermost row and the denominator in the crank row of the carriage.[9]

Most interesting was extracting square roots from numbers. First of all an operator had to prepare the machine to carry out this operation by: 1) placing the carriage on the right-hand side (the word Species disappeared and the word Radices was visible through an aperture; numerical apertures of the second row of the carriage opened), 2) moving the carriage (using two handles on the right and left-hand sides of the machine) from the top to the bottom (the inscription Extraction appeared in an aperture on the machine, 3) removing the main crank in the middle of the carriage. Then the machine was ready to perform.

[9] The way of carrying out four basic arithmetic operations using Stern's machine was very similar to operating Staffel's calculating machine (described below).

To calculate the square root of a given number, the operator set this number in the uppermost row and zeros in the first and second rows of the carriage except the position of units in the second row, where the number 1 was set. At the apertures for ordinary numbers of the uppermost row, there were various signs dividing this row into sections (there were signs at units, hundreds, tens of thousands, millions, and so on). Identical signs were on small folding cranks so that each crank corresponded to two wheels of the uppermost row (the first crank from the right corresponded to units and tens, the second one – hundreds and thousands, and so on). The last sign, at the given number, indicated the crank from which the operation had to start. For example, if the number was 144 (ended on the wheels of the second sign) the operation started with the second crank from the right (having the same sign). The folding crank indicated this way was unfolded and the carriage was moved to the left until this crank stopped in front of the last sign of the given number. Then the operator rotated this crank as long as the number on the uppermost row, in front of the rotating crank, became smaller than or, at least, equal to the number positioned in front of the same crank in the second row of the carriage. Next, this crank was folded and the crank on the right of it was unfolded. By pressing a flap on the right-hand side of the carriage, the carriage moved by itself to the right- hand side, until it was stopped by a folded crank, just in front of the previous section and the same operation was performed. This was repeated for all sections of the given number. After completing the operation, if a given number was a full square, it was replaced by zeroes and the square root in the crank row in the carriage appeared. Otherwise, except for the whole number root, an additional fraction resulted (the numerator on the uppermost row and the denominator in the second row in the carriage).

The machine was also prepared for approximating the square roots in decimal fractions. For example, to compute the square root of 7 approximated with two decimal digits, zeros were set in two sections (4 wheels) and the given number 7 was set in the third section, which was on the 5$^{\text{th}}$ wheel of the uppermost row. On the machine there was a small hand to distinguish between the number actually given and the zeroes attached to it. The given number 7 was under the third sign, so the operator had to unfold the third crank and perform the operations as described above. The result appeared on 3 crank wheels as the number 264. Cutting off two digits for a decimal fraction (as the hand indicated), the result was understood as 2.64. In addition, in the uppermost row there was the number 304, as a numerator, and in the second row of the carriage 529, as a denominator of the ordinary fraction.

Above, the structure of Stern's machine and the way of using it was presented. However, there is one part of the machine which was described, but the purpose of designing it has not been explained yet – two rows of wheels with Roman numerals. Stern put these wheels into the machine for checking (testing) the results of arithmetical operations. The way of performing such tests was presented in the case of multiplying (division was tested in a similar way).

While carrying out the multiplication, the digits of one of the factors (set in the lowermost row) disappeared and were replaced by zeroes one by one. To make visible, after the work, which factor was a part of the problem, the operator set it in advance in a Roman numbered row located above the apertures of the lowermost row. After completing the operation, the result (the product) was on the uppermost row; in the lowermost row there are only zeroes. The operator shifted as many zeroes to the number 9 as the number of digits of the factor in the Roman numbered row, except for the first digit, being meaningful, on the right-hand side of the factor, where zero remains. Then the carriage was moved to the left and stopped by itself at the last number 9. After that the rotation of the main crank lasted as long as the number appeared was equal to the Roman numeral right above it (the same which had previously disappeared). At that time, the operator pressed a flap on the right-hand side of the carriage, the carriage moved to the right and the rotations proceeded further, as before, until the given factor fully appeared in its first place, that is, in the lowermost row. After this work, if it turned out that there were as many digits in the factor in the lowermost row as the number of zeroes in the uppermost row, at the right-hand side, and the numbers following them were equal to the numbers in the crank row of the carriage, then it was clear that the product (the result of the process of multiplication) was true, otherwise it was false.

In the case of testing the result of division the operator proceeded in a similar way, but only to retain digits of the dividend (which disappeared during the work, having been replaced by zeroes) the row of wheels with Roman numerals above the row of the dividend's digits was used.

Stern's machines were highly valued, among others by the Royal Warsaw Society of the Friends of Science, but they were never manufactured, maybe because of the intricate mechanisms which resulted in high costs of production. Stern did not have a sufficient amount of money to begin mass-production of his machine, but the machine was finally produced and used.

3. Between mathematics and machines – the invention of Chaim Zelig Słonimski

Continuing after Abraham Stern was his son-in-law Chaim Zelig Słonimski. CHAIM ZELIG SŁONIMSKI was born on 31st March, 1810 in Białystok, (Eastern Poland). He was a deeply knowledgeable Talmudist and a self-educated scientist. Słonimski had wide interests; he was interested in philosophy, astronomy, physics and mathematics. He was the first to begin writing and publishing science books in Hebrew to enlighten the Jewish population in Eastern Europe. He introduced to Hebrew an entire vocabulary of technical terms. Słonimski was a born popularizer; at the age of 23, he composed a brief practical guide on the foundations of mathematics. The first part of the guide, dedicated to algebra, was published in 1834. In 1835, inspired by the general interest in the passing of Halley's Comet, he published a book on astronomy, "Comet", describing Halley's Comet and explaining the laws of Kepler. In 1838 he published another book on astronomy in which he described his own research on the calculations of eclipse dates and on composing the Hebrew calendar. Later in life, he started publishing a popular science magazine "Hazefirah", the first Hebrew journal with an emphasis on science, which continued after his death, till 1931. He was also the author of a biography of Alexander von Humboldt. Słonimski died on May 15th, 1904 in Warsaw.

Słonimski was a talented inventor. He invented several devices and processes of various sorts. In 1853 he invented a chemical process for plating iron vessels (dishes) with lead, and in 1856 an electrochemical device for sending quadruple telegrams (the system of multiple telegraphy perfected by Lord Kelvin in 1858 was based on Słonimski's discovery). Among other of Słonimski's inventions, calculating machines were worth noting. He invented and produced two calculating machines, one for addition and subtraction, and the other one for multiplication. The most interesting is the second one, which was based on a theorem of number theory called Słonimski's Theorem.

Słonimski's Theorem Let Z be any natural number and $z_1, z_2, z_3, z_4, ...$ be the (decimal) digits of this number (denoted from the right to the left). If we write down the number Z and its multiples $2Z, 3Z, 4Z, 5Z, 6Z, 7Z, 8Z$ and $9Z$ in such a way that single digits, decimals, hundreds and so on form vertical lines, then the last vertical line passing by the last digit z_1 of the number Z will contain *the second* digits of multiples $2z_1, 3z_1, 4z_1, 5z_1, 6z_1, 7z_1, 8z_1$ and $9z_1$. But in every other line the situation is different, for example the line passing by z_ε does not contain the second digits of multiples $2z_\varepsilon, 3z_\varepsilon, 4z_\varepsilon$, $5z_\varepsilon, 6z_\varepsilon, 7z_\varepsilon, 8z_\varepsilon$ and $9z_\varepsilon$. To obtain these digits a special sequence (called a "complementary sequence") must be added to the sequence of multiples of z_ε. This complementary sequence depends on the digits after the number Z. There are only **twenty-eight** different complementary sequences. To obtain digits of the vertical line passing by z_ε, the sequence multiples of z_ε should be added to complementary sequences corresponding to the digits, which follow z_ε in number Z.

To understand Słonimski's theorem let us consider for example the number $Z = 1246$. Then $z_1 = 6, z_2 = 4, z_3 = 2$ and $z_4 = 1$. If we write down Z and its multiples $2Z, 3Z, 4Z, 5Z, 6Z, 7Z, 8Z$ and $9Z$ as follows:

		z_4	z_3	z_2	z_1
Z		1	2	4	6
$2Z$		2	4	9	2
$3Z$		3	7	3	8
$4Z$		4	9	8	4
$5Z$		6	2	3	0
$6Z$		7	4	7	6
$7Z$		8	7	2	2
$8Z$		9	9	6	8
$9Z$	1	1	2	1	4

then the last vertical line passing by the last digit 6 of the number 1246 will contain *the second* digits of multiples $6, 12, 18, 24, 30, 36, 42, 48$ and 54 (see the column marked in the table above). The theorem states that "[...] in every other line the situation is different, for example the line passing by z_ε does not contain the second digits of multiples $2z_\varepsilon, 3z_\varepsilon, 4z_\varepsilon, 5z_\varepsilon, 6z_\varepsilon, 7z_\varepsilon, 8z_\varepsilon$ and $9z_\varepsilon$. To obtain these digits a special sequence (called a "complementary sequence") must be added to the sequence of multiples of z_ε." So, in every other line decimals of multiples are carried to the next left column. To illustrate this, let us examine the table below where digits in brackets were carried to the next left column (missing in the columns of the table above), where they were printed in bold:

		z_4	z_3	z_2	z_1
Z		1	2	4	6
$2Z$	0	$02 + \mathbf{0} = (0)2$	$04 + \mathbf{0} = (0)4$	$08 + \mathbf{1} = (0)9$	(1)2
$3Z$	0	$03 + \mathbf{0} = (0)3$	$06 + \mathbf{1} = (0)7$	$12 + \mathbf{1} = (1)3$	(1)8
$4Z$	0	$04 + \mathbf{0} = (0)4$	$08 + \mathbf{1} = (0)9$	$16 + \mathbf{2} = (1)8$	(2)4
$5Z$	0	$05 + \mathbf{1} = (0)6$	$10 + \mathbf{2} = (1)2$	$20 + \mathbf{3} = (2)3$	(3)0
$6Z$	0	$06 + \mathbf{1} = (0)7$	$12 + \mathbf{2} = (1)4$	$24 + \mathbf{3} = (2)7$	(3)6
$7Z$	0	$07 + \mathbf{1} = (0)8$	$14 + \mathbf{3} = (1)7$	$28 + \mathbf{4} = (3)2$	(4)2
$8Z$	0	$08 + \mathbf{1} = (0)9$	$16 + \mathbf{3} = (1)9$	$32 + \mathbf{4} = (3)6$	(4)8
$9Z$	1	$09 + \mathbf{2} = (1)1$	$18 + \mathbf{4} = (2)2$	$36 + \mathbf{5} = (4)1$	(5)4

In such a way, passing over the first row (for Z), the bold digits form complementary sequences as follows: $(0, 0, 0, 0, 0, 0, 0, 0)$ for z_1, $(1, 1, 2, 3, 3, 4, 4, 5)$ for z_2, $(0, 1, 1, 2, 2, 3, 3, 4)$ for z_3 and $(0, 0, 0, 1, 1, 1, 1, 2)$ for z_4. Now the question arises: how many complementary sequences may occur, regardless of the digits of number Z? Słonimski found that exactly 28 different complementary sequences can occur.[10] That is the content of Słonimski's theorem presented above.

This theorem was derived from the Farey sequence.[11] Słonimski does not seem to have published the theorem. He presented it to the St. Petersburg Academy but he never proved it himself. However, a German mathematician August Leopold Crelle, who was familiar with the theorem because of Słonimski's personal communication during his visit to Berlin in 1844, proved Słonimski's Theorem and published the result in his own journal [Crelle, 1846]. Using his theorem, Słonimski composed a table with 280 columns, each of them containing 9 numbers. This table was the main component of the multiplication machine which showed products of all ranks for a given number.[12]

Słonimski's machine was a box sized 40 cm × 33 cm × 5 cm. There were some cylinders inside, which could both revolve around the axis and move along it. The table of digits derived from Słonimski's theorem was placed (engraved) on the main cylinders. There were two small cylinders beside it

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[10] Crelle marks the complementary sequences with and shows it as a table in the proof of Słonimski's theorem published in [Crelle, 1846].

[11] Farey's sequence of order n is a sequence of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to n, arranged in order of increasing size. Each Farey sequence starts with the value 0, denoted by the fraction 0/1, and ends with the value 1, denoted by the fraction 1/1. F1 = 0/1, 1/1, F2 = 0/1, 1/2, 1/1, F3 = 0/1, 1/3, 1/2, 2/3, 1/1, F4 = 0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1,. . .

[12] The tables are in the book [Knight, 1847].

with digits from 0 to 9 on one of them and letters a, b, c, d together with digits 1 to 7 on the other one. The cylinders were driven with the use of handles fastened to the shaft end. While the small cylinders were immobile, the main cylinders were moved along their axis with toothed gearing, driven with screws mounted on the cover of the machine. There were also eleven rows of apertures on the cover. By these apertures the signs engraved on the cylinders were visible.

The use of Słonimski's machine was very simple. The multiplicand was set on the lowermost (the first) row of apertures with handles mounted on the cover. After that, both letters and numbers appeared in the apertures of the second and third row. Their combination formed the code which informed the operator which screw should be turned (and which cylinder was to be shifted). Then in the rows of the 4^{th}–11^{th} apertures appeared the resulting numbers. In the 4^{th} row was the product of multiplication by 2, the 5^{th} row by 3, the 6^{th} row by 4 etc. Finally, the products of all ranks were displayed. After adding them on the paper, the desired product was obtained. Needless to say, the convenience of this method was rather questionable, and it is no wonder that there is no evidence of its systematic practical usage. But Słonimski's machine got high recognition during his lifetime. On 8^{th} August, 1844 he demonstrated his device to the Royal Prussian Academy of Sciences in Berlin. Słonimski's work was highly appreciated.

The next year, on April 4, 1845, he presented the machine and explained its design to the Academy of Sciences in St. Petersburg during a seminar at the department of physics and mathematics. The academician V. A. Bunyakovski and the scientific secretary P. N. Fuss (Voss) composed a very positive official review of the invention. They emphasized the solid mathematical ground of the presented work because the discovery of the basic feature of numbers was the principlal but not the only condition for composing this calculating machine. In the review the shrewdness of Słonimski was appreciated, because he arranged the aforementioned tables and invented also the code which the operator used to calculate the products. So the surface of cylinders was covered with a complicated system of 2280 numbers and 600 letters. In November 1845 Słonimski received a 10 year patent for his invention.[13] He was also awarded the Demidov prize of the Second grade amounting to 2000 rubles [Trzesicki, 2006].[14]

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13 He also applied for patents in the U.S. and Britain, but unsuccessfully.

14 The Demidov family were Russian industrial magnates of the 18^{th} and 19^{th} centuries. The family established a foundation in support of science and education. The Second grade prize amounted to 2500 Rubles.

Summing up, Słonimski's machine was a simple device, whose construction was based on a theorem in number theory. This theorem, named after its inventor, enabled Słonimski to arrange a table of numbers, which was the basis of construction for the calculating machine. Thanks to the theorem Słonimski's machine had a very simple construction and was cheap. At that time only a few calculating machines existed which were based on such a good theoretical background. That was the "mathematical art" of the device, but unfortunately the machine did not survive to our times.

4. The machine of Izrael Abraham Staffel

Another machine which did not survive to the present day is an invention of a clockmaker from Warsaw, Abraham Staffel. ABRAHAM IZRAEL STAFFEL (1814-1885) was born in Warsaw. At the age of only 19 he opened a clockmaster's shop, where he worked till his death. Most of his life Staffel spent on developing various inventions. He designed an automatic taximeter for cabs which was controlled automatically: it started during the getting on of passengers and stopped after their getting off. In 1851 Staffel presented at an exhibition in London a probe for determining the contents of alloys based on Archimedes law. It was used for testing the authenticity of coins. Staffel designed also: an anemometer (which besides showing the direction also measured the force of the wind), a device for destroying locusts, a press for printing multicolor stamps, a machine for preventing the forging of documents and securities, a series of fans (or rather air conditioning) installed in many buildings, hospitals, and in The Royal Castle in Warsaw. Staffel was also the designer of a "small amusing underground train going from the kitchen to the dining room". Despite the fact that he was a well-known and appreciated inventor, he had financial troubles all his life. Staffel died in poverty after a long disease in 1885.

Abraham Staffel designed and built also calculating machines. For the first time he presented a machine for four basic arithmetical operations, exponentiation, and extracting square roots, in 1845 at the industrial exhibition in Warsaw. Unfortunately this machine didn't survive to our times, so its construction and way of performing operations can be found only in the contemporary press and in reports on exhibitions.

The mechanism of the machine Staffel put in a box sized 20 inches $\times$ 10 inches $\times$ 8 inches. There were 13 apertures for showing the digits of the result on the case. Below, there was a cylinder with 7 rollers placed on it. There were apertures for putting numbers on the rollers. At the bottom of the machine there were 7 apertures for setting the digits of the multiplier and for showing the result of division. On the cover of the machine there were also a crank and a hand to select the type of operation (by setting the handle on one of the inscriptions: *extractio*, *substractio/divisio and additio/multiplio*).

The modern (for those days) construction of the machine enabled performing not only simple calculations but also calculating more complicated expressions like, for example:[15]

To calculate the above expression the operator put number a on the rollers of the machine, turned the hand on the engraving *substractio/divisio* and turned the crank. Then he put number b on the rollers and turned the crank and finally put number c on the same place. After turning the crank the sum $a + b + c$ was visible in the upper row of apertures. To continue calculation the operator put the hand on the inscription *substractio/divisio* and putting numbers d and e one by one turned the crank in the opposite direction than in the case of addition. The partial result appeared in the upper row of apertures. Then the operator put the hand on the *substractio/divisio* and performed to calculate the product in the bracket. In order to do that he set number g on the cylinder and the number h in the lower row of seven apertures and turned the crank until the digits in the apertures of the lower row all became zeros. After that the value of the expression appeared in the upper row of apertures. Then, after setting the hand on the *substractio/divisio* the operator calculated in the same way the product $m \times m$ and obtained the value of the numerator of the above fraction. To complete the calculation he put the number n on the cylinder and zeros in the lowermost row of apertures (the hand there was still on the inscription *substractio/divisio*). After turning the crank in the lowermost row of apertures the value of whole expression appeared and in the upper row of apertures there was still the value of the numerator of this expression.

Staffel's machine could serve as a tool for extracting square roots. To calculate the square root of a given number, the operator set this number in the upper row of apertures, zeros on the cylinder, zeros in the lowermost row except the position of units in the row where number 1 was set and put in the handle on the inscription *extractio*. No description of performing calculations has survived to our times, but the procedure was probably similar to extracting square roots on Stern's machine.

[15] This example and the description of using the machine was taken from [TI1, 1867].

Staffel introduced a few improvements to his machine. In the case of division, if the result wasn't the integral then the value of the numerator of the result was in the upper row of apertures and the denominator was in the lower one. In the machine a ring was built which indicated a mistake in the case when the result of subtraction was negative (when the numbers were set in the wrong order) and when the divisor is bigger than the dividend while performing division.

The machine of Abraham Staffel was presented in at least three exhibitions. At the exhibition in Warsaw, which was mentioned above, Staffel received a silver medal for his invention. Articles in contemporary newspapers pointed out that the machine considerably shortened the calculation (tables comparing the time of calculations in seconds were published). In 1846 Staffel presented his machine to the Russian Academy of Sciences in St. Petersburg. Two famous mathematicians, V. Bunyakovski and B. Jacobi, gave it a very positive opinion and Staffel was awarded a Demidov prize amounting to 1500 rubles[16] but he never patented it. The machine was also presented at The Great Exhibition in London in 1851 in one group with the arithmometer of Xavier Thomas de Colmar. Two machines were awarded: Staffel's and Colmar's. After the London Exhibition a brief note in Scientific American appeared saying [SA1, 1851]:

> An extraordinary calculating machine, says the London Times, is now placed in the Russian Court. It is the invention of a Polish Jew, named Staffel, a native of Warsaw, and works addition, subtraction, multiplication and division, with a rapidity and precision that are quite astonishing.

At the end of his life Staffel handed over his invention to the Russian Academy of Science. After the collapse of tsarism the collection of Academy broke down. Probably Staffel's machine was destroyed then and did not survive to our times.

5. Conclusion

The calculating machines described above, despite the recognition they gained, were not mass-produced. The following question which arises is: why were these machines not universally used in manufactures, banks, and scientific institutions, as happened in the case of Xavier Thomas de Colmar's arithmometers? Probably the main reason was the complexity of the

[16] Compare with the prize which Słonimski received.

machines' construction. The production of single parts of the machines was very problematic and expensive, so would result in very high prices for the machines. Moreover the machines described above were invented by Jewish artisans living in the region of Poland under the rule of Russia who had not much freedom of activity and not enough money to realize their ideas.

Abraham Stern, Chaim Słonimski, and Izrael Staffel were Jewish artisans who lived at the same time in the same place. Perhaps it was no coincidence. It should be supposed that there was a dynamic Jewish artisan community in Warsaw in the 19th century, though there is no evidence for personal relations either between Staffel and Stern or between Staffel and Słonimski. There is also no evidence that Staffel investigated Stern's or Słonimski's machines.

It is worth underlining that inventors in Polish territory (particularly the Jews) very rarely traveled abroad and therefore did not have contact with inventions from Western Europe at that time. As was mentioned above, the Jews had very limited access to the knowledge of technical and scientific achievements (most of them could read neither in Polish nor in Russian). In spite of that, these Jewish artisans succeeded in creating numerous interesting inventions including calculating machines, which were comparable with calculating machines produced in Western Europe at the time. Their construction was rewarded during exhibitions in the Russian Empire and abroad. Stern's, Słonimski's, and Staffel's calculating machines were not the oldest constructions of such type in Europe, but taking into account the hard conditions in which they worked they can be considered as forming an interesting part of the history of mechanical computing in Europe.[17]

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Izabela Bondecka-Krzykowska
Adam Mickiewicz University in Poznań
Faculty of Mathematics and Computer Science
ul. Umultowska 87, 61–614 Poznań, Poland
e-mail: `izab@amu.edu.pl`

Joanna Golińska-Pilarek and Taneli Huuskonen

LOGIC OF DESCRIPTIONS. A NEW APPROACH TO THE FOUNDATIONS OF MATHEMATICS AND SCIENCE

Abstract. We study a new formal logic LD introduced by Prof. Grzegorczyk. The logic is based on so-called *descriptive equivalence*, corresponding to the idea of shared meaning rather than shared truth value. We construct a semantics for LD based on a new type of algebras and prove its soundness and completeness. We further show several examples of classical laws that hold for LD as well as laws that fail. Finally, we list a number of open problems.

Keywords: non-classical logic, logic of descriptions, equivalence connective, paradoxes of implication

1. Introduction

Logic arose from philosophical and linguistic reflections that began in ancient Greece and later spread throughout Europe. In the 20$^{\text{th}}$ century, *formal logical systems*, especially for classical two-valued logic, achieved perfection and became the gold standard in the foundations of mathematics (and of science in general). However, every now and then a philosopher, a logician or a mathematician has expressed doubts and objections concerning this standard. These objections have been made on various grounds, and many so-called "non-classical" logics have been proposed to rectify the perceived faults, such as modal, intuitionistic, conditional, relevant, paraconsistent, free, quantum, fuzzy, independence-friendly, and so on. Nevertheless, none of these logics has been generally accepted as the right one, and a resolution to the arguments about their practical and philosophical merits and drawbacks is nowhere in sight.

Historically, logic was born out of attempts to explain the structure of human reasoning. It should be emphasized that ancient logicians did not aspire to create an abstract model of human thought, akin to modern attempts at passing the Turing test. The lofty goal of their reflections on

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logical principles was to find their way into the very essence of reality. This goal, however utopian it may have been, was consistently pursued by the philosophers who brought forth the logic revolution. However, the invention of formal methods in mathematics – and hence a means to achieve unprecedented rigour – led to logic being "taken over" by engineers and computer scientists. This observation is not meant as a criticism. The results obtained in the field of mathematical logic, as well as its fruitful applications in information technology are impressive indeed. Nevertheless, despite the great success of logic in these areas, we may still ask whether the formal systems commonly used in mathematical logic can serve as adequate tools for understanding human reasoning.

Prof. Grzegorczyk treats this question in his recent article [2011], which can be described as a manifesto calling for the creation of new logical principles suitable for *scientific description of reality* and for the revision of the current standard; that is, various versions of classical two-valued logic.

One of Prof. Grzegorczyk's objections to classical logic is the fact that it "restricts itself to considering only one, admittedly the most important, parameter of the content of a claim, namely its *truth value*" [2011, p. 446], which – as the author points out – is the source of the paradoxical nature of certain tautologies involving implication and equivalence, such as *false implies everything, anything implies the truth, any true sentences are equivalent regardless of their content.* As the author explains, such tautologies are useful in formal deductions in the technical sense, but do not otherwise contribute to understanding. Of course, the paradoxes of material implication have been widely discussed elsewhere, and several non-classical logics have been created in order to solve them. However, the main point of the paper does not involve material implication as such, but rather the problematic nature of material equivalence, which is clearly seen in the following example. In mathematics, one may say that the equations $x + 2 = 3$ and $1-x = 0$ "mean the same" or "say the same thing in different ways" because they are logically equivalent; that is, their truth values are the same for any given value of x. On the other hand, any two true propositions are equivalent to each other. So, if we consistently speak of equivalent propositions as "meaning the same", we end up claiming that "$2 + 2 = 4$" means the same as "Warsaw lies on the Vistula river." According to Prof. Grzegorczyk, this shows that

> Such combined propositions, established as equivalences according to classical formal logic, do not lead to anything interesting. The paradoxical nature of the classical concept of equivalence arises because we may expect equivalence

> to open the possibility of linking the content of one claim with that of the other. We would like equivalent sentences not only to be equally true, but also TO SPEAK ABOUT THE SAME SUBJECT. It seems (from a philosophical point of view) that claims that are not connected by a common subject cannot be treated as fully equivalent. [2011, p. 447]

To remedy the ills of classical equivalence and to avoid the above-mentioned paradox, we should carefully distinguish between two kinds of equivalence, which are:

1. "truth-functional equivalence" – the condition that the truth values of two propositions are the same; this is classical equivalence $\leftrightarrow$ ("coarse, even cynically paradoxical").
2. "descriptive equivalence"[1] – the condition that the meanings of two propositions are the same ("more subtle, but not totally determined, allowing for an intuitive interpretation of being connected by a shared subject").

Introducing a new connective involves describing its usage, which naturally leads one to consider a new logical formalism in which the classical equivalence connective $\leftrightarrow$ has been replaced with a descriptive equivalence connective, which, according to Prof. Grzegorczyk, better reflects human ways of thinking.

We will use the symbol $\equiv$ to denote the new connective. The new logic, denoted here by LD, is defined by rules of inference and a set of axioms. In the article [Grzegorczyk, 2011], a number of important questions concerning the new logic are raised. Firstly, do the new equivalence and the corresponding implication coincide with their respective classical counterparts? Secondly, can we define a semantics for which the given syntactic proof system is sound and complete?

In the present article, we analyze the logic LD as presented in [Grzegorczyk, 2011]. We will show that the descriptive equivalence connective is indeed different from the classical one. In fact, our further results show that the new logic is substantially different from many other known ones, representing various kinds of non-classical logics. One of our central results is the construction of a semantics for LD, which, in turn, allows us to prove further, rather peculiar, properties of LD, shedding some light on the obscure secrets of descriptive equivalence.

[1] In a draft version of the article [Grzegorczyk, 2011], the new connective is called the *equimeaning* connective, while in its final, published version, it is named the *perceptive equivalence* connective. Later on, Prof. Grzegorczyk came to the conclusion that the most fitting name for the new connective is the *descriptive equivalence* connective, and this is the terminological convention we will follow.

It is worth noting that the distinction between classical and descriptive equivalence is essentially the same as the distinction between the truth value and the meaning of a sentence. Roman Suszko, among others, argued for the need, and even necessity, to consider the latter distinction when building the semantical basis for a logical system, and he introduced so-called *non-Fregean logic* as a formalization of this idea (see [Suszko, 1968]). The sentential version of non-Fregean logic (called *Sentential Calculus with Identity*, SCI) is obtained from classical sentential logic by adding a new *identity connective*. Suszko's philosophical motivations for creating SCI were similar to those of Prof. Grzegorczyk for creating LD. By coincidence, both of them chose the symbol $\equiv$ for essentially the same purpose: to denote descriptive equivalence in LD and identity of sentences in SCI. However, their intuitions and underlying philosophical assumptions have led to two very different formalisms. The logic SCI is based on classical logic, which is simply extended by adding new axioms expressing the properties of sentential identity. On the other hand, LD is built from the ground up to reflect the interactions between descriptive equivalence and the basic connectives of negation, disjunction and conjunction, introducing counterparts of many classical laws involving equivalence and omitting others. Nevertheless, the logics of Suszko and Grzegorczyk have many common elements, which can be seen especially in our construction of a semantics for LD.

The paper is organized as follows. In Section 2, we present the language and the Hilbert-style axiomatization of LD with examples of LD-provable formulas. We present a semantics and prove its soundness and completeness in Section 3. In Section 4, we discuss some interesting properties of LD, in particular classical results that fail for LD and the independence of the axioms. In Section 5, we study two proposed alternative formulations of LD, showing how they fail to fulfill the philosophical motivations behind LD. Conclusions and open problems are presented in Section 6. To avoid cluttering the main text with excessive tables, we present most example models in the Appendix.

Our results rely heavily on computer software for semi-automatic proof generation and model checking, written by the second author. The software and associated files are available upon request.

2. Logic LD: axiomatization

Logic LD belongs to the family of propositional logics. The vocabulary of the logic LD consists of the following pairwise disjoint sets of symbols:

- $\mathbb{V} = \{p_0, p_1, p_2, \ldots\}$ – an infinite countable set of propositional variables,
- $\{\neg, \vee, \wedge, \equiv\}$ – propositional operations of negation $\neg$, disjunction $\vee$, conjunction $\wedge$, and descriptive equivalence $\equiv$.

In practice, we will use the symbols p, q, r instead of the "official" subscripted ones.

As usual in propositional logics, we define the set of LD-formulas as the smallest set that contains all the propositional variables and is closed under all the propositional operations. The logic LD is given by the Hilbert-style axiomatization. Below we list the axioms and rules of inference of LD in their original forms from [Grzegorczyk, 2011].

Axioms:[2]

Ax1 $p \equiv p$

Ax2 $(p \equiv q) \equiv (q \equiv p)$

Ax3 $(p \equiv q) \equiv [(p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r))]$

Ax4 $(p \equiv q) \equiv (\neg p \equiv \neg q)$

Ax5 $(p \equiv q) \equiv [(p \equiv q) \wedge ((p \vee r) \equiv (q \vee r))]$

Ax6 $(p \equiv q) \equiv [(p \equiv q) \wedge ((p \wedge r) \equiv (q \wedge r))]$

Ax7 $(p \vee q) \equiv (q \vee p)$

Ax8 $(p \vee (q \vee r)) \equiv ((p \vee q) \vee r)$

Ax9 $p \equiv (p \vee p)$

Ax10 $(p \wedge q) \equiv (q \wedge p)$

Ax11 $(p \wedge (q \wedge r)) \equiv ((p \wedge q) \wedge r)$

Ax12 $p \equiv (p \wedge p)$

Ax13 $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$

Ax14 $(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$

Ax15 $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$

Ax16 $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$

Ax17 $\neg\neg p \equiv p$

Ax18 $\neg(p \wedge \neg p)$

Observe that among the eighteen axioms of LD, only one axiom, namely Ax18, does not involve the descriptive equivalence connective. Moreover, the rest of the axioms can be divided into three groups, with Ax3 playing a double role. First, the axioms Ax1–Ax3 express the basic properties of

[2] Here we adopt a revised form of the axiom Ax3, presented in Errata 2012 to [Grzegorczyk, 2011]. We will discuss the original form in Section 5.

descriptive equivalence, namely its reflexivity, symmetry, and transitivity. Axioms Ax3–Ax6 formulate the idea that equals can be substituted for each other. Axioms Ax7–Ax17 state some basic properties of equivalence of compound formulas built with the classical connectives of negation, conjunction, and disjunction, which are: associativity, commutativity, and idempotency of conjunction and disjunction (axioms Ax7–Ax12), distributivity of conjunction (resp. disjunction) over disjunction (resp. conjunction) (axioms Ax13 and Ax14), involution of negation that additionally satisfies de Morgan laws (axioms Ax15–Ax17).

Rules of inference:

$$(\mathsf{MP_{LD}}) \quad \frac{\varphi \equiv \psi, \varphi}{\psi} \qquad\qquad (\mathsf{Sub}) \quad \frac{\varphi(p_0, \ldots, p_n)}{\varphi(p_0/\psi_0, \ldots, p_n//\psi_n)}$$

$$(\wedge_1) \quad \frac{\varphi, \psi}{\varphi \wedge \psi} \qquad\qquad (\wedge_2) \quad \frac{\varphi \wedge \psi}{\varphi, \psi}$$

For technical reasons, we impose the additional restriction that the rule (Sub) may be applied only to axioms. As in the classical case, this restriction is not essential when no additional assumptions are used in the proof.

The rules (Sub), $(\wedge_1)$, and $(\wedge_2)$ are standard in classical logic. However, the crucial feature of LD is that it contains a *modus ponens*-type rule only with respect to descriptive equivalence, while the classical modus ponens rule is not present. Furthermore, LD does not have any rule for introduction or elimination of disjunction or negation. As we will show later, only a disjunction introduction rule is derivable in LD.

An LD-formula φ is said to be LD-*provable* ($\vdash \varphi$ for short) whenever there exists a finite sequence $\varphi_1, \ldots, \varphi_n$ of LD-formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and each φ_i, $i \in \{1, \ldots, n\}$, is an axiom or follows from earlier formulas in the sequence by one of the rules of inference. If X is any set of LD-formulas, then φ is said to be LD-*provable from* X ($X \vdash \varphi$ for short) whenever there exists a finite sequence $\varphi_1, \ldots, \varphi_n$ of LD-formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and for each $i \in \{1, \ldots, n\}$, φ_i is an axiom or $\varphi_i \in X$ or φ_i follows from earlier formulas in the sequence by one of the rules of inference.

Now, it is worth noting that the logic LD is consistent, as the interpretation of $\equiv$ as the usual classical equivalence yields that all the axioms are classical tautologies and all the rules preserve classical validity. Hence, one of all possible models of LD is the two-element Boolean algebra of the classical propositional logic. This implies also that the formula $p \equiv \neg p$ is not LD-provable.

The axiomatization of LD enables us to prove many of the classical laws, but not all. In particular, the formula $\varphi \vee \neg\varphi$ is LD-provable.

$\vdash (\varphi \vee \neg\varphi)$, for any LD-formula φ.

(1) $\neg(\varphi \wedge \neg\varphi)$ — (Sub) to $\mathsf{Ax18}$ for p/φ

(2) $\neg(\varphi \wedge \neg\varphi) \equiv (\neg\varphi \vee \neg\neg\varphi)$ — (Sub) to $\mathsf{Ax16}$ for p/φ, $q/\neg\varphi$

(3) $\neg\varphi \vee \neg\neg\varphi$ — $(\mathsf{MP_{LD}})$ to (1) and (2)

(4) $(\neg\varphi \vee \neg\neg\varphi) \equiv (\neg\neg\varphi \vee \neg\varphi)$ — (Sub) to $\mathsf{Ax7}$ for $p/\neg\varphi$, $q/\neg\neg\varphi$

(5) $\neg\neg\varphi \vee \neg\varphi$ — $(\mathsf{MP_{LD}})$ to (3) and (4)

(6) $\neg\neg\varphi \equiv \varphi$ — (Sub) to $\mathsf{Ax17}$ for p/φ

(7) $(\neg\neg\varphi \equiv \varphi) \equiv [(\neg\neg\varphi \equiv \varphi) \wedge ((\neg\neg\varphi \vee \neg\varphi) \equiv (\varphi \vee \neg\varphi))]$
— (Sub) to $\mathsf{Ax5}$ for $p/\neg\neg\varphi$, q/φ, r/φ

(8) $(\neg\neg\varphi \equiv \varphi) \wedge ((\neg\neg\varphi \vee \neg\varphi) \equiv (\varphi \vee \neg\varphi))$ — $(\mathsf{MP_{LD}})$ to (6) and (7)

(9) $(\neg\neg\varphi \vee \neg\varphi) \equiv (\varphi \vee \neg\varphi)$ — $(\wedge_2)$ to (8)

(10) $\varphi \vee \neg\varphi$ — $(\mathsf{MP_{LD}})$ to (5) and (9)

Also the following formula is provable in LD:

$$\neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi^{3}.$$

Below we sketch its proof.

$\vdash \neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi$, for all LD-formulas φ and ψ.

(1) $(\neg\varphi \vee \psi) \vee \neg(\neg\varphi \vee \psi)$ — (Sub) to $\mathsf{Ax18}$ for $p/(\neg\varphi \vee \psi)$

(2) $(\neg\varphi \vee \neg(\neg\varphi \vee \psi)) \vee \psi$ — from (1) by $\mathsf{Ax7}$ and $\mathsf{Ax8}$

(3) $\neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi$ — from (2) by $\mathsf{Ax16}$

It is also easy to see that the following rule is a *derived* LD-rule:

$$(\mathsf{tran}) \qquad \frac{\varphi \equiv \psi, \psi \equiv \vartheta}{\varphi \equiv \vartheta},$$

which can be proved as follows:

(1) $\varphi \equiv \psi$

(2) $\psi \equiv \vartheta$

(3) $(\varphi \equiv \psi) \equiv [(\varphi \equiv \psi) \wedge ((\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta))]$
— (Sub) to $\mathsf{Ax3}$ for p/φ, q/ψ, r/ϑ

(4) $(\varphi \equiv \psi) \wedge ((\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta))$ — $(\mathsf{MP_{LD}})$ to (1) and (3)

(5) $(\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta)$ — $(\wedge_2)$ to (4)

[3] If the usual definition of the classical implication is assumed, that is, $(p \to q) \equiv (\neg p \vee q)$, then the formula $\neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi$ can be abbreviated as $(\varphi \wedge (\varphi \to \psi)) \to \psi$.

(6) $[(\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta)] \equiv [(\psi \equiv \vartheta) \equiv (\varphi \equiv \vartheta)]$
(Sub) to Ax2 for $p/(\varphi \equiv \vartheta)$, $q/(\psi \equiv \vartheta)$

(7) $(\psi \equiv \vartheta) \equiv (\varphi \equiv \vartheta)$ $(\mathsf{MP}_{\mathsf{LD}})$ to (5) and (6)

(8) $(\varphi \equiv \vartheta)$ $(\mathsf{MP}_{\mathsf{LD}})$ to (2) and (7)

However, a distinguishing feature of LD is that the algebra of its formulas does not form a Boolean algebra, since neither the absorption laws

$$p \vee (p \wedge q) \equiv p \quad \text{and} \quad p \wedge (p \vee q) \equiv p,$$

nor the boundness laws

$$(p \vee \neg p) \equiv (q \vee \neg q) \quad \text{and} \quad \neg(p \wedge \neg q) \equiv \neg(q \wedge \neg q),$$

are adopted as axioms, nor – as we will show later – are they LD-provable. One of the motivations for not allowing these laws is that we do not wish to treat two tautologies as identical in meaning if their contents are completely different. For instance, the sentences, “Professor Grzegorczyk is in the next room or he is not there” and “President Obama is in the next room or he is not there” are both tautologies and hence logically equivalent, but they are derived from claims concerning different persons, so their contents are different. To quote Prof. Grzegorczyk: “Of course, mathematicians do not concern themselves with anything outside imagined reality, where everything consistent is acceptable. They may thus consider all tautologies to have the same meaning. However, a philosopher ought to be more careful.”[4]

A further interesting feature of the logic LD is concerned with possible derived rules allowed in LD. As we showed, the formula $\neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi$ is provable in LD, but the corresponding rule, that is, the classical rule of modus ponens:

$$\frac{\varphi, (\neg\varphi \vee \psi)}{\psi},$$

is not derivable in LD in the sense that the premises may be satisfied by a valuation that fails to satisfy the conclusion, as we will show in Section 4.

We end this section with a short discussion of the redundancy of the LD-axiomatization presented above. It can be easily seen that some LD-axioms are redundant, since they follow from the others. For example, the reflexivity of $\equiv$ follows from its symmetry and transitivity and the fact that every formula is equivalent to some other formula. The proof can be formalized easily enough.

[4] Personal communication, 2012-01-21

(1) $\neg\neg p \equiv p$ Ax17

(2) $(\neg\neg p \equiv p) \equiv ((\neg\neg p \equiv p) \wedge ((\neg\neg p \equiv p) \equiv (p \equiv p)))$
(Sub) to Ax3 $p/\neg\neg p, q/p, r/p$

(3) $(\neg\neg p \equiv p) \wedge ((\neg\neg p \equiv p) \equiv (p \equiv p))$ $\mathsf{MP}_{\mathrm{LD}}$ to (1) and (2)

(4) $(\neg\neg p \equiv p) \equiv (p \equiv p)$ $(\wedge_2)$ to (3)

(5) $p \equiv p$ $\mathsf{MP}_{\mathrm{LD}}$ to (1) and (4)

Moreover, it seems intuitively plausible that Ax5 and Ax6 should be provable from each other using DeMorgan laws. This is indeed the case, even though the formal proofs turn out to be quite lengthy. In the same way, we can eliminate half of the axioms Ax7 through Ax16. More precisely, let $\mathsf{LD}_{\mathtt{red}}$ be obtained from LD by removing Ax1, Ax5, Ax7, Ax8, Ax9, Ax14, and Ax16. Then, the following holds:

Proposition 1

The axioms Ax1, Ax5, Ax7, Ax8, Ax9, Ax14, and Ax16 are provable in $\mathsf{LD}_{\mathtt{red}}$.

Proof.

Ax1: Already proved above.
Ax5: We have

$$\begin{aligned}(p \equiv q) &\equiv (\neg p \equiv \neg q)\\ &\equiv [(\neg p \equiv \neg q) \wedge ((\neg p \wedge \neg r) \equiv (\neg q \wedge \neg r))]\\ &\equiv [(\neg p \equiv \neg q) \wedge (\neg(p \vee r) \equiv \neg(q \vee r))]\\ &\equiv [(p \equiv q) \wedge ((p \vee r) \equiv (q \vee r))].\end{aligned}$$

Even though this outline seems simple and convincing enough, as it is easy to check that each step involves only substitutions that are directly justified by axioms and do not occur in the scope of a disjunction, expanding it to a full formal proof is surprisingly tedious. Our computer-generated proof consists of 8 axioms and 75 applications of rules. One could plausibly find a significantly shorter proof, but we assume it would still consist of several dozen lines, as intuitively obvious substitutions sometimes require rather complicated formal manipulations.

We will prove Ax16 next to be able to use it in the remaining proofs.

Ax16:
$$\begin{aligned}\neg(p \wedge q) &\equiv \neg(\neg\neg p \wedge \neg\neg q)\\ &\equiv \neg\neg(\neg p \vee \neg q)\\ &\equiv (\neg p \vee \neg q).\end{aligned}$$

$$
\begin{array}{lrcl}
\mathsf{Ax7}: & (p \vee q) & \equiv & \neg\neg(p \vee q) \\
 & & \equiv & \neg(\neg p \wedge \neg q) \\
 & & \equiv & \neg(\neg q \wedge \neg p) \\
 & & \equiv & \neg\neg(q \vee p) \\
 & & \equiv & (q \vee p). \\
\mathsf{Ax8}: & (p \vee (q \vee r)) & \equiv & \neg\neg(p \vee (q \vee r)) \\
 & & \equiv & \neg(\neg p \wedge \neg(q \vee r)) \\
 & & \equiv & \neg(\neg p \wedge (\neg q \wedge \neg r)) \\
 & & \equiv & \neg((\neg p \wedge \neg q) \wedge \neg r) \\
 & & \equiv & \neg(\neg(p \vee q) \wedge \neg r) \\
 & & \equiv & \neg\neg((p \vee q) \vee r) \\
 & & \equiv & ((p \vee q) \vee r). \\
\mathsf{Ax9}: & (p \equiv (p \vee p)) & \equiv & (\neg p \equiv \neg(p \vee p)) \\
 & & \equiv & (\neg p \equiv (\neg p \wedge \neg p)),
\end{array}
$$

and the last equivalence is obtained from Ax12 by substitution.

$$
\begin{array}{lrcl}
\mathsf{Ax14}: & ((p \vee (q \wedge r)) & \equiv & ((p \vee q) \wedge (p \vee r))) \\
 & & \equiv & (\neg(p \vee (q \wedge r)) \equiv \neg((p \vee q) \wedge (p \vee r))) \\
 & & \equiv & ((\neg p \wedge \neg(q \wedge r)) \equiv (\neg(p \vee q) \vee \neg(p \vee r))) \\
 & & \equiv & ((\neg p \wedge (\neg q \vee \neg r)) \equiv ((\neg p \wedge \neg q) \vee (\neg p \wedge \neg r))),
\end{array}
$$

and again, the last element of the equivalence chain is obtained by substitution, this time from Ax13. ■

The mostly computer-generated full formal proofs and program sources are available from the authors upon request. In Section 4 we will show that all axioms of $\mathsf{LD}_{\mathtt{red}}$ are independent from each other.

3. Logic LD: semantics

The logic LD is originally given by Hilbert-style axiomatization. So, a natural problem to be solved is to provide a sound as well as *complete* semantics for LD. In this section, on the basis of some modifications of non-Fregean models for the logic SCI, we define a suitable class of structures for LD, and then we prove its soundness and completeness. First, we introduce some useful notions.

A structure $(U, \oplus, \otimes)$ is said to be a *distributive bisemilattice* whenever the following hold, for all $a, b, c \in U$ and for any $\odot \in \{\otimes, \oplus\}$:

- $a \odot b = b \odot a$, (commutativity of $\otimes, \oplus$)
- $a \odot (b \odot c) = (a \odot b) \odot c$, (associativity of $\otimes, \oplus$)
- $a \odot a = a$, (idempotency of $\otimes, \oplus$)
- $a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c)$, (distributivity of $\otimes$ over $\oplus$)
- $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$. (distributivity of $\oplus$ over $\otimes$)

A *de Morgan bisemilattice* is a structure $(U, \sim, \oplus, \otimes)$ such that $(U, \oplus, \otimes)$ is a distributive bisemilattice and for all $a, b \in U$, the following hold:

- $\sim\sim a = a$,
- $\sim(a \oplus b) = \sim a \otimes \sim b$.

A *Grzegorczyk algebra* is a structure $(U, \sim, \oplus, \otimes, \circ)$ such that $(U, \sim, \oplus, \otimes)$ is a de Morgan bisemilattice and for all $a, b, c \in U$, the following hold:

- $a \circ b = b \circ a$,
- $a \circ b = \sim a \circ \sim b$,
- $a \circ b = (a \circ b) \otimes ((a \circ c) \circ (b \circ c))$,
- $a \circ b = (a \circ b) \otimes ((a \oplus c) \circ (b \oplus c))$,
- $a \circ b = (a \circ b) \otimes ((a \otimes c) \circ (b \otimes c))$.

Fact 2

A structure $(U, \sim, \oplus, \otimes, \circ)$ is a Grzegorczyk algebra if and only if the following conditions hold, for all $a, b, c \in U$:

(LD1) $a \circ b = b \circ a$,

(LD2) $a \circ b = (a \circ b) \otimes ((a \circ c) \circ (b \circ c))$,

(LD3) $a \circ b = \sim a \circ \sim b$,

(LD4) $a \circ b = (a \circ b) \otimes ((a \oplus c) \circ (b \oplus c))$,

(LD5) $a \circ b = (a \circ b) \otimes ((a \otimes c) \circ (b \otimes c))$,

(LD6) $a \oplus b = b \oplus a$,

(LD7) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(LD8) $a \oplus a = a$,

(LD9) $a \otimes b = b \otimes a$,

(LD10) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$,

(LD11) $a \otimes a = a$,

(LD12) $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$,

(LD13) $a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c)$,

(LD14) $\sim(a \oplus b) = \sim a \otimes \sim b$,

(LD15) $\sim(a \otimes b) = \sim a \oplus \sim b$,

(LD16) $\sim\sim a = a$.

It is worth emphasizing the following fact:

Fact 3

Boolean algebras, Kleene algebras, and de Morgan algebras are Grzegorczyk algebras.

However, the converse of the above does not hold. The class of Grzegorczyk algebras is quite extensive and contains subclasses that form bases for semantics of various non-classical logics of different types. Grzegorczyk algebras will be a basis for structures of LD.

An LD-*structure* is of the form $(U, \sim, \oplus, \otimes, \circ, D)$, where:

- U, D are non-empty sets such that $D \subseteq U$,
- $(U, \sim, \oplus, \otimes, \circ)$ is a Grzegorczyk algebra,
- For all $a, b \in U$, the following hold:
  $(a \otimes b) \in D$ if and only if $a \in D$ and $b \in D$,
  $(a \circ b) \in D$ if and only if $a = b$,
  $\sim(a \otimes \sim a) \in D$ and $(a \otimes \sim a) \notin D$.

Let $\mathcal{M} = (U, \sim, \oplus, \otimes, \circ, D)$ be an LD-structure. A *valuation* on $\mathcal{M}$ is any mapping $v\colon \mathbb{V} \to U$ such that for all LD-formulas φ and ψ:

- $v(\neg\varphi) = \sim v(\varphi)$,
- $v(\varphi \wedge \psi) = v(\varphi) \otimes v(\psi)$,
- $v(\varphi \vee \psi) = v(\varphi) \oplus v(\psi)$,
- $v(\varphi \equiv \psi) = v(\varphi) \circ v(\psi)$.

A formula φ is said to be *satisfied in an* LD-*structure by a valuation* v if and only if $v(\varphi) \in D$. It is *true in* $\mathcal{M}$ whenever it is satisfied in $\mathcal{M}$ by all the valuations on $\mathcal{M}$, and it is LD-*valid* if it is true in all LD-structures.

We may think of LD-structures as variants of non-Fregean structures, introduced by Suszko in [Suszko, 1971]. The universe of a non-Fregean structure consists of the correlates of sentences in a given language, and its elements are known as *situations* or *states of affairs.* The correlates of true sentences are *factual situations* and the correlates of false sentences (ones whose negations are true) are *counterfactual situations.* Unlike Suszko, we do not insist on logical two-valuedness but allow for sentences that are neither true nor false; their correlates are *undetermined situations.* The set D is the set of factual situations.

Example 4

Let $U = \{0, 1, 2, 3\}$, $D = \{2, 3\}$, and define the operations as follows:

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	3	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	3

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	1	2	3
3	0	0	3	3

$\oplus$	0	1	2	3
0	0	0	3	3
1	0	1	2	3
2	3	2	2	3
3	3	3	3	3

It can be verified that the above tables indeed define a Grzegorczyk algebra, but the absorption laws do not hold, as $1 \oplus (1 \otimes 0) = 1 \otimes (1 \oplus 0) = 0 \neq 1$, for instance.

First, we will prove that LD is sound with respect to the class of all LD-structures as defined above. Thus, we need to show that all LD-axioms are LD-valid and all LD-rules preserve LD-validity. To be more precise, a rule of the form $\frac{\varphi_1,\ldots,\varphi_n}{\psi_1,\ldots,\psi_m}$, for $n, m \leq 2$, is called *weakly* LD*-correct* whenever the LD-validity of $\varphi_1, \ldots, \varphi_n$ implies the LD-validity of $\psi_1, \ldots, \psi_m$, and *strongly* LD*-correct* whenever for every LD-structure $\mathcal{M}$ and every valuation v on $\mathcal{M}$ such that $\mathcal{M}, v \models \varphi_1, \ldots, \varphi_n$, it holds that $\mathcal{M}, v \models \psi_1, \ldots, \psi_m$.

Proposition 5

All the LD-rules except (Sub) are strongly LD-correct. Moreover, (Sub) is weakly LD-correct.

Proof.

The proofs of the weak correctness of (Sub) and the strong correctness of $(\wedge_1)$ and $(\wedge_2)$ are easy to carry out. So by way of example, we will show the strong correctness of the rule $(\mathsf{MP}_{\mathrm{LD}})$.

Let $\mathcal{M} = (U, \sim, \oplus, \otimes, \circ, D)$ be an LD-structure and v a valuation on $\mathcal{M}$ such that $\mathcal{M}, v \models \varphi \equiv \psi$ and $\mathcal{M}, v \models \varphi$. By the assumption, $v(\varphi \equiv \psi) \in D$ and $v(\varphi) \in D$. Since $v(\varphi \equiv \psi) \in D$, we have also $v(\varphi) \circ v(\psi) \in D$, so $v(\varphi) = v(\psi)$. Therefore $v(\psi) \in D$. Hence, the rule $\mathsf{MP}_{\mathrm{LD}}$ is strongly LD-correct. ■

Proposition 6

All the LD-axioms are LD-valid.

Proof.

Let $\mathcal{M} = (U, \sim, \oplus, \otimes, \circ, D)$ be an LD-structure and let v be a valuation on $\mathcal{M}$. We need to show that if ϕ is an LD-axiom, then $v(\phi) \in D$.

First, by the definition of an LD-structure, we have the following: $v(p) = v(p)$ iff $v(p) \circ v(p) \in D$ iff $v(p \equiv p) \in D$. Hence, the axiom $\mathsf{Ax2}$ is LD-valid.

By Fact 2(LD1), for all $a, b \in U$, $(a \circ b) = (b \circ a)$. Therefore, $(v(p) \circ v(q)) = (v(q) \circ v(p))$. On the other hand, by the definition of an LD-structure, we have also: $(v(p) \circ v(q)) = (v(q) \circ v(p))$ iff $(v(p) \circ v(q)) \circ (v(q) \circ v(p)) \in D$ iff $v((p \equiv q) \equiv (q \equiv p)) \in D$. Thus, axiom Ax2 is LD-valid. In a similar way, we can prove LD-validity of axioms Ax3–Ax16. Generally, for $i \in \{3, \dots, 17\}$, LD-validity of axiom Axi follows from condition (LDi-1) of Fact 2. Furthermore, by the definition of an LD-structure, axiom Ax18 is obviously LD-valid. ■

Propositions 5 and 6 yield soundness of the logic LD with respect to the class of all LD-structures:

Proposition 7 (Soundness of LD)

Every LD-provable formula is LD-valid.

Now, we will proceed to completeness. Let R be the following binary relation on the set of all LD-formulas:

$$\varphi \, R \, \psi \text{ if and only if } \varphi \equiv \psi \text{ is provable in LD.}$$

Fact 8

The relation R is an equivalence relation on the set of all LD-formulas. Moreover, R is compatible with all LD-connectives.

Proof.

Let φ, ψ, ϑ be any LD-formulas. Clearly, $\varphi \equiv \varphi$ is provable in LD, so R is reflexive. Assume $\varphi \, R \, \psi$, that is, $\varphi \equiv \psi$ is provable in LD. By axiom Ax2, $((\varphi \equiv \psi) \equiv (\psi \equiv \varphi))$ is provable in LD. Thus, by the assumption, so is $\psi \equiv \varphi$, which implies $\psi \, R \, \varphi$. Hence, R is symmetric. Now, assume that $\varphi \, R \, \psi$ and $\psi \, R \, \vartheta$, which means that $\varphi \equiv \psi$ and $\psi \equiv \vartheta$ are provable in LD. Since R is symmetric, $\psi \, R \, \varphi$, so $\psi \equiv \varphi$ is provable in LD. By axiom Ax3, LD proves:

$$(\psi \equiv \varphi) \equiv [(\psi \equiv \varphi) \wedge ((\psi \equiv \vartheta) \equiv (\varphi \equiv \vartheta))].$$

Applying the rule $(\mathsf{MP}_{\mathrm{LD}})$, and then $(\wedge_2)$, we obtain that LD proves:

$$(\psi \equiv \vartheta) \equiv (\varphi \equiv \vartheta).$$

Applying again the rule $(\mathsf{MP}_{\mathrm{LD}})$ and the assumption $\psi \equiv \vartheta$, we have that $\varphi \equiv \vartheta$ is provable in LD, which implies $\varphi \, R \, \vartheta$. Hence, R is transitive.

Assume then that $\varphi_1 \, R \, \varphi_2$ and $\psi_1 \, R \, \psi_2$. Then $\neg\varphi_1 \, R \, \neg\varphi_2$ by Ax4, so R is compatible with $\neg$. Moreover,

$$\varphi_1 \equiv \psi_1 \, R \, \varphi_2 \equiv \psi_1 \, R \, \psi_1 \equiv \varphi_2 \, R \, \psi_2 \equiv \varphi_2 \, R \, \varphi_2 \equiv \psi_2,$$

so by the transitivity of R, it follows that R is also compatible with $\equiv$. The proofs of compatibility with $\wedge$ and $\vee$ are similar. ■

By the above fact, we can define a structure $\mathcal{M}^{\mathrm{LD}} = (U, \sim, \oplus, \otimes, \circ, D)$ as follows:

- $U = \{\, |\varphi|_R : \varphi$ is an LD-formula$\,\}$, that is, U is the set of equivalence classes of R on the set of all LD-formulas,
- $D = \{\, |\varphi|_R : \varphi$ is provable in LD$\,\}$, that is, D is the set of equivalence classes of R on the set of all provable formulas in LD,
- For all $|\varphi|_R, |\psi|_R \in U$:

$$\sim|\varphi|_R \stackrel{\mathrm{df}}{=} |\neg\varphi|_R, \qquad |\varphi|_R \circ |\psi|_R \stackrel{\mathrm{df}}{=} |\varphi \equiv \psi|_R,$$
$$|\varphi|_R \otimes |\psi|_R \stackrel{\mathrm{df}}{=} |\varphi \wedge \psi|_R, \qquad |\varphi|_R \oplus |\psi|_R \stackrel{\mathrm{df}}{=} |\varphi \vee \psi|_R.$$

Proposition 9

The structure $\mathcal{M}^{\mathrm{LD}}$ is an LD-structure.

Proof.

First, we will show that $\mathcal{M}^{\mathrm{LD}} = (U, \sim, \oplus, \otimes, \circ)$ is a Grzegorczyk algebra. By Fact 2, it suffices to show that $\mathcal{M}^{\mathrm{LD}}$ satisfies all the conditions (LD1), (LD2), ..., (LD16). Let φ, ψ, ϑ be any LD-formulas.

*Proof of (*LD1*)*

$$|\varphi|_R \circ |\psi|_R = |\varphi \equiv \psi|_R = |\psi \equiv \varphi|_R = |\psi|_R \circ |\varphi|_R.$$

Proof of (LD2)

$$\begin{aligned}(|\varphi|_R \circ |\psi|_R) &= |\varphi \equiv \psi|_R \\ &= |(\varphi \equiv \psi) \wedge ((\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta))|_R \\ &= |\varphi \equiv \psi|_R \otimes |(\varphi \equiv vartheta) \equiv (\psi \equiv \vartheta)|_R \\ &= (|\varphi|_R \circ |\psi|_R) \otimes ((|\varphi|_R \circ |\vartheta|_R) \circ (|\psi|_R \circ |\vartheta|_R)).\end{aligned}$$

Proof of (LD3)

$$\begin{aligned}|\varphi|_R \circ |\psi|_R &= |\varphi \equiv \psi|_R = |\neg\varphi \equiv \neg\psi|_R \\ &= |\neg\varphi|_R \circ |\neg\psi|_R = \sim|\varphi|_R \circ \sim|\psi|_R.\end{aligned}$$

Proof of (LD4)

$$\begin{aligned}|\varphi|_R \circ |\psi|_R &= |\varphi \equiv \psi|_R \\ &= |(\varphi \equiv \psi) \wedge ((\varphi \vee \vartheta) \equiv (\psi \vee \vartheta))|_R \\ &= |\varphi \equiv \psi|_R \otimes |(\varphi \vee \vartheta) \equiv (\psi \vee \vartheta)|_R \\ &= (|\varphi|_R \circ |\psi|_R) \otimes ((|\varphi|_R \oplus |\vartheta|_R) \circ (|\psi|_R \oplus |\vartheta|_R)).\end{aligned}$$

Proof of (LD5)

$$\begin{aligned}|\varphi|_R \circ |\psi|_R &= |\varphi \equiv \psi|_R \\ &= |(\varphi \equiv \psi) \wedge ((\varphi \wedge \vartheta) \equiv (\psi \wedge \vartheta))|_R \\ &= |\varphi \equiv \psi|_R \otimes |(\varphi \wedge \vartheta) \equiv (\psi \wedge \vartheta)|_R \\ &= (|\varphi|_R \circ |\psi|_R) \otimes ((|\varphi|_R \otimes |\vartheta|_R) \circ (|\psi|_R \otimes |\vartheta|_R)).\end{aligned}$$

Proof of (LD6)

$$|\varphi|_R \oplus |\psi|_R = |\varphi \vee \psi|_R = |\psi \vee \varphi|_R = |\psi|_R \oplus |\varphi|_R.$$

Proof of (LD7)

$$\begin{aligned}|\varphi|_R \oplus (|\psi|_R \oplus |\vartheta|_R) &= |\varphi \vee (\psi \vee \vartheta)|_R = |(\varphi \vee \psi) \vee \vartheta|_R \\ &= (|\varphi|_R \oplus |\psi|_R) \oplus |\vartheta|_R.\end{aligned}$$

Proof of (LD8)

$$|\varphi|_R \oplus |\varphi|_R = |\varphi \vee \varphi|_R = |\varphi|_R.$$

Proof of (LD9)

$$|\varphi|_R \otimes |\psi|_R = |\varphi \wedge \psi|_R = |\psi \wedge \varphi|_R = |\psi|_R \otimes |\varphi|_R.$$

Proof of (LD10)

$$\begin{aligned}|\varphi|_R \otimes (|\psi|_R \otimes |\vartheta|_R) &= |\varphi \wedge (\psi \wedge \vartheta)|_R = |(\varphi \wedge \psi) \wedge \vartheta|_R \\ &= (|\varphi|_R \otimes |\psi|_R) \otimes |\vartheta|_R.\end{aligned}$$

Proof of (LD11)

$$|\varphi|_R \otimes |\varphi|_R = |\varphi \wedge \varphi|_R = |\varphi|_R.$$

Proof of (LD12)

$$\begin{aligned}|\varphi|_R \otimes (|\psi|_R \oplus |\vartheta|_R) &= |\varphi \wedge (\psi \vee \vartheta)|_R = |(\varphi \wedge \psi) \vee (\varphi \wedge \vartheta)|_R \\ &= (|\varphi|_R \otimes |\psi|_R) \oplus (|\varphi|_R \otimes |\vartheta|_R).\end{aligned}$$

Proof of (LD13)

$$\begin{aligned}|\varphi|_R \oplus (|\psi|_R \otimes |\vartheta|_R) &= |\varphi \vee (\psi \wedge \vartheta)|_R = |(\varphi \vee \psi) \wedge (\varphi \vee \vartheta)|_R \\ &= (|\varphi|_R \oplus |\psi|_R) \otimes (|\varphi|_R \oplus |\vartheta|_R).\end{aligned}$$

Proof of (LD14)

$$\begin{aligned}{\sim}(|\varphi|_R \oplus |\psi|_R) &= {\sim}|\varphi \vee \psi|_R = |\neg(\varphi \vee \psi)|_R = |\neg\varphi \wedge \neg\psi|_R \\ &= |\neg\varphi|_R \otimes |\neg\psi|_R = {\sim}|\varphi|_R \otimes {\sim}|\psi|_R.\end{aligned}$$

Proof of (LD15)

$$\begin{aligned}{\sim}(|\varphi|_R \otimes |\psi|_R) &= {\sim}|\varphi \wedge \psi|_R = |\neg(\varphi \wedge \psi)|_R = |\neg\varphi \vee \neg\psi|_R \\ &= |\neg\varphi|_R \oplus |\neg\psi|_R = {\sim}|\varphi|_R \oplus {\sim}|\psi|_R.\end{aligned}$$

Proof of (LD16)

$$\sim(\sim|\varphi|_R) = \sim|\neg\varphi|_R = |\neg\neg\varphi|_R = |\varphi|_R.$$

Hence, we have shown that $(U, \sim, \oplus, \otimes, \circ)$ is a Grzegorczyk algebra. Now, we will prove that $\mathcal{M}^{\mathsf{LD}}$ satisfies all other conditions required in the definition of LD-structures. Clearly, U and D are non-empty sets such that $D \subseteq U$. By the definition of $\mathcal{M}^{\mathsf{LD}}$, for any formula φ: $|\varphi|_R \in D$ if and only if φ is provable in LD. Let $|\varphi|_R, |\psi|_R \in U$. Then, $|\varphi|_R \otimes |\psi|_R = |\varphi \wedge \psi|_R \in D$ iff $\varphi \wedge \psi$ is provable in LD iff φ and ψ are provable in LD iff $|\varphi|_R \in D$ and $|\psi|_R \in D$. Therefore, $|\varphi|_R \otimes |\psi|_R \in D$ if and only if $|\varphi|_R \in D$ and $|\psi|_R \in D$. Furthermore, we have also: $|\varphi|_R \circ |\psi|_R = |\varphi \equiv \psi|_R \in D$ iff $\varphi \equiv \psi$ is provable in LD iff $\varphi R \psi$ iff $|\varphi|_R = |\psi|_R$. Hence, $|\varphi|_R \circ |\psi|_R \in D$ if and only if $|\varphi|_R = |\psi|_R$. By axiom Ax18, for any formula φ, $\neg(\varphi \wedge \neg\varphi)$ is provable in LD, so $\sim(|\varphi|_R \otimes \sim|\varphi|_R) \in D$. On the other hand, for any formula φ, $\varphi \wedge \neg\varphi$ is not provable in LD, since otherwise by Proposition 7, it would be true in all LD-structures, which is impossible. Therefore, we obtain: $(|\varphi|_R \otimes \sim|\varphi|_R) \notin D$. Hence, $\mathcal{M}^{\mathsf{LD}}$ is an LD-structure. ■

From now on, the structure $\mathcal{M}^{\mathsf{LD}}$ is referred to as *canonical*.

Proposition 10 (Completeness of LD)

For every LD-formula φ, if φ is LD-valid, then it is LD-provable.

Proof.

Let φ be an LD-valid formula, and let v be the valuation on $\mathcal{M}^{\mathsf{LD}}$ such that $v(\psi) = |\psi|_R$ for every ψ. It is easy to check that v is indeed a valuation. Now, by the assumption, $v(\varphi) \in D$, and hence φ is LD-provable. ■

Finally, by Propositions 7 and 10, we obtain:

Theorem 11 (Soundness and Completeness of LD)

For every LD-formula φ, the following conditions are equivalent:

1. φ is LD-provable.
2. φ is LD-valid.

By the completeness theorem, LD-structures will be referred to as LD-*models*.

Next, we consider LD-consistency.

Definition 12

Let S be a set of LD-formulas.

1. S is LD-*satisfiable* if there are an LD-model $\mathcal{M}$ and a valuation v on M such that for every $\varphi \in S$, it holds that $\mathcal{M}, v \models \varphi$.

2. S is LD-*inconsistent* if there is some formula φ such that $S \vdash \varphi \wedge \neg\varphi$. Otherwise, S is LD-*consistent.*

Proposition 13

A set S of LD-formulas is LD-satisfiable if and only if S is LD-consistent.

Proof.

Assume first that S is LD-satisfiable. Let $\mathcal{M}$ be an LD-model and let v be a valuation on $\mathcal{M}$ such that $\mathcal{M}, v \models S$. Let φ be an LD-formula. Suppose $S \vdash \varphi \wedge \neg\varphi$. Then, $\mathcal{M}, v \models \varphi \wedge \neg\varphi$, which means that $v(\varphi \wedge \neg\varphi) \in D$, so $v(\varphi) \in D$ and $v(\neg\varphi) \in D$, which contradicts the definition of an LD-structure. Hence, S is LD-consistent.

Assume then that S is LD-consistent. We build a model in the same way as the canonical model above. So, let R be the binary relation on the set of all LD-formulas defined as: $\varphi R \psi$ if and only if $S \vdash \varphi \equiv \psi$. As before, R is an equivalence relation compatible with all connectives, and hence we can define a Grzegorczyk algebra $(U, \sim, \oplus, \otimes, \circ)$ from R exactly as in the definition of the canonical model. The earlier proof works almost verbatim. Let further $D = \{\, |\varphi|_R : S \vdash \varphi \,\}$. Again, the proof of the required properties of D is otherwise essentially the same as before, but showing that there is no φ such that both $|\varphi|_R \in D$ and $\sim|\varphi|_R \in D$ require some extra care. Assume towards a contradiction that φ is a counterexample. Then it follows from the definitions that $S \vdash \varphi$ and $S \vdash \neg\varphi$, which contradicts the assumption that S is LD-consistent. ∎

4. Some interesting properties

In [Grzegorczyk, 2011], Prof. Grzegorczyk raises important questions about the relationship between equality of descriptions and material equivalence, as well as the corresponding implications. Let us introduce the following definitions:

- $(p \to q) \stackrel{\text{df}}{=} (\neg p \vee q)$ (classical implication)
- $(p \leftrightarrow q) \stackrel{\text{df}}{=} (p \to q) \wedge (q \to p)$ (classical equivalence)
- $(p \Rightarrow q) \stackrel{\text{df}}{=} (p \equiv (p \wedge q))$ (descriptive implication)

Now, the questions about relationships between descriptive and classical equivalences as well as between descriptive and classical implications can be formalized as follows:

(Q1) Is the formula $(p \equiv q) \equiv (p \leftrightarrow q)$ provable in LD?

(Q2) Is the formula $(p \Rightarrow q) \equiv (p \rightarrow q)$ provable in LD?

Both questions have negative answers, as the following proposition shows.

Proposition 14

1. The formula $(p \equiv q) \equiv (p \leftrightarrow q)$ is not provable in LD.
2. The formula $(p \Rightarrow q) \equiv (p \rightarrow q)$ is not provable in LD.

Proof.

Let $(U, \sim, \oplus, \otimes, \circ, D)$ be as in Example 4 above, and let $v(p) = v(q) = 2$. Then

$$v(p \rightarrow q) = {\sim} v(p) \oplus v(q) = {\sim} 2 \oplus 2 = 1 \oplus 2 = 2,$$

but

$$v(p \Rightarrow q) = v(p) \circ (v(p) \otimes v(q)) = 2 \circ (2 \otimes 2) = 2 \circ 2 = 3.$$

In the same way, we see that $v(p \leftrightarrow q) = 2$ but $v(p \equiv q) = 3$. ■

Next, we present some LD-provable formulas and derived rules as well as classical results that fail in LD. Due to the excessive lengths of the formal proofs, we omit the details of most of them, showing only outlines. The models we use as counterexamples are listed in the Appendix.

Even though there are no explicit rules concerning disjunction, there is a derived disjunction introduction rule.

Proposition 15

The following rule is strongly LD-correct:

$$\frac{\varphi}{\psi \vee \varphi}$$

Proof.

Assume $\mathcal{M}, v \models \varphi$. The formula $\psi \vee \neg\psi$ is LD-valid, so we get $\mathcal{M}, v \models \varphi \wedge (\psi \vee \neg\psi)$. Further,

$$\begin{aligned} \varphi \wedge (\psi \vee \neg\psi) &\equiv (\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) \\ &\equiv ((\varphi \wedge \psi) \vee \varphi) \wedge ((\varphi \wedge \psi) \vee \neg\psi) \\ &\equiv ((\varphi \vee \varphi) \wedge (\psi \vee \varphi)) \wedge ((\varphi \wedge \psi) \vee \neg\psi), \end{aligned}$$

whence $\mathcal{M}, v \models \psi \vee \varphi$. ■

Proposition 16

The formula $(p \equiv p) \equiv (q \equiv q)$ is not LD-provable.

Proof.

See Example 36 in the Appendix. ■

Proposition 17

The formula $\neg(p \equiv \neg p)$ is not LD-provable.

Proof.

See Example 38 in the Appendix. Now, actually for any $a \in U$, it holds that $\sim(a \circ \sim a) = 2 \notin D$, so there is no valuation v such that

$$\mathcal{M}, v \models \neg(p \equiv \neg p).$$ ■

Note that there cannot be any LD-model $\mathcal{M}$ and valuation v such that $\mathcal{M}, v \models p \equiv \neg p$, as $v(p \wedge \neg p) \notin D$ but $v(p \vee \neg p) \in D$.

Fact 3 shows that many familiar types of algebras are also Grzegorczyk algebras. However, there are Grzegorczyk algebras that do not belong to any of those types, and they often seem to be complicated and difficult to understand intuitively. However, they do illustrate various unexpected aspects of LD. For instance, we can show the failure of the classical modus ponens rule, despite the provability of the corresponding formula, by constructing a suitable model and choosing a valuation that satisfies the premises but not the supposed conclusion. Moreover, it follows directly from the definition of an LD-model that if a formula φ is satisfied in a model $\mathcal{M}$ by a valuation v, then $\neg\varphi$ is not satisfied by v in $\mathcal{M}$, but the converse does not hold, as we saw above. This trait of LD contrasts strongly with most other well-known logics, that is, ones that follow the negation clause of Tarski's truth definition. On the other hand, LD is also quite unlike intuitionistic logic, as there is no negation introduction rule, but the axioms include DeMorgan's laws and double negation is treated classically. It follows from this combination of DeMorgan's laws, a classical conjunction, and a non-standard negation, that also disjunction behaves in an unexpected way. Indeed, the formula $p \equiv \neg p$ is unsatisfiable and $(p \equiv \neg p) \vee \neg(p \equiv \neg p)$ is a tautology, but $\neg(p \equiv \neg p)$ is not provable. Hence, the connection between the truth values of a disjunction and the disjuncts is less definite than in classical logic. This property is, at least to a degree, in accordance with the philosophical motivations that LD is based on, as the LD-provability of a formula gives us information not only of its necessary truth, but also of its connection to the axioms.

The soundness of LD with respect to the class of LD-models allows us to derive further unprovability results.

Proposition 18

The following formulas are not provable in LD:

1. $(\varphi \vee (\varphi \wedge \psi)) \equiv \varphi$.
2. $(\varphi \wedge (\varphi \vee \psi)) \equiv \varphi$.
3. $(\varphi \vee \neg\varphi) \equiv (\psi \vee \neg\psi)$.
4. $(\varphi \wedge \neg\varphi) \Rightarrow \psi$.
5. $\varphi \Rightarrow (\psi \vee \neg\psi)$.

Moreover, the following rule is not strongly LD-correct:

$$(\vee_1) \quad \frac{\varphi, \neg\varphi \vee \psi}{\psi}.$$

Proof.

For (1) and (2), see Example 34 in the Appendix. For (3), (4), and (5), see Example 35. For $(\vee_1)$, see Example 37. ■

Note that the formulas (3), (4), and (5) are instances of the classical paradoxes of equivalence and implication: "any true statements are equivalent to each other", "false implies everything", and "the truth is implied by anything", respectively. Hence their failure indicates that LD indeed avoids these paradoxes.

Now, we can prove that all the axioms of $\mathrm{LD}_{\mathtt{red}}$ are independent of each other. By a *quasi* LD*-structure* we will mean a structure $\mathcal{M} = (U, \sim, \otimes, \oplus, \circ, D)$ such that U, D are nonempty sets, $D \subseteq U$, $\sim$ is a unary operation on U and $\otimes, \oplus, \circ$ are binary operations on U. The notions of valuation, satisfaction, and the truth in a quasi LD-structure are defined in the same way as for LD-models.

Proposition 19

Let $S = \{\mathsf{Ax2}, \mathsf{Ax3}, \mathsf{Ax4}, \mathsf{Ax6}, \mathsf{Ax10}, \mathsf{Ax11}, \mathsf{Ax12}, \mathsf{Ax13}, \mathsf{Ax15}, \mathsf{Ax17}, \mathsf{Ax18}\}$. Then, for each $\varphi \in S$, there is a quasi LD-structure $\mathcal{M}_\varphi$ such that $\mathcal{M}_\varphi \not\models \varphi$ but for each $\psi \in S \setminus \varphi$, it holds that $\mathcal{M}_\varphi \models \psi$.

Proof.

We will list the structures in the Appendix. ■

So, the set S forms an independent set of axioms for LD.

5. Alternative versions of LD

During discussions about LD in seminar meetings, some alternative forms of Ax3 were suggested. One of the motivations for adopting Ax3 was to express the transitivity of descriptive equivalence, but at least superficially, Ax3 appears to say something stronger. Therefore, it was natural to consider alternative versions and their respective consequences. As we mentioned above in Section 2, the original version of Ax3 as given in [Grzegorczyk, 2011] was different, but our version is the intended one, as published in Errata [2012]. We will now discuss the logics obtained by replacing Ax3 with two alternative forms: Ax3* and Ax3$'$. The axiom Ax3* is the original version presented in [Grzegorczyk, 2011], whereas Ax3$'$ was proposed by Stanisław Krajewski, and it is already mentioned in the Errata. These axioms have the following forms:

Ax3* $(p \equiv q) \equiv [(p \equiv r) \equiv (q \equiv r)]$,

Ax3$'$ $[(p \equiv q) \wedge (q \equiv r)] \Rightarrow (p \equiv r)$.

Recall that $\Rightarrow$ is an LD-implication defined as:

$$\varphi \Rightarrow \psi \stackrel{\text{df}}{=} \varphi \equiv (\varphi \wedge \psi).$$

Thus, the explicit form of axiom Ax3$'$ is:

$$[(p \equiv q) \wedge (q \equiv r)] \equiv [(p \equiv q) \wedge (q \equiv r) \wedge (p \equiv r)].$$

By LD* (resp. LD$'$) we will denote the logic obtained from LD by replacing the axiom Ax3 with Ax3* (resp. Ax3$'$). It is again easy to see that both logics are consistent, since under the interpretation of $\equiv$ as the usual classical equivalence, both axioms Ax3* and Ax3$'$ are classical tautologies.

First, we will discuss the logic LD*. We can actually prove Ax3 in LD*, as the following proof shows:

(1) $[(p \equiv q) \equiv ((p \equiv r) \equiv (q \equiv r))] \equiv$
$\equiv [((p \equiv q) \equiv ((p \equiv r) \equiv (q \equiv r)))\wedge$
$\wedge(((p \equiv q) \wedge (p \equiv q)) \equiv (((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)))]$
(Sub) to Ax6 for $p/(p \equiv q)$, $q/((p \equiv r) \equiv (q \equiv r))$, $r/(p \equiv q)$

(2) $(p \equiv q) \equiv ((p \equiv r) \equiv (q \equiv r)]$ Ax3*

(3) $((p \equiv q) \equiv ((p \equiv r) \equiv (q \equiv r)))\wedge$
$\wedge(((p \equiv q) \wedge (p \equiv q)) \equiv (((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)))$
($\mathsf{MP}_{\mathrm{LD}}$) to (1) and (2)

(4) $((p \equiv q) \wedge (p \equiv q)) \equiv (((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q))$ ($\wedge_2$) to (3)

(5) $[((p \equiv q) \wedge (p \equiv q)) \equiv (((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q))] \equiv$
$\equiv \{(((p \equiv q) \wedge (p \equiv q)) \equiv (p \equiv q)) \equiv$
$\equiv ((((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv (p \equiv q))\}$
(Sub) to Ax3* for $p/((p \equiv q) \wedge (p \equiv q))$,
$q/(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q))$, $r/(p \equiv q)$

(6) $(((p \equiv q) \wedge (p \equiv q)) \equiv (p \equiv q)) \equiv$
$\equiv ((((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv (p \equiv q))$
($\mathsf{MP}_{\mathrm{LD}}$) to (4) and (5)

(7) $(p \equiv q) \equiv ((p \equiv q) \wedge (p \equiv q))$ (Sub) to $\mathsf{Ax12}$ for $p/(p \equiv q)$

(8) $[(p \equiv q) \equiv ((p \equiv q) \wedge (p \equiv q))] \equiv [((p \equiv q) \wedge (p \equiv q)) \equiv (p \equiv q)]$
(Sub) to $\mathsf{Ax2}$ for $p/(p \equiv q)$, $q/((p \equiv q) \wedge (p \equiv q))$

(9) $((p \equiv q) \wedge (p \equiv q)) \equiv (p \equiv q)$ ($\mathsf{MP}_{\mathrm{LD}}$) to (7) and (8)

(10) $(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv (p \equiv q)$ ($\mathsf{MP}_{\mathrm{LD}}$) to (6) and (9)

(11) $(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))$
(Sub) to $\mathsf{Ax10}$
for $p/((p \equiv r) \equiv (q \equiv r))$, $q/(p \equiv q)$

(12) $[(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv (p \equiv q)] \equiv$
$\equiv \{[(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))] \equiv$
$\equiv [(p \equiv q) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))]\}$
(Sub) to $\mathsf{Ax3}^*$ for $p/(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q))$,
$q/(p \equiv q)$, $r/((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))$

(13) $[(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))] \equiv$
$\equiv [(p \equiv q) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))]$
($\mathsf{MP}_{\mathrm{LD}}$) to (10) and (12)

(14) $(p \equiv q) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))$ ($\mathsf{MP}_{\mathrm{LD}}$) to (11) and (13).

Thus, we get the following proposition.

Proposition 20

Every LD-formula φ that is provable in LD is also provable in LD^*.

It turns out that replacing $\mathsf{Ax3}$ with $\mathsf{Ax3}^*$ defeats the purpose of introducing a new connective, as the following proposition shows:

Proposition 21

The following rules are strongly correct in LD^*:

$$\frac{p \leftrightarrow q}{p \equiv q} \qquad \frac{p \equiv q}{p \leftrightarrow q}$$

Proof.

First, it is easy to see that $(p \equiv p) \equiv (q \equiv q)$ holds for any p, q, as both sides are equal to $(p \equiv q) \equiv (p \equiv q)$, by $\mathsf{Ax3}^*$ and symmetry. Let us denote this common value by 1. Then,

$$\begin{aligned} (p \equiv 1) &\equiv ((p \equiv 1) \equiv (1 \equiv 1)) \\ &\equiv ((p \equiv 1) \equiv 1), \end{aligned}$$

whence $p \equiv (p \equiv 1)$. In particular, if φ is provable in LD^*, then so is $\varphi \equiv 1$. Hence,

$$\begin{aligned} p \vee 1 &\equiv p \vee (p \vee \neg p) \\ &\equiv p \vee \neg p \\ &\equiv 1. \end{aligned}$$

Moreover,

$$\begin{aligned} p &\equiv (p \equiv 1) \\ &\equiv (p \equiv 1) \wedge ((p \wedge 1) \equiv (1 \wedge 1)) \\ &\equiv p \wedge (p \wedge 1), \end{aligned}$$

so $(p \wedge 1) \equiv p$ for all p.

If we substitute 1 for q in axioms $\mathsf{Ax5}$ and $\mathsf{Ax6}$ and simplify, we get, respectively,

$$\begin{aligned} p &\equiv p \wedge ((p \wedge r) \equiv r), \\ p &\equiv p \wedge (p \vee r). \end{aligned}$$

So, it follows that

$$(p \wedge q) \equiv [(p \wedge q) \wedge ((p \wedge q) \equiv p) \wedge ((p \wedge q) \equiv q)].$$

By applying DeMorgan's laws and some further manipulations, we also get

$$(\neg p \wedge \neg q) \equiv [(\neg p \wedge \neg q) \wedge ((p \wedge q) \equiv p) \wedge ((p \wedge q) \equiv q)].$$

So,

$$(p \leftrightarrow q) \equiv [(p \leftrightarrow q) \wedge ((p \wedge q) \equiv p) \wedge ((p \wedge q) \equiv q)].$$

From this and the transitivity of $\equiv$, the claim follows.

On the other hand, we can prove the tautology $q \leftrightarrow q$ as in the classical case, and hence $(q \leftrightarrow q) \equiv 1$ and further $r \equiv (r \equiv (q \leftrightarrow q))$, for any r. Moreover,

$$(p \equiv q) \equiv [(p \equiv q) \wedge ((p \leftrightarrow q) \equiv (q \leftrightarrow q))],$$

and therefore $(p \equiv q) \equiv ((p \equiv q) \wedge (p \leftrightarrow q))$. ■

So, LD^* is effectively just an unnecessarily complex reformulation of classical propositional logic. It also follows that the converse of Proposition 20 does not hold.

Let us now consider $\mathsf{Ax3}'$. We can define an LD'-model by changing the definition of an LD-model appropriately, that is, by replacing the condition $a \circ b = (a \circ b) \otimes ((a \circ c) \circ (b \circ c))$ with $(a \circ b) \otimes (b \circ c) = (a \circ b) \otimes (b \circ c) \otimes (a \circ c)$. Now, every LD'-provable formula is true in every LD'-model, which can be proved essentially in the same way as in the case of LD. However, the converse implication does not hold. Let φ be the formula $(p \equiv q) \equiv ((p \wedge p) \equiv q)$. The structure presented in Example 39 satisfies all LD'-axioms and rules but does not satisfy φ, which means that φ is not LD'-provable. On the other hand, φ is clearly true in every LD'-model, as $\equiv$ is interpreted as equality. Moreover, the philosophical motivations for LD suggest that φ should be true. Therefore, we will not study LD' any further.

6. Conclusions

We have defined the logic LD in terms of syntactic deduction rules and axioms, defined a corresponding semantics, and proved a soundness and completeness theorem. We have given several examples of classical laws that hold in LD as well as laws that fail in LD. We have shown that the original axiomatization is redundant and found an independent set of axioms. We have considered two proposed alternative forms of $\mathsf{Ax3}$ and found both of them unsatisfying as replacements for $\mathsf{Ax3}$.

In studying the properties of LD, we have found computer-assisted methods indispensable. We developed the following tools for LD:

- Two proof checkers, accepting two different input syntaxes and using different methods for checking, generally being as independent as possible apart from the fact that both were written by the second author.
- An automatic proof builder, which accepts a set of targets and some intermediate steps and attempts to output a full formal proof of the targets, in a format suitable for either checker or LaTeX.
- A model checker, which inputs a finite structure in the signature of LD as well as some deduction rules and formulas, and checks whether the structure obeys the rules and satisfies the formulas.

As we mentioned before, the programs and sample input and output files are available on request.

There are several interesting questions about LD that we have not answered here.

1. Is LD decidable?
   We conjecture LD has the finite model property, formulated in the following way due to the non-classical negation:
   For every formula φ such that there are an LD-model $\mathcal{M}$ and a valuation v on $\mathcal{M}$ such that $\mathcal{M}, v \not\models \varphi$, there are a finite LD-model $\mathcal{M}'$ and a valuation v on $\mathcal{M}'$ such that $\mathcal{M}', v' \not\models \varphi$.
   For logics with a Tarskian negation, this formulation is, of course, equivalent with the usual one. In the case of LD, however, this is the form that we need. Indeed, if the conjecture is true, we can prove the decidability of LD in the usual way.
2. If LD is decidable, what is the complexity of deciding whether an LD-formula is provable?
   Our conjecture about the finite model property, mentioned above, is based on a construction of a finite model whose size is doubly exponential in the size of the formula. If the construction is correct, there is an obvious decision algorithm that runs in doubly exponential space and hence triply exponential time: simply search for a small enough counterexample.
3. Is the classical modus ponens rule weakly correct for LD?
   We have a counterexample showing that MP is not strongly correct. However, we do not know whether it is weakly correct.
4. Are there other interesting variants of LD?
   We showed that LD^* is too strong and LD' too weak to formalize the motivating philosophical ideas. However, $\mathsf{Ax3}'$ appears plausible in its own right, and instead of replacing $\mathsf{Ax3}$ with $\mathsf{Ax3}'$, one could simply add $\mathsf{Ax3}'$ to LD. So far, our preliminary results suggest such an extension would be similar to LD, with only minor technical differences.
5. Can one generate LD-proofs fully automatically in practice?
   Our prover needs a human-generated outline consisting of intermediate steps, which it then attempts to expand to a full proof by applying some derived rules. If the outline is not sufficiently detailed, the prover fails. The non-classical nature of negation and disjunction prevents a straightforward implementation of a tableau-based prover. So far, we have not found a practical proof strategy for LD. Of course, a brute-force search is possible in principle.
6. Is there a normal form for LD-formulas?
   A suitable normal form may simplify the task of finding an automatic proof system, among other things.

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Appendix

In this section, we will list the models that show the unprovability claims made in the main text.

Example 22

Here is the simplest possible LD-model, unique up to isomorphism.

$U = \{0, 1\}$, $D = \{1\}$

$\sim$	0	1
	1	0

$\circ$	0	1
0	1	0
1	0	1

$\otimes$	0	1
0	0	0
1	0	1

$\oplus$	0	1
0	0	1
1	1	1

Example 23

This model shows that Ax2 is independent of $\mathsf{LD}_{\mathtt{red}}$. That is, the formula $(p \equiv q) \equiv (q \equiv p)$ is not true in it, but all other axioms of $\mathsf{LD}_{\mathtt{red}}$ are.

$U = \{0, 1, 2, 3, 4, 5\}$, $D = \{3, 4, 5\}$; $v(p) = 0$, $v(q) = 2$.

$\sim$	0	1	2	3	4	5
	5	4	3	2	1	0

$\circ$	0	1	2	3	4	5
0	3	1	1	1	1	1
1	1	4	1	2	1	2
2	2	1	4	1	2	1
3	1	2	1	4	1	2
4	2	1	2	1	4	1
5	1	1	1	1	1	3

$\otimes$	0	1	2	3	4	5
0	0	1	2	1	2	1
1	1	1	1	1	1	1
2	2	1	2	1	2	1
3	1	1	1	3	3	3
4	2	1	2	3	4	3
5	1	1	1	3	3	5

$\oplus$	0	1	2	3	4	5
0	0	2	2	4	4	4
1	2	1	2	3	4	3
2	2	2	2	4	4	4
3	4	3	4	3	4	3
4	4	4	4	4	4	4
5	4	3	4	3	4	5

Example 24

The axiom $(p \equiv q) \equiv [(p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r))]$ (Ax3) is independent.

$U = \{0, 1, 2, 3\},\ D = \{2, 3\};\ v(p) = 0,\ v(q) = 0,\ v(r) = 0.$

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	3
3	0	0	3	3

$\oplus$	0	1	2	3
0	0	0	3	3
1	0	1	3	3
2	3	3	2	3
3	3	3	3	3

Example 25

The axiom $(p \equiv q) \equiv (\neg p \equiv \neg q)$ (Ax4) is independent.

$U = \{0, 1, 2, 3\},\ D = \{2, 3\};\ v(p) = 1,\ v(q) = 1.$

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	0	0	0
1	0	3	0	0
2	0	0	2	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	3
3	0	0	3	3

$\oplus$	0	1	2	3
0	0	0	3	3
1	0	1	3	3
2	3	3	2	3
3	3	3	3	3

Example 26

The axiom $(p \equiv q) \equiv [(p \equiv q) \wedge ((p \wedge r) \equiv (q \wedge r))]$ (Ax6) is independent.

$U = \{0, 1, 2, 3\},\ D = \{2, 3\};\ v(p) = 3,\ v(q) = 3,\ v(r) = 1.$

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	3	0	0	0
1	0	2	0	0
2	0	0	2	0
3	0	0	0	3

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

Example 27

The axiom $(p \wedge q) \equiv (q \wedge p)$ (Ax10) is independent.

$U = \{0, 1, 2, 3\},\ D = \{2, 3\};\ v(p) = 0,\ v(q) = 1.$

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	0	0	0
1	0	2	0	0
2	0	0	2	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	0	2	2
3	1	1	3	3

$\oplus$	0	1	2	3
0	0	0	2	2
1	1	1	3	3
2	2	2	2	2
3	3	3	3	3

Example 28

The axiom $(p \wedge (q \wedge r)) \equiv ((p \wedge q) \wedge r)$ (Ax11) is independent.

$U = \{0, 1, 2, 3, 4, 5\},\ D = \{3, 4, 5\};\ v(p) = 0,\ v(q) = 0,\ v(r) = 4.$

$\sim$	0	1	2	3	4	5
	5	4	3	2	1	0

$\circ$	0	1	2	3	4	5
0	3	2	2	2	2	2
1	2	3	2	2	2	2
2	2	2	3	2	2	2
3	2	2	2	3	2	2
4	2	2	2	2	3	2
5	2	2	2	2	2	3

$\otimes$	0	1	2	3	4	5
0	0	0	2	2	1	1
1	0	1	2	2	1	1
2	2	2	2	2	2	2
3	2	2	2	3	3	3
4	1	1	2	3	4	4
5	1	1	2	3	4	5

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	4
1	1	1	2	3	4	4
2	2	2	2	3	3	3
3	3	3	3	3	3	3
4	4	4	3	3	4	5
5	4	4	3	3	5	5

Example 29

The axiom $p \equiv (p \wedge p)$ (Ax12) is independent.

$U = \{0, 1, 2, 3\},\ D = \{2, 3\};\ v(p) = 0.$

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	1	1	1
1	1	2	1	1
2	1	1	2	1
3	1	1	1	2

$\otimes$	0	1	2	3
0	1	1	1	1
1	1	1	1	1
2	1	1	2	2
3	1	1	2	3

$\oplus$	0	1	2	3
0	0	1	2	2
1	1	1	2	2
2	2	2	2	2
3	2	2	2	2

Example 30

The axiom $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$ (Ax13) is independent.

$U = \{0, 1, 2, 3\},\ D = \{2, 3\};\ v(p) = 1,\ v(q) = 1,\ v(r) = 2.$

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	0	0	0
1	0	2	0	0
2	0	0	2	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	2
3	0	0	2	3

$\oplus$	0	1	2	3
0	0	1	3	3
1	1	1	3	3
2	3	3	2	3
3	3	3	3	3

Example 31

The axiom $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$ (Ax15) is independent.

$U = \{0, 1, 2, 3\},\ D = \{2, 3\};\ v(p) = 0,\ v(q) = 1.$

$\sim$	0	1	2	3
	2	3	0	1

$\circ$	0	1	2	3
0	2	0	0	0
1	0	2	0	0
2	0	0	2	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

Example 32

The axiom $\neg\neg p \equiv p$ (Ax17) is independent.

$U = \{0, 1, 2, 3, 4, 5, 6, 7\},\ D = \{4, 5, 6, 7\};\ v(p) = 1.$

$\sim$	0	1	2	3	4	5	6	7
	7	6	5	4	3	1	2	0

$\circ$	0	1	2	3	4	5	6	7
0	4	0	0	0	0	0	0	0
1	0	4	0	0	0	0	0	0
2	0	0	4	0	0	0	0	0
3	0	0	0	4	0	0	0	0
4	0	0	0	0	4	0	0	0
5	0	0	0	0	0	4	0	0
6	0	0	0	0	0	0	4	0
7	0	0	0	0	0	0	0	4

$\otimes$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	1	1	1	1
2	0	0	2	2	2	2	2	2
3	0	1	2	3	3	3	3	3
4	0	1	2	3	4	4	4	4
5	0	1	2	3	4	5	4	5
6	0	1	2	3	4	4	6	6
7	0	1	2	3	4	5	6	7

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	3	3	4	5	6	7
2	2	3	2	3	4	5	6	7
3	3	3	3	3	4	5	6	7
4	4	4	4	4	4	5	6	7
5	5	5	5	5	5	5	7	7
6	6	6	6	6	6	7	6	7
7	7	7	7	7	7	7	7	7

Example 33

The axiom $\neg(p \wedge \neg p)$ (Ax18) is independent.

$U = \{0, 1, 2, 3\},\ D = \{3\};\ v(p) = 1.$

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	3	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	3

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

Example 34

In this model, the absorption law $(p \wedge (p \vee q)) \equiv p$ does not hold.

$U = \{0, 1, 2, 3\},\ D = \{2, 3\};\ v(p) = 1,\ v(q) = 0.$

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	3	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	3

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	1	2	3
3	0	0	3	3

$\oplus$	0	1	2	3
0	0	0	3	3
1	0	1	2	3
2	3	2	2	3
3	3	3	3	3

Example 35

Here the formulas $(p \vee \neg p) \equiv (q \vee \neg q)$, $(p \wedge \neg p) \Rightarrow q$, and $\neg q \Rightarrow (p \vee \neg p)$ are not true.

$U = \{0, 1, 2, 3\},\ D = \{2, 3\};\ v(p) = 1,\ v(q) = 0.$

∼	0	1	2	3
	3	2	1	0

∘	0	1	2	3
0	3	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	3

⊗	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

⊕	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

Example 36

In this model, the formula $(p \equiv p) \equiv (q \equiv q)$ is not true.

$U = \{0, 1, 2, 3, 4, 5\},\ D = \{3, 4, 5\};\ v(p) = 0,\ v(q) = 1.$

∼	0	1	2	3	4	5
	5	4	3	2	1	0

∘	0	1	2	3	4	5
0	5	0	0	0	0	0
1	0	3	0	0	0	0
2	0	0	5	0	0	0
3	0	0	0	5	0	0
4	0	0	0	0	3	0
5	0	0	0	0	0	5

⊗	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	0	0	2
2	0	2	2	0	0	2
3	0	0	0	3	3	3
4	0	0	0	3	4	3
5	0	2	2	3	3	5

⊕	0	1	2	3	4	5
0	0	2	2	3	3	5
1	2	1	2	5	5	5
2	2	2	2	5	5	5
3	3	5	5	3	3	5
4	3	5	5	3	4	5
5	5	5	5	5	5	5

Example 37

In this model, the modus ponens rule is not correct. That is, the formula $p \wedge (p \rightarrow q)$ is satisfied by a valuation that does not satisfy q.

$U = \{0, 1, 2, 3, 4, 5\},\ D = \{4, 5\};\ v(p) = 4,\ v(q) = 3.$

∼	0	1	2	3	4	5
	5	4	3	2	1	0

∘	0	1	2	3	4	5
0	5	0	0	0	0	0
1	0	5	0	0	0	0
2	0	0	5	0	0	0
3	0	0	0	5	0	0
4	0	0	0	0	5	0
5	0	0	0	0	0	5

⊗	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	0	1	1
2	0	1	2	0	1	2
3	0	0	0	3	3	3
4	0	1	1	3	4	4
5	0	1	2	3	4	5

⊕	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	4	4	5
2	2	2	2	5	5	5
3	3	4	5	3	4	5
4	4	4	5	4	4	5
5	5	5	5	5	5	5

Example 38

In this example, the formula $\neg(p \equiv \neg p)$ is not true, even though one can prove a contradiction from $p \equiv \neg p$. Thus, there cannot be a negation introduction rule.

$U = \{0, 1, 2, 3\},\ D = \{3\};\ v(p) = 0.$

∼	0	1	2	3
	3	2	1	0

∘	0	1	2	3
0	3	2	2	2
1	2	3	2	2
2	2	2	3	2
3	2	2	2	3

⊗	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

⊕	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

Example 39

This example shows that the formula $(p \equiv q) \equiv ((p \wedge p) \equiv q)$ is not LD$'$-provable.

$U = \{0,1,2,3,4,5\},\ D = \{3,4,5\};\ v(p) = 2,\ v(q) = 0.$

$\sim$	0	1	2	3	4	5
	5	4	3	2	1	0

$\circ$	0	1	2	3	4	5
0	5	1	0	0	1	1
1	1	5	4	0	1	1
2	0	4	5	1	0	0
3	0	0	1	5	4	0
4	1	1	0	4	5	1
5	1	1	0	0	1	5

$\otimes$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	0	0	1
2	0	1	1	0	0	1
3	0	0	0	4	4	4
4	0	0	0	4	4	4
5	0	1	1	4	4	5

$\oplus$	0	1	2	3	4	5
0	0	1	1	4	4	5
1	1	1	1	5	5	5
2	1	1	1	5	5	5
3	4	5	5	4	4	5
4	4	5	5	4	4	5
5	5	5	5	5	5	5

Joanna Golińska-Pilarek
Institute of Philosophy, University of Warsaw, Poland
j.golinska@uw.edu.pl

Taneli Huuskonen
Department of Mathematics and Statistics, University of Helsinki, Finland
huuskone@cc.helsinki.fi

Stanisław Krajewski

EMERGENCE IN MATHEMATICS?

Abstract. Emergence is difficult to define. In mathematics, a subjective, indeed psychological, definition of emergence seems reasonable. The following conditions seem necessary: the appearance of surprising properties that are inescapably surprising, even for the expert. It is another matter whether this is a sufficient condition for the presence of emergence. To the canonical examples of emergence – life, mind, (self)consciousness – some mathematical examples can be added: fractals (already proposed by other authors) and the emergence of undecidability (and incompleteness) of natural numbers when they are considered as a structure with both addition and multiplication.

1. Emergence, an Introduction

The notion of emergence has been proposed as a means of describing situations in the material world in which growing complexity either causes the appearance of essentially new features or provides an occasion for essentially new features to appear. To be sure, it is not arbitrary new features that are meant here, since *some* new features must necessarily appear when any change takes place; rather, what is meant are essential, fundamentally new features: new higher-level qualities that are "ungraspable," that is, cannot be grasped (or understood) from a lower level. Unfortunately, it is not obvious what these "levels" are. It is also far from clear what "ungraspability" should mean: it can be explained as irreducibility, non-educibility, indeterminateness, or unexpected character of the phenomenon. Fortunately, some examples are beyond doubt. Thus, the emergence of life, of the mind, and of consciousness illustrate higher-level, irreducible, absolutely new features – or rather dimensions – that cannot be grasped from a lower level, that is, from the level of inanimate matter or unconscious life. The notion of emergence was introduced in the 1920s. In recent decades it has been revived. It is claimed that emergence would be found in various connected structures that could be regarded as separate entities: a book by Steven Johnson is titled *Emergence: The Connected Lives of Ants, Brains, Cities,*

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and Software. Other scholars refer to the internet and other self-organizing structures, and more metaphysically minded authors use the concept of emergence to talk about spirituality – even, as does Ken Wilbur, spirituality on a cosmic scale. The construction of hierarchies of emergence can result in a variety of religious claims. Unsurprisingly, such claims are divergent. Thus, Philip Clayton in the book *Mind and Emergence: From Quantum to Consciousness* writes that Terrence Deacon draws Buddhist conclusions, Harold Morowitz finds Jewish-Spinozian ones, while Nancy Murphy, Niels Gregorsen and Clayton himself interpret such ultimate emergence in a Christian framework.

Emergence can also be described by saying that "the whole is more than the sum of its parts," but this is hardly helpful. Every whole built from parts – not just collected as is the case with sets in the distributive sense – is more than the totality of its parts. What sets emergence apart is that it occurs when novelty is essential – another concept difficult to define – and is not reducible to the connections linking the parts. However these terms just used, "essential novelty" and "(ir)reducibility," seem just as difficult to explain as the term "emergence" itself. My aim in this paper is to analyze a seemingly simpler, but nonetheless elusive subject, the appearance of emergence, if there is any, in mathematics.[1]

2. The Psychological Character of Emergence in Mathematics

If a mathematical structure is enriched, it becomes more complicated and one can detect in it new features and new phenomena. Sometimes these can represent something genuinely new. If, for instance, integers are extended to rational numbers the ordering of numbers becomes dense, and if we further extend the structure to real numbers a new feature characterizes the ordering – it becomes continuous. It is rather doubtful, however, whether such phenomena, ubiquitous in mathematics, can be described as "emergence." Indeed, I feel it would trivialize the concept of emergence if it were used in such cases. Similarly doubtful is the application of the concept of emergence when, in the spirit of Nicholas Bourbaki, structures are gradually expanded – from sets to ordered sets to added algebraic structures to topo-

[1] This paper is based on a Polish paper „Emergencja w matematyce?", in *Struktura i emergencja* (red. Michał Heller i Janusz Mączka), Biblos, Kraków 2006, 110–118, published also as Chapter 1 of my book *Czy matematyka jest nauką humanistyczną?*, Copernicus Center Press, Kraków 2011, 11–20.

logical ones to their combinations, etc. The problem is, then, what novel phenomena can rightly be described as "emergent." Are there any identifiable features of mathematical situations that are necessary for emergence to occur?

In order to move toward a solution let us consider an example that at first glance seems promising. Passing from finite to infinite structures is very natural in mathematics, or rather in modern mathematics, as until the 19th century this step was considered unacceptable. Georg Cantor, however, has introduced us to the realm of actually infinite sets. We consider longer and longer sequences and naturally move to considering transfinite sequences, which can be manipulated in much the same way as the finite ones. Infinite sets, too, are treated in the same way as finite sets, or to be more specific, operations such as taking the union of a set of sets or the powerset of a given set, which raise no reservations in the case of finite sets, are also executed on actually infinite sets. In fact, the possibility of the unrestricted extension of those operations to the realm of infinite sets can be seen as the essence of the Cantorian revolution in mathematics. It is certain that new features, new regularities appear in the realm of infinite sets. The main novelty can be seen in the presence of, well, infinity! Are we dealing with "emergence" here? The answer is not easy. While we have no criterion for recognizing emergence, it seems that the mechanism producing the new situation is relatively clear: the potential infinity, or the potential to extend the finite beyond any limit, leads to a jump into the actual infinity. The situation is by no means simple, but the novelty is created by our decision to perform a jump. Therefore, we cannot say that the result is unexpected. And it seems to me that in order to talk about emergence it is necessary to face something truly unexpected. Some surprise is needed, the appearance on the higher level of something that is not simply higher, more complex, but is also *astonishing* – an unforeseen feature or regularity, impossible to anticipate at the lower level.

A necessary feature has just been formulated that must characterize a situation in which emergence occurs, namely, the presence of something unexpected, a *surprise*. This criterion is not very clear, and what may seem worse, it is subjective. Is it useful? Is there anything surprising on the level of infinite sets with respect to the level of finite sets? Of course, new features appear. One of them is noteworthy, the possibility that a part be equal to the whole, or rather as big as the whole, in the sense that there is a one-to-one correlation of the part with the whole. This is surprising at first. A moment later, however, the surprise disappears. The numeric equivalence of integers with the even numbers is so simple that no surprise remains. More generally, the one-to-one correspondence of a set and its proper part can be treated as a characterization of infinity ("Dedekind infinite sets"). Other equivalences proved by Cantor, like the denumerability of the set of rational numbers, are only slightly more difficult. Their proofs are transparent, as is the proof of the uncountability of the set of real numbers (if the existence of this set as a completed entity is assumed). Once understood they no longer cause surprise.

It can be realized now that another necessary criterion has just been formulated: to witness emergence one must not only feel a surprise, but also sense a surprise that is impossible to overcome, an *inescapable surprise*. Now this criterion is not just subjective, it seems purely psychological. Is this acceptable? My thesis is that the psychological nature of emergence is unavoidable at least in the case of mathematics. I leave out the question whether the psychological definition is appropriate for science as well, although my guess is that some psychological dimension is inevitable.

In the case of infinity, is the feeling of unexpected developments present or not? It is possible to show more and more difficult theorems about infinite sets: statements that need a lot of effort to understand. A real expert in the field can, however, understand them so well that the difference between them and the basic properties is only one of quantity. To witness emergence we need, I believe, to face a qualitative difference, which even for an expert indicates a different order of events. To answer whether the expert perceives a qualitative difference there, we must refer to a psychological approach. Certainly, it would be good to have an objective definition of qualitative as distinct from merely quantitative difference in the degree of surprise provoked by theorems. Yet to formulate such a definition seems to be just as difficult as is the definition of emergence.

Therefore, having accepted the psychological nature of emergence, we are looking for an example of the situation in which a new property appears, one that is unexpected and one where the feeling of surprise is lasting (independent of the level of expertise and familiarity with the subject) so that we can be sure that we face a genuinely new quality. If the structure gets richer, can some new unanticipated properties appear? Properties that are unexpected in an inescapable manner? To repeat, I do not mean here just any new property, but an essentially new property or regularity. The requirement that such a property must be inescapably unexpected defines a necessary condition for emergence. If sufficiently many examples can be found then, hopefully, one would be able to say whether this is a sufficient condition as well.

3. Mathematical examples

Mathematics of the last several decades provides some highly interesting examples. Fractals are generally known by now, even among the mathematically illiterate. Iterating simple functions produces unanticipated, immensely complicated structures that are self-similar: zooming in we encounter the same structure again and again. The complication is infinite. What is even more relevant in our context is that some such iterations provide a huge variety of patterns, or "landscapes." I believe that even the best experts are repeatedly and inescapably surprised when they study the successive regions of the Mandelbrot set. Our psychological criterion for the presence of emergence is satisfied.

Another example is provided by the mathematics of deterministic chaos, initiated over a hundred years ago and developed in a deeper way only relatively recently. Even in completely deterministic processes tiny changes in the initial conditions can produce huge differences in results. This can explain why surprise is an inescapable property of models of weather. Some authors call this impossibility to predict the result emergence. More systematically, similar processes are studied in the theories of complexity, in which various phenomena of growing complexity are investigated. Some of them are generated by iterated applications of very simple rules. For example, John Conway's game Life is played on an infinite plane grid of regular squares where an initial arbitrary finite pattern of black squares among the remaining white ones is consecutively modified according to fixed rules that establish the color of a given square based on the colors of its immediate neighbors at the previous step. The resulting patterns undergo an evolution, and the process can be somewhat similar to the movement of schematic organisms that grow, shift, multiply, get "old," etc. A closer investigation has been made possible due to computers. A similar realm, that of finite automata, has been classified by Stephen Wolfram. In his impressive and highly unconventional book, *A New Kind of Science*, published in 2002, he attempts to demonstrate that all sorts of physical phenomena can be represented as iterations of simple algorithms. Wolfram considers rows of black and white cells that change according to fixed rules; in each step the color of a cell depends only on the colors of itself and its immediate neighbors in the previous step. Even in this simple space some rules determine extremely complicated behavior that is neither periodic nor completely random. Wolfram is ready to say that the whole universe is an automaton, or a gigantic computer. This view can be seen as the ultimate expression of the Pythagorean approach to nature. In the last analysis, our universe and everything in it would be an automaton. This includes us, our brains and ourselves – or "our selves."

While the above vision looks extreme, the fact is that some finitely describable structures can contain more complex, and actually arbitrarily complex, phenomena. The existence of such universal structures has been known in the foundations of mathematics since 1936, when Alan Turing defined what we today call a universal Turing machine. With a (coded) natural number as input, it can imitate another Turing machine, and each Turing machine can be simulated that way, given an appropriate parameter. This looks like emergence but actually the matter is no longer surprising as soon as we realize that the parameter encodes the program of the given machine that is to be simulated. Therefore, according to our criteria, emergence is not taking place here. Still, the example indicates that perhaps one can find emergence in mathematical structures analyzed with the help of methods employed in modern mathematical logic.

4. Logical foundations of mathematics

The so-called nonstandard models come to mind when mathematical logic is evoked in our context. Theories in first order logic have unintended models; that is, there exist mathematical structures that satisfy all axioms of a given theory but are not isomorphic with the "standard" model that served as the source of the axioms. In the case of set theory we are faced with the Skolem paradox: whereas set theory (formulated in first order logic) is supposed to describe all sets, of all possible cardinalities, it admits countable models that can be constructed from natural numbers. Each of them does contain higher infinities in the sense of the model, since there is no function in the sense of the model inside the model that would establish a one-to-one correlation of the sets of different cardinalities. The sets are countable only from the outside. This situation is well understood by logicians, for whom the initial paradox disappears.

Set theory was conceived as maximal, referring to all sets. At the other end there are theories of natural numbers $0, 1, 2, 3, \ldots$. All of them, if in first order logic, admit uncountable models. This is as surprising as Skolem's paradox. (It can be mentioned here that the first nonstandard model for arithmetic was also constructed by Skolem.) And actually there are many complicated nonstandard models of arithmetic, countable and uncountable. Each of them contains infinite numbers, that is, numbers bigger than each standard number $0, 1, 2, 3, \ldots$. But, again, this infinity is perceived only from outside; inside the model all elements have the same status as the standard numbers. It follows, and this is well understood by logicians, that it is impossible to express the notion of a standard number inside the model.

Nonstandard models have some properties that can be linked to emergence: they are unexpected, at least initially. However, the surprise disappears for anyone who is initiated into the theory of models of first order logic. One quickly gets used to the fact that first order logic emerges as too weak to describe the intended model properly. In addition, nonstandard models of arithmetic introduce a more sophisticated view of the aforementioned passage from the finite to the infinite. Mathematicians are free to produce all sorts of abstract models; they can be made of anything and are considered acceptable as long as they satisfy all the axioms. One of the most fruitful methods is due to Henkin: models can be constructed from abstract linguistic entities, and if we begin with an arbitrary consistent set of sentences we can add all the necessary individual constants, identify some of them, and obtain a model of these sentences. The existence of an immense variety of models ceases to be startling. More to the point, the surprise caused by non-standard models does not seem inescapable. It is, therefore, doubtful that we are really facing emergence here.

And yet in this area of mathematics, or rather the logical foundations of mathematics, there exists a phenomenon that is, in my view, fully worthy of the name "emergence." It can be found in the realm of natural numbers: not in nonstandard models, however, but in the familiar standard model. The concept of a natural number seems very ... natural. It seems that the operation of successor, "+1", describes the concept. We begin with 0 and iterate the operation indefinitely. One can remark that the concept of unlimited iteration is itself very close to the concept of a natural number. Yet, still, the resulting set of natural numbers, N, seems transparent. We have always known that there are many difficult problems involving natural numbers, yet their totality seems transparent enough to assume that there exists, at least in principle, a procedure to decide whether a given statement is true or not. And, indeed, there is such a procedure if the language is first order and its vocabulary consists of standard logical concepts (sentential connectives and quantifiers binding variables ranging over natural numbers) and the successor operation. In fact a natural set of axioms is complete; each sentence in this language can be either logically derived or refuted from the axioms. What is more, a slight extension of the theory preserves decidability. Namely, when the operation of addition is added, one gets Presburger Arithmetic. In 1929, Mojżesz Presburger demonstrated that the first order sentences true in $(N, +)$ form a complete, decidable theory, which can be axiomatized by a series of natural axioms. Incidentally, when only multiplication is considered, the resulting theory is also decidable. (This was proved by, again, Skolem.) It would seem, then, that nothing unexpected can happen, and that the elementary theory of natural numbers, taken with addition and multiplication, is decidable, as should be also the theory extended by more complicated operations, like exponentiation. That was indeed the expectation of Hilbert and all logicians until 1930. But they were wrong.

Indeed, this is common knowledge now: that standard arithmetic involving both addition and multiplication is not decidable, and that it admits no complete axiomatization as long as the set of axioms is required to be recursive. This was demonstrated by Gödel in his epoch-making paper of 1931. Actually, Gödel proved much more. His result is not only about first order logic, but about arbitrary means of effective listing of, among other things, arithmetical sentences in the first order language referring to addition and multiplication. No axiomatization, formal system, computer, or Turing machine can produce all such true, and only true, sentences.

The advent of undecidability as a consequence of one simple step consisting of piecing together multiplication and addition, should be called emergence. This is surprising; and it was surprising to all experts when it was discovered. What is more, it remains surprising. This claim may be controversial, so it requires an explanation. The contrary view would be based on the argument that the phenomenon of undecidability of arithmetic is well known now, as are many related results. Mathematicians and logicians have got used to the fact discovered by Gödel and they know that the structure of natural numbers, considered with addition and multiplication, is so involved that one can represent in it all recursive functions, and this can be proved to be sufficient to represent also some non-recursive sets. It can be added here, that the natural question, of what happens if exponentiation is added, and then further functions, admits an impressive answer: nothing new is happening. Exponentiation can be defined in Peano Arithmetic (and even in weaker theories), and all primitive recursive functions can be defined as well, as was shown by Gödel in his paper. If so, does the fact that we have learned so much eliminate the initial surprise?

My view is that it does not. The reason is that it is hard to explain why this undecidability occurs. It seemed that the iteration of the successor operation defines the natural numbers. And it still seems so. What happens when addition and multiplication are added? More can be expressed. The natural numbers, so utterly simple, become suddenly very complicated. The complexity is objective; it has nothing to do with our way of approaching it.

The numbers emerge as being anything but simple. Whatever we propose as a definition of them is necessarily inadequate. No definition, no program is sufficient. This much is known; but it is very difficult to overcome the initial surprise and to agree that no definition, no finite description can be given. After all, it seems we do *know* what the natural numbers are. Why is there no comprehensive, adequate definition of natural numbers? After all, we do give definitions that seem to grasp our intuitive concept of natural number; we formulate Peano axioms, define second order arithmetic, etc. What are those definitions, if they are not adequate? The answer is that they are definitions good enough for us, but not comprehensive enough. Apparently, they function as definitions only in conjunction with some background knowledge that is not explicit. That is, some implicit resources are inevitable. In the case of natural numbers defined by means of the successor operation, the intuitive knowledge is applied, as mentioned before, in the idea of unlimited iteration of the operation. To understand what this means, the concept of potential infinity must be available, and, indeed, some understanding of the natural numbers.

It is impossible to describe the entirety of our concept of natural numbers in a way comprehensive enough that it can be communicated to another human being or to an artificial being, say a robot, without additional assumptions about the tacit background knowledge of the recipient. Where *our* tacit knowledge comes from is an interesting issue. Usually biological evolution is offered as the source. Some intuitive common knowledge is needed to understand mathematical definitions of even the simplest concepts. As a matter of fact, they can turn out to be not so simple. The necessity of some background knowledge was obvious for traditional philosophers and also suits those modern attempts that try to uncover hidden assumptions, like phenomenology, at least since Husserl's *Lebenswelt*, and the later philosophy of Wittgenstein. The awareness that tacit background knowledge must exist is also present in analytic philosophy that, like the work of Putnam, overcomes the naïve temptation to formalize everything. One can also say that it should be obvious that in order to formalize anything, something unformalized must be left as the fundament. Gödel's theorem seems not only to confirm that intuition but also to indicate that it is necessarily so.

Interestingly enough, the thesis about the unavoidability of tacit knowledge even in apparently simple mathematics remains in place even if we take into account the possibility that the human mind is equivalent to a machine. This possibility is not excluded by Gödel's results despite claims to the contrary made by many authors who have not understood what Gödel himself noticed: that the existence of such a machine, equivalent to the mind in the realm of arithmetical sentences, does not contradict his theorem. If it exists then there must remain something beyond the transparent, understandable fragments of the program. This "something" may be located in some common knowledge that makes it possible to say that the program is correct, or it may be some innate feature or some property of hardware.

Whatever the nature of the background knowledge is, and however hard we try to understand it, it seems to me that we are unable to imagine the resulting complexity of numbers taken with both addition and multiplication if our point of departure is solely our intuitive understanding and the naïve definition of natural numbers. Thus we can say that we are facing genuine emergence.

Additionally, emergence in a loose and rather metaphorical sense can be also seen in two other aspects of the logical foundations of mathematics indicated by Gödel's proof. Thanks to his proof one can refer to "Gödelian emergence." Let us consider an axiomatic theory in a broad sense – we require only that it is rich enough to make Gödel's proof applicable. Given such a theory, if we assume the consistency of the theory then automatically the arithmetical sentence expressing the consistency of the theory can be assumed to be true. The sentence can have a simple form; namely, it can state that "there is no solution of a Diophantine equation p = 0", where p is a polynomial with integer coefficients." (The polynomial p is defined specially for the theory in question.) What is more, this is still true even if the theory is about a completely different area with no direct connection to arithmetic. As soon as we agree that the theory is consistent we can also assume that a certain specific and highly unreadable equation has no integer solutions and, still more, that this statement (stating that the equation has no solution) is not derivable in the original theory (if the standard coding procedures are used). Some kind of inexhaustibility of mathematics can be seen here; in particular, if a theory is intended to include the *whole* of mathematics, an appropriate statement about the non-existence of integer solutions of a certain equation refutes the intention. This inexhaustibility, that is, the process of going beyond any framework that we can propose that is supposed to capture mathematics, can be seen as a sort of emergence.

A related interpretation was proposed by Michael Dummett. He brings Gödel's results as an argument in favor of the intuitionist concept of number. Incompleteness, he says, is the result of the internal unclarity of the meaning of number. We know what numbers are but we cannot escape ambiguity because the principle of mathematical induction must be true for all well-defined properties of numbers, and the concept "well-defined property of number" has no fixed reference but is indefinitely extendible. According to

Dummett, Gödel has shown that the class of principles used to recognize the truth of sentences involving quantification over natural numbers cannot be precisely defined; it must be seen as an indefinitely extendible class. This conclusion is in accordance with the vision advocated by mathematical intuitionists: the class of intuitionistically acceptable proofs grows in time because we understand our mathematical constructions better and better. This extendibility of the notion of a mathematical proof and of the concept of a well-defined property can be seen as an indication that the concepts are creative. There is something new arising, something impossible to anticipate, and, therefore, we are witnessing here emergence or something akin to it.

The two prime examples of emergence in mathematics considered here, the fractals and the undecidability of the structure of the natural numbers taken with both addition and multiplication, have an interesting similarity to examples from the material world. The properties of fractals and numbers are considered as objectively existing properties of structures that are completely independent of us. In these examples, mathematics looks strikingly similar to science. It is probable that examples of emergence in mathematics can be relevant for the philosophy of natural sciences.

Stanisław Krajewski
Institute of Philosophy
University of Warsaw
`stankrajewski@uw.edu.pl`

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Witold Marciszewski

CONFRONTATION OF REISM WITH TYPE-THEORETICAL APPROACH AND EVERYDAY EXPERIENCE

1. Grzegorczyk's project for merging reism with type theory

Quine's famous essay "On What There Is" starts from the following remark.

> A curious thing about the ontological problem is its simplicity. It can be put in three Anglo-Saxon monosyllables" 'What is there?' It can be answered, moreover, in a word – 'Everything' – and everyone will accept this answer as true.

Obviously, such accord is only apparent, since each philosophical doctrine offers its individual answer on what this *everything* does comprise. However, the strategies of justifying their claims have something in common. Each philosopher tries to distinguish a category of entities whose existence would be most certain, beyond any doubt, and then – on this basis – to demonstrate the existence of other kinds of entities. Let the elements of such a basic category be called *prime existents*[1] The history of philosophy can by summarized through mentioning kinds of entities which were acknowledged as prime existents by particular thinkers: by Plato – universal ideas; by Aristotle – substantial individuals; by Democritus – atoms; by Descartes – his own mind; by Hume – sense data; by Brentano (in one phase of his development) – mental phenomena, etc.

What in the Aristotelian idiom is called *individual substances* can be translated into vernacular English as individual things or, even simpler, *things* – Latin *res*.[2]

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[1] In the use of the term "individual substances" I follow Moran's [2000, p. 35] account on Brentano's ontology.

[2] Dictionaries contain a long list of everyday meanings of "thing". Among them are

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Thus Aristotelian ontology might be called reism, but in fact this designation has come into use owing to Tadeusz Kotarbiński. He termed so a specially strict form of reism which restricted the realm of existence to those things which are physical tridimensional bodies in space and time; hence another term he used to denote his view is *somatism*. Still another name is *concretism*.

There are a number of Andrzej Grzegorczyk'a writings in which he introduces himself as a telling follower of reism. However, his involvement is different from that of Kotarbiński. The latter held firmly his tenet as the last decisive word of philosophical wisdom, and focussed his attention on defending principles of such materialistically oriented reism. With Grzegorczyk there is no stress on materialistic orthodoxy; instead, he cares for the culture of rational thinking, and sees reism as a possibly useful tool for this purpose. In pursuing this goal, he is like an earnest researcher who gets deeply engaged in a thought experiment; that of adopting reism for the dissemination of logical culture. In this enterprise, he looks attracted by reism, and with empathy embraces it as if his own position. However: with the proviso that one succeeds in transforming Kotarbiński's rigid reism into a more flexible tool of efficient thinking.

In his book devoted to applications of logic in real human life, under the expressive title "Logic – a Human Affair" [1997], Grzegorczyk endorses reism but with the said proviso. This runs as follows (p. 12):

> The style of writing I have chosen in this book may be called **reistic**. It is reism in a liberal sense. The sense will be explained in a moment.

When employing the term *liberal*, Grzegorczyk displays the awareness that in an innovative way he is combining Kotarbiński's manifesto with his own epistemological and ontological vision, and his logical expertise. Grzegorczyk's own explanation of liberalizing is to the effect that the reistic style of describing the world can be exhibited in a more technical way, to wit, with recourse to Russell's simple theory of types. Its core gets explained by Grzegorczyk in a way quite similar to the formulation found in Kurt Gödel's article "Russell's mathematical logic" [1944, p. 126] (I quote Gödel to give – by the way – some taste of the classics of mathematical logic). Gödel's text runs as follows.

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the following: a separate and self-contained entity, an action, any attribute or quality considered as having its own existence, a piece of information. It is the first item in this enumeration that gives rise to the philosophical reistic meaning of "thing".

> By the theory of simple types I mean the doctrine which says that the objects of thought [...] are divided into types, namely: individuals, properties of individuals, relations between individuals, properties of such relations, etc., with a similar hierarchy for extensions [i.e., classes, called also sets].

Consequently, the theory of types admits of quantifying variables of any type, not only those of the type of individuals. These classics are disregarded by Kotarbiński [1957] who puts the following rigid restriction on quantifying variables other than those ranging over individuals; he writes what follows.

> The [acceptable by reism] system [of quantification logic] is devised from a standpoint which does not admit of binding by quantifiers other variables than individual ones. This restraint is to prevent the use of the existential quantifier with respect to predicate variables. For this would entail the existence of some entities other than individuals; namely, sets denoted by predicates, while – in fact – individuals are the only existents. [p. 158, ad hoc translation by W. M.]

When taking into account that the very core of type theory consists in allowing quantification within each type – sets (as extensions of predicates), sets of sets, etc. – Grzegorczyk's type-theoretical enhancement of reism seems to be like reforming one party's dictatorship through converting it into pluralistic democracy; it would be rather turning the system upside down instead of a limited correction.

However, the term "reism" can be used in a legitimate way, following its Latin etymology, without sticking to the orthodoxy established by Kotarbiński. A way out should be found with the catchword "Back to Aristotle", following Grzegorczyk's [1959, p. 8] suggestion that the Russellian theory of types may be seen as a modern counterpart of the Aristotelian idea of categories of being.

Let's note how these theories are related and complete each other. The theory of types has the enormous advantage that successive types are being introduced in a systematic manner, according to a uniform procedure, and without any upper limit (up to infinity). However, being a mere formal system, a syntactic framework, this theory does not tell what entities constitute the type of basic elements, called *individuals*, that is, the lowest type. As individuals there may be taken numbers, atoms, apples, points, situations, Platonian ideas, events, sense-data, minds, mental states, etc. What will be chosen depends on an intended semantic interpretation of the basic set.

Now let's go back to Aristotle, and ask about his interpretation of the lowest type, i.e., lowest category. He establishes a hierarchy of types as well, but much restricted in number. For he takes into account only those which we can perceive in the limited range of our experiencing reality. Thus his categorial framework is less embracing, as being concerned with one actual world, not with a multitude of possible worlds (each having basic elements of its own). In this world there is no uniform systematic procedure of entering new types on the basis of preceding steps, but each step has to be considered separately, with ontological intuition.

Why does the Aristotelian system of categories deserve to be called reistic? Let us start from noticing that Aristotle distinguishes the primary kind of reality from secondary kinds, and this primary (so to speak, complete) attaches to the lowest category alone, while different secondary kinds (less and less remote from completeness) get distributed among remaining types. In such a framework, the reistic component consists in identifying the basic elements (that is, completely existent) with things. They are called *primary substances* and form a universe which includes things in the modern reistic sense.

The Aristotelian categorization can be connected with a type-theoretical categorial framework owing to the Russellian distinction of *complete* and *incomplete* expressions. The former denote higher types (classes, properties etc.) while the latter denote individuals. This semantic distinction can be transferred to the ontological level, and be applied to respective entities as denoted by said kinds of expressions. Thus things will be called complete entities, while classes, or properties – incomplete entities. An alternative stipulation is due to Alexius Meinong who for the kind of reality possessed by properties employed the term “subsistence” (German *Bestand*).[3]

Now the difference between the original strict reism and liberal reism can be concisely expressed as follows. Liberal reism – in the Aristotelian vein – acknowledges two modes of reality, complete and incomplete, or (in another idiom) existence and subsistence, attached to the lowest type (individual things) and the higher types (classes etc.), respectively. On the other hand, strict reism denies the very idea of subsistence (or, incompleteness in being) as one of the justified modes of reality, claiming instead that only individual things, conceived as physical bodies, constitute the whole of the real world.

Despite all these divergences, there is a feature common to type theory and strict reism. It is the notion of the individual as being at the bottom of the ladder of types. Significantly, *we are forced to consider individuals as*

[3] The prefix “sub” (meaning in Latin “under”) in “subsistence” hints at a secondary sort of actuality. More on this notion – see Bergmann [1964], Findlay [1963], Simons [1992].

the basis of our logic when we use quantifiers and predicates.[4] This feature is characteristic of the whole of modern logic, including type theory as its eminent representative; strangely enough, it is alien to the Aristotelian syllogistic, which in a rather Platonian manner is oriented toward universals. Such a preference for individuals in constructing logic appears to be the rational nucleus in the programme of reism.

2. Tenets of strict reism and its perplexities in the face of scientifiic theories

The confrontation I have in mind occurs with respect to any scientific theory which involves abstract theoretical concepts, remote from our everyday perceptions of such tridimensional bodies as trees, stones, buildings. Theoretical concepts to be taken into account may by light, heat, magnetism, electricity, gravity, entropy, and so on. In the present Section the main tenets of strict reism are formulated, and discussed with reference to some scientific concepts which prove very resistant to the demands of reistic correctness. This is why one can speak of perplexities.[5]

Kotarbiński takes advantage of the list of ontological categories proposed by Wilhelm Wundt at the end of the 19th century. It includes: things, states of affairs, relations and properties. This original list does not include events and processes; presumably, they are subsumed into the states of affairs; it is clear, however, that it was Kotarbiński's intention to treat them as non-things. His strict reism consists in affirming the existence of things and denying any existence to the remaining items from the list. This can be summed up with the following statements.

[SR1] Any object, that is, whatever does exist, is a thing.

[SR2] No object is a state of affairs (including events and processes), relation or property.

[SR3] x is a thing if and only if x is a resistant and extended object (a material body).

On account of SR3, Kotarbiński also termed his view *pansomatism*; the term derives from the Greek "pan" (all) and "soma" (body). Thus, pansomatism

[4] This maxim is borrowed from Peter Simons [2005, p. 43]

[5] In the paragraphs of this Section which are concerned with reistic tenets, I follow Jan Woleński's [2012] article on Reism in Stanford Encyclopedia of Philosophy: http://plato.stanford.edu/entries/reism/ (I do not use quotation marks since the citation is abbreviated, and also in other ways non-literal). When speaking of perplexities, I follow my own line of reasoning.

claims that all objects (i.e., whatever exists) are material bodies. He used interchangeably the terms "reism", "pansomatism" and "concretism", the last indicating that things should be construed as concrete objects, which is to say, individuals, as opposed to the rest of the categories listed in SR2.

Some comments on each of the points will be in order. As to SR1, the phrase "any object" should be understood as "whatever exists", and the copula "is" as one to express equivalence.

As to SR2, a serious puzzle arises about the question of how we should discern those properties, states of affairs, etc., which are possible and attach to some things, from those which are impossible. The impossible ones are defined as those which are not able to exist. Correspondingly, the possible properties should be defined as being able to exist, and consequently, those who actually attach to something should be regarded as belonging to reality in the way termed subsistence, or incomplete existence, as considered above (§1). Subsistents are no figments, but in the reistic setting there is no way to tell and justify the difference.

Special attention is due to processes as a category of particular importance in sciences. What about such processes as waves in some medium, as water, air etc. where "water", "air" etc. are mass terms? Do mass terms really denote things in the sense of strict reism?[6]

Even worse, what about the non-mechanical type of wave, like electromagnetic waves, which do not require any medium? Instead, they consist of periodic oscillations in electrical and magnetic fields generated by charged particles, and can travel through a vacuum.[7]

Should a vacuum be counted along with things? Democritus would presumably answer in the affirmative, but what about Kotarbiński? Putting scientific doubts aside, we encounter similar perplexities when experiencing processes without any individual thing as their substratum in our everyday experience; such are river, wind, light, heat etc. As not being properties of anything, should they be regarded as things, or rather as no-things?

Among items omitted in SR2, but also thought with reism as nonexistents, there are sets or classes (the two terms are used in the present context interchangeably, according to stylistic convenience). Implicitly, they belong to the forbidden zone, since classes are defined by relevant properties, hence

[6] A penetrative discussion of the peculiarities of mass terms is found in Simons' [2005] paper "Mass Logic".

[7] Even in the Newtonian theory of light, which is closer to reism because of defining rays as streams of particles (hence microscopic solids), there appear concepts from the reistically forbidden language (italicized in the following quotation), to wit: "*Sine* of *incidence* is in a given *ratio* to sine of *refraction*, for every ray considered apart." – Newton's "Opticks", Proposition VI, Theorem V.

having to share with the latter the status of nonexistents. Integer positive numbers, in turn, when conceived as properties of classes, have even a stronger reason for non-existence.[8]

Concerning SR3, the vagueness of the word "resistant" leads to some questions formerly raised in the comment on SR2. Are mechanical waves resistant? True, liquids and gases are resistant, but the waves themselves as propagated in such media are just transitory configurations of particles, being more like geometrical forms than tough pieces of matter. Strict reism leads to even more troubling puzzles when it comes to considering gravity; these cases will be discussed, as especially interesting, in the next Section.

Reism as a system of ontology gives rise to its semantic counterpart – to the effect that expressions referring to properties, relations, classes etc. are not regarded as genuine names; they are called therefore apparent names or onomatoids. In these terms the semantic thesis of reism is stated as the following principle.

[SR4] Only sentences with *genuine names* have a meaning. Those with apparent names are meaningless, unless they are translatable into sentences containing genuine names alone. Owing to such translation apparent names can be wholly *eliminated* from the language. If we use them, this is only for practical reasons without any theoretical necessity (e.g., for the sake of greater conciseness).

For instance, the sentence “Wisdom is a property of some people” has sense owing to the fact that it can be translated into the sentence “Some people are wise” where no apparent names (i.e. “property” and “wisdom”) occur. If such elimination is not available, then the utterance containing apparent names is devoid of any sense.

SR4 when taken jointly with SR1 and SR3, would have disastrous consequences for the whole of our science, since mathematics as well as natural and social sciences ought to be regarded then as meaningless. Even the simplest arithmetical sentences, say (A) “$1 > 0$”, must be regarded by a reist as nonsensical, as the apparent names “1” and “0” cannot be eliminated by replacing A with any sentence about bodies; that is entities existing in time and space. Also set-theoretical utterances would lose the rank of meaningful sentences; nobody can manage to get rid of the word “class”, e.g. in the following string of words: “There is a class of such classes that no one of them is its own element”.

[8] The verdict concerning the non-existence of classes is explicitly stated, e.g., with Kotarbiński [1957, p. 157f].

The lack of sense in the language of natural and social sciences would be equally evident. Consider the following sentence: "The maximal speed of communicating the content of a message cannot exceed the speed of light". According to SR4, the apparent names "speed", "content", "light" should get eliminated in favour of the names of tough resistant bodies – as sole constituents of a sentence which would express exactly the same thought. Is it a tractable task? Such an attempt does not seem to have any chance, even for the most sophisticated followers of reism. But, ultimately, the answer is up to them.

3. The small case study of gravity – to exemplify the notion of abstract constituents

Let us compare three assertions concerning gravity: one due to Newton [1687], another one to Leibniz (as an opponent of Newtonian theory on the grounds of natural philosophy), and still another to Kotarbiński – to be labelled, respectively, as AN, AL and AK (A for "assertion").

AN: There exists the force of gravity.
— On the premise that multiple application of the law of gravity in all areas of the universe has not yielded any counterexamples.

AL: There does not exist a force of gravity.
— On the premises that (L1) the acting of such force would have been an action at a distance, while (L2) there cannot be any action at a distance.

AK: There does not exist a force of gravity.
— On the premises that (K1) gravity would have been a relation between bodies, while (K2) there cannot exist any relation between any objects (compare SR2 above in §2).

AL and AK look like identical assertions, but in the context of their premises they obtain different meanings. The AL denial of gravity is supported by a view belonging to the philosophical foundations of physics; with Leibniz this view included the basic regulative principle that it is factually impossible for any body to exert an action at a distance. On the other hand, the rejection of gravity by AK derives from the ontological tenet that any relation is condemned to non-existence, hence gravity too.

Now, the homework to be done by the followers of reism would consist in getting rid of (what they think as) apparent names (italicized below in LG) from Newton's Law of Gravity.

> **LG:** Every *point mass* in the universe attracts every other point mass with a *force* that is directly proportional to the *product* of their *masses* and inversely proportional to the *square* of the *distance* between them.

Let us recall the scientific meaning of terms used in the above statement.

> *Mass* is the quantity of matter in a body regardless of its volume or of any forces acting on it. The term should not be confused with weight, which is the measure of the force of gravity acting on a body. Under ordinary conditions the mass of a body is regarded to be constant, while its weight not, since the force of gravity varies from place to place.
>
> A *point mass* means a point particle with a nonzero mass and no other properties or structure (likewise, in the theory of electromagnetism there appears the notion of particles with a nonzero charge).
>
> *A force* is that which can cause an object with mass to change its velocity, i.e., to accelerate, or which can cause a flexible object to deform.

In order to account for meanings of the terms italicized in the definitions above, we shall need a special notion; it seems not to have appeared so far in the literature, but shall prove indispensable in confronting reism with the reality handled by science.

Let's note that when speaking of mass, point mass, and force, we mean some constitutive parts of bodies. Let us call them *constituents*. They are parts since they are somehow *in* bodies, being different from the whole of the body in question. They are constitutive since they are necessary to constitute a body. However, they are parts which are not able to exist in separation from their wholes, while, for instance, a car's wheels can exist separated from the car. In this sense, we say that a wheel is a *concrete constituent* of car. Consequently, it is in order to say that the mentioned masses, forces etc. are *abstract constituents* of bodies. A more familiar example, taken from everyday experience, is that of perceiving surfaces. The surface of the moon is a constituent of the moon, but it cannot exist in separation from the moon itself; hence it is no concrete constituent, but an abstract one.

Another handy expression to account for this ontological relation is the Platonian term *participation* (Gr. *metechein*, but taken in a sense which is reverse (not to say "perverse")) to that in Plato's "Parmenides" and other dialogues (cp. Scheffel [1975]). To wit, with Plato individual concrete things participate in universals, while in the here – adopted usage (akin to Aristotle's) universals participate in individuals. Such a reversal reveals in what sense the view here expounded can be regarded as liberal reism: reism for that anti-Platonic point, and liberal for not denying reality to universals conceived as abstract constituents.

The notion of abstract constituent fits into the categorial framework of the theory of types, since abstract constituents form a kind of property (the main idea of the theory of types has been sketched in §1). Thus the surface of the moon is a property of the moon as a solid, and the line being a perimeter of this plane is the plane's property. Hence an abstract constituent of a whole which is of type n, belongs itself to type $n+1$, like in the case of properties of bodies (as their abstract constituents). Let this be exemplified by a rough (with ad hoc numbering) calculation of types of objects involved in the process of gravitational attraction.

> Let's agree that any body in the universe belongs to the lowest type, labelled with number 1. Hence its mass, being its abstract constituent, is of type 2.
>
> A point mass being an abstract constituent of mass, is of type 3.
>
> Gravitational attraction acting with a certain force is a relation between point masses, hence it is of type 4.
>
> Force of attraction is a property of attraction processes, hence it is of type 5.

Every next type number marks a higher level of abstraction in the above sequence of abstract constituents. With this fact in mind, a kind of arguing "ad hominem" (i.e., appealing to personal considerations) will be in order. A person who considers that there is in the universe the force of attraction, will be ready to acknowledge that there are abstract constituents in reality. On the other hand, one who claims that no abstract objects can participate in reality is bound to deny reality to the force of gravity, independently of how great is the scientific merit of this concept.[9]

The same is to be said of many other abstract theoretical concepts in science. Each of us is free in her/his worldview either to believe that there are abstract constituents, or to believe that they are not the case, being just mental figments which in a mysterious way prove astonishingly fruitful in perceiving and mastering physical reality.

[9] Such a radical attitude is typically represented by Nelson Goodman [1967, p. 214] in the following statement. "[I] do not presume to restrict the scientist. The scientist may use platonic class constructions, complex numbers, divination by inspection of entrails, or any claptrappery that he thinks may help him get the results he wants. But what he produces then becomes raw material for the philosopher, whose task is to make sense of all this: to clarify, simplify, explain, interpret in understandable terms."

4. A type-theoretical approach to analysing systems, esp. domains of discourse

No philosopher should be expected to hold the same views in every period of his philosophical development; this would contradict the very idea of intellectual evolution. On the other hand, in such evolution there is usually an aspect of continuity. and that ought to be also taken into account by commentators. This I shall try with respect to Grzegorczyk's [1963, Polish] paper on *applications of the logical method of formally analysing domains of discourse* in the sciences, technology and economy".

The concept of a domain of discourse is patterned on that described by Grzegorczyk in his textbook [1974] on mathematical logic; there are found typical examples of *domains* or *systems* or else (still another term) *structures* studied in arithmetic and algebra with the use of notions provided by logic (allied with set theory); hence the use of the phrase "the logical method". The paper [1963] is meant to extend this method over other, possibly all, domains of discourse – with the purpose of making their concepts more precise.[10]

Before looking at how the Author deals with some examples of domains (structures, systems), it will be in order to consider a methodological reflexion closing his paper. This is worth special attention, since there shines through it a conflict between the reistic and the pragmatic approach in doing science. Let the core of the latter be summed up by the Chinese proverb *Black cat or white cat: if it can catch mice, it's a good cat.* Suppose that reism is a white (this means somehow a nicer) cat, and there is a theory disapproved by reism which nicely proves its mettle (catches mice); such a pragmatic reason justifies employing that theory as a good black cat.

The said reflexion is occasioned by using the term "internal states" by Grzegorczyk in describing such structures as machines and organisms, though speaking about states of affairs is by reism forbidden as meaningless. Now, in the light of this Chinese wisdom, let us read the following passage of his article [1963, p. 73] which might be entitled: *a pragmatic justification of the acknowledgement of abstract constituents.*

> Somebody may try to challenge the introducing of internal states, saying that the mode of existence of such objects is suspect from a philosophical point of

[10] A trait of continuity can be seen not only in the mentioned connection between publications of 1963 and 1974, but also in the fact that the same didactic intention guides the study of 1963 and a later book [2007] – both dealing with applications of logic in human affairs.

> view (for instance, may be, from the reistic point). However [...] with respect to the description of a particular phenomenon it is often very convenient to introduce parameters whose mode of existence encounters numerous difficulties from an ontological point of view. A classical example is provided with the concept of the geometrical point, which is philosophically hardly conceivable in reistic language, but it functions as the basis of calculus, and the whole of technological applications of mathematics rests on this concept. [Ad hoc translation by W. M.]

In spite of being aware of the conflict between a pragmatic approach and reistic orthodoxy, the Author does not give up his methodological pragmatism. The core of his pragmatism consists in treating properties as if they were individuals, if only this proves efficient in analysing a system. Let me explain this with the help of the notion of abstract constituents (as discussed above in §3).

And thus, e.g., the visible surface of the moon is its property whose reality consists in being an abstract constituent. It is the moon's constituent since it belongs to the moon as a solid, and it is abstract as there cannot be surfaces outside solids. The moon's surface, as its property, in turn, has the properties of having such and such shape, of being colored etc. Let's consider its color, say, gold. The property of being gold, as not being able to occur outside a surface, is – in this example – an abstract constituent of the lunar surface. Goldness, in its turn, may be more or less vivid, more or less deep, and so on. Such properties form a certain set of abstract constituents of colors; we call them abstract (let me recall this once more) since, for instance, vividness of color cannot occur independently outside a color.

Such a lunar story in a simple way exemplifies the type-theoretical hierarchy of properties (see Gödel's definition in §1). One recognizes properties of an individual self-contained thing (the moon's solid), then properties of these properties, which again possess properties of their own. According to the strict original reism, such higher-type objects are mere mental figments which cannot form any non-empty set. Grzegorczyk's [1963] practice of system analysis demonstrates the pragmatic unavoidability of higher-type entities. He considers non-empty sets of elements of different types, without any fear of climbing higher and higher up this ladder, and owing to such a procedure he obtains a cognitively fruitful, hence pragmatically recommendable, picture of a domain of reality.

Such a domain is specified by the following components: (1) the set, or more sets, of individuals as basic elements, sometimes with some individuals distinguished by their names, (2) the list of properties and relations to be predicated of individuals, (3) specified apart, there are many-one relations, that is, functions; those in mathematics are necessary for computing, while in empirical systems – for establishing constant dependencies, for instance, causal connections, as a basis for making predictions. The most schematic presentation of such a system is as follows (the label SS abbreviating "the Schema of a System").

SS: $$\langle X_1, \ldots, X_k;\ R_1, \ldots, R_m;\ F_1, \ldots, F_n \rangle.$$

Among the examples analysed, there is a piece of painting. In the first segment in SS, Grzegorczyk puts the following sets of basic elements, that is, individuals: (X_1) the set of colored spots which constitute the given picture, (X_2) the set of all possible shapes which can attach to colors, (X_3) the set of all possible colors.

What is curious in this description is its overtly anti-reistic feature. Its basic elements, or individuals, are no things – in the sense of tridimensional bodies. The analysis starts from abstract constituents of the piece of painting: spots, colors, geometric shapes. Such a proceeding can be justified only by stipulating that instead of an absolute concept of the individual, one deals with a relative one, that is to say, relative to the system under study. In a full order of types, one that starts from self-contained individuals is basic; the colors and shapes would be objects of a higher order. However, if they are what the analysis starts from, then basic elements are allowed to be treated as relative to the system in question. Such a proceeding can be compared to a quick going up the stairs, when one jumps two steps at once, with one leap; then the second step is for him like the first. In the further description of the given painting there appear relations between the mentioned abstract constituents (playing the role of individuals), for instance the ordering relations of being a clearer color and being a more saturated color; also the function to attribute a shape to a spot, and so on.

Another case discussed by the Author is language as a system whose basic set (that of all individuals taken into account) consists of symbols; as being written tokens. Symbols are things in the strict reistic sense, hence no tactics of reisation is here necessary. However, in describing a language one needs another liberalizing move, to wit, a hierarchy of sets (classes): the set of all admissible (in the language of question) concatenations of symbols divides into certain categories (parts of speech), each category being a class which divides into subclasses, etc., and thus we obtain a set of categories, hence a higher type set containing other sets as its elements.

Special interest is due to the structure called by the author machine or automaton. It nicely fits into the idea of Turing's machine (whose enormous impact upon current scientific thinking is beyond any question). This structure includes two sets of basic elements: (i) internal states of the machine and (ii) the tokens, hence material objects, which it produces. While the latter are things in the reistic sense, the former belong to the category of states (possessed by things) which strict reism excludes from the scope of reality. Thus one treats internal states of a machine as basic elements on a par with physical tokens produced by the machine; the same applies to "living machines", namely organisms.

A still further departure from strict reism is found in Grzegorczyk's description of the process of production as performed by machines. He acknowledges the process itself as an object to be described, and defines its basic set as the collection of all possible conditions needed to produce an output. In this way, such abstract entities as are conditions of a process get promoted to the rank of basic elements; that is, individuals, in the structure being described.

There are even more abstract entities in Grzegorczyk's repertory of examples, to wit: (1) games, (2) the whole of some country's economy, (3) the mental life of a human. As basic sets of individual elements we have, respectively, (1') a class of game situations, (2') a class of human economic activities, (3') classes of human reactions (to stimuli from a certain class), and classes of human dispositions (to act accordingly to certain conditions).

When dealing with such analysis of systems that reveals their highly abstract constituents, one may understand the pangs of reistic conscience as testified by the Author in the passage (see above) which I titled "a pragmatic justification of the acknowledgement of abstract constituents". In fact, there occurs a confrontation of reistic tenets with an actual practice of research in which the type theoretical approach proves necessary. This approach commits us to treat abstract constituents as legitimate elements of empirical reality.

There is another source of such commitment, namely resorting to what Grzegorczyk calls *everyday experience*. In his intention it should have supported tenets of reism. However, some second thoughts lead to the realization that abstract constituents of bodies are present to us even in our everyday experiences.

5. How does the acknowledgement of abstract constituents comply with everyday experience?

Andrzej Grzegorczyk belongs to those philosophers who in high esteem hold what they call everyday experience.[11] His view can be truly rendered by the maxim which is due to Thomas Reid: *I acknowledge that a man cannot perceive an object that does not exist.*[12]

This maxim provides us with a relevant context to explain the sense of the expressions “object” and “there is” which in the foregoing narrative were used without such a reflexion. For the sake of academic communication it is recommendable to follow the usage practised by the classics, and these are – in our issue – Russell and Meinong. Both use the term “object” (German “Gegenstand”) for absolutely everything. Some objects exist, some subsist, and some neither exist nor subsist (as for the concept of subsistence, see § 1 in the text referring to note 3, and in §2 a comment on SR2).

In using the phrase “there is”, a reasonable strategy seems to be the following: *let the quantifier expression “there is/are” involve – in its domain of quantification – any object which either exists or subsists.* Thus, when suitably paraphrasing Reid’s principle, we could obtain a handy idiom to render his idea more exactly: A man cannot perceive an object that does not exist or does not subsist.

In a more explicit way, the thus modified Reid’s Maxim (RM for short) will be rendered by the conditional RM or, equivalently, RM*.

RM: If an object neither exists nor subsists, then it cannot be perceived.

The same in the form of sentential schema: $(\neg e \land \neg s) \Rightarrow \neg p)$.

RM*: If an object can be perceived, then it either exists or subsists.

This is not a suggestion which might be welcome for strict reism, since any idea of something like subsistence is, obviously, alien to it. But what about liberalized reism? Could it acknowledge this idea as its own? To address this issue, let us refer to Grzegorczyk’s [1997] statement which I label with the letters ECR to mean *Epistemological Criterion of Reality.*

> ECR: [1] We grasp reality directly only in our human every-day experience. Hence the first task of a philosophical system is to produce a philosophical

[11] This orientation is shared by quite a number of authors who highly appreciate what they call common sense, or appeal, as did prominently Gilbert Ryle [1949], to the arbitration of a natural language.

[12] Quoted after Simons [1992], p. 159.

> language suitable for a consistent and coherent description of this macro-every-day-reality. This reality is the first thing, which may be meant as something that is **given** and that we should report on. [p. 8]
>
> [2] The reality of every-day experience comprises: things *with* different properties, connected by different relations, making up different sets. [p. 12, numbering and italics by WM]

In the original text, part 2 is underlined with bold type to hint at the importance of this point. In context, however, one does not find any comment to explain which of many meanings of "thing" in ordinary English is the one the Author has in mind. Fortunately, we find an explanation with Grzegorczyk [1959]. This runs as follows:[13]

> An object is said to be a thing if it is tangible, spatial, weighing and lasting, as are tables, stones, trees, houses, people, animals. [p. 10]

The key role in ECR/2 seems to be played by the monosyllable "with". Should it mean that the properties of things, as well as relations between things, hence their abstract constituents, belong to the reality grasped (according to ECR/1) by everyday experience? By using "with" instead of "and" the Author might have meant that properties etc. are not perceived in the same way as are things, but nevertheless they together do belong to the field of perception. And this way of their belonging gets, in the present discussion, rendered by "subsistence".

Before considering how far this may comply with Grzegorczyk's intention, let's consult our own bodily senses and our inborn common sense, with the help of the following story.

> Nice instances of abstract constituents can be found in the heavens. There we observe objects from a constant angle, without any opportunity to change it and to see the object from another side, and so to obtain additional information as a premise of inference. To wit, in the reasoning which occurs with changing the angle, we infer that the front of a building, observed a moment before from another angle, is only one of the surfaces of the solid in question, and thereby our experience, having been merged with an inference, involves some foreign elements. However, due to our choosing some heavenly objects for observation, we can obtain pure, most indubitable, sense data; now even a radical empiricist cannot impair the sensory trustworthiness of our experience.

[13] Not having found anything like this statement in the book of 1997, I make use of a much earlier text, but then the question arises: whether the later Grzegorczyk would have agreed with himself earlier? For the moment, the question must remain open, hence the interpretation which follows should be regarded as hypothetical.

Nobody will doubt that when observing the moon, the sun, or other heavenly entities, we perceive them not as solids but as twodimensional planes. If one wishes to stick to the reistic orthodoxy, one is committed to assert that planes do not exist. A plane has, for instance no weight, which according to Grzegorczyk (following Kotarbiński on this issue) is necessary for existing as a thing. But how to account then for the fact that the moon's twodimensinal surface is given in our every day (and even everynight) experience? No other way out seems to be available than the admitting that planes are not things to exist inedependently but are abstract constituents which are present in things. Theirs is not complete existence but some sub-ordinate way of being, which appropriately can be (following Meinoing's usage) termed as subsistence. A case like that of the moon makes us aware that to be subsistent, likewise to be existent, is sufficient to be given in experience, including the everyday mode of experiencing. And this is the moral Reid's maxim RN* is to tell us.

Could Grzegorczyk give a nod when listening to this story? In his texts, I did not encounter any explicit utterances in this matter, but his intention of liberalizing reism would find the second realization, besides that of adopting the theory of types. Moreover, these two steps would complete each other. If one rejected RM*, then only the objects of a lower type could be experienced, since those from higher levels would be pure figments. Then how should we distinguish, for instance, such geometric objects as planes, lines or points from such as rectangular circles?

Grzegorczyk notes in his paper of 1963 (quoted above in §4) that the concept of the geometric point functions as the basis of calculus, and the whole of technological applications of mathematics rests on this concept. It is a good pragmatic reason to see points as some constituents of reality, in contradistinction to rectangular circles. How to account for this pragmatically justified differentiation? The suggestion which results from this paper's discussion is to the effect that rectangular circles do not exist, solids belong to existents, while points represent the category of subsistents as being abstract constituents of something that exists.

This might be the standpoint of liberal reism. It would remain reism because of the firmly held priority of individual things as forming the basic type. And would become liberalized owing to admitting abstract constituents into the sphere of reality in the role of subsistent abstract objects; this should be justified by their pragmatic fruitfulness as well as their ability to be perceived in everyday experience.

And this may be a fitting answer to Quine's question: *What is there?*.

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Krystyna Misiuna

PHILOSOPHICAL IMPORTANCE OF ANDRZEJ GRZEGORCZYK'S WORK ON INTUITIONISTIC LOGIC

Abstract. We compare Grzegorczyk's semantics for Heyting's intuitionistic predicate calculus with the Intuitionistic Kripke Models. The main problem with model-theoretic semantics for intuitionistic logic is that the concept of truth which implicitly is contained in this logic is different than the classical absolute concept of truth. Intuitionistic 'truth' is temporal. We compare Kripke's and Grzegorczyk's account of intuitionistic 'truth'. The main advantage of Grzegorczyk's semantics is simply the absence of the truth relation which occurs awkwardly in Kripke's semantics. Grzegorczyk replaces the truth relation with the fundamental pragmatic relation of forced assertion between an information state and a statement. Grzegorczyk investigated the relation of assertion in several subsequent papers and defined at least five types of this relation. He also discerned the classical and constructivist assertions. We argue that Grzegorczyk's semantics for Heyting's intuitionistic predicate calculus might be regarded as a predecessor of different present semantics which have arisen in the contemporary informational turn in logic.

1. Introduction

The most important of A. Grzegorczyk's papers on intuitionistic logic come from the years 1964–1971. In [Grzegorczyk, 1964], A. Grzegorczyk published his profound paper establishing the completeness of Heyting's propositional and predicate calculus with respect to the interpretation of this calculus based on the concept of forced assertion. Although A. Grzegorczyk's [Grzegorczyk, 1964] paper adds one more interpretation of intuitionistic calculus to the interpretations existing before, like that of Gödel [Gödel, 1933], Kolmogorov [Kolmogorov, 1932], Jaśkowski [Jaśkowski, 1936], Tarski [Tarski, 1938], Kleene [Kleene and Vesley, 1965], Beth [Beth, 1959], and Kripke [Kripke, 1963; 1965], his interpretation has interesting philosophical consequences, and expresses his deep understanding of intuitionism. In his [Grzegorczyk, 1967] paper, A. Grzegorczyk divides all formal interpretations of intuitionistic calculus into two groups: recursive interpretations

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and topological ones, and places his own interpretation amid the interpretations belonging to the topological group, along with the interpretations of Jaśkowski [Jaśkowski, 1936], Tarski [Tarski, 1938], Beth [Beth, 1959] and Kripke [Kripke, 1963; 1965]. The importance of A. Grzegorczyk's work on intuitionistic logic consists not only in establishing the mathematical result, but foremost is a deep philosophical understanding of intuitionistic logic as a logic of investigation. This has been done in his [Grzegorczyk, 1964] paper, together with subsequent works: his [Grzegorczyk, 1967] and [Grzegorczyk, 1968] papers, as well as in his [Grzegorczyk, 1971] paper. In his [Grzegorczyk, 1968] paper, A. Grzegorczyk has written:

> S. A. Kripke on the Oxford 1963 Colloquium spoke about interpretation of intuitionistic logic in [3]. I did not attend this meeting; but I also published similar ideas in 1964 in [2] developing a bit more the philosophical interpretation [Grzegorczyk, 1968, 86].

In this passage, A. Grzegorczyk mentions Kripke's famous paper on intuitionistic logic and his own 1964 paper. Although both semantics for intuitionistic predicate logic belong to the topological type, there are certain important differences between them, which are not always noted in relevant literature and appreciated. It is true that both semantics prove completeness theorems for Heyting's intuitionistic predicate calculus, but they are based on different concepts and different formalisms.

2. Grzegorczyk's Semantics vs. Intuitionistic Kripke Models

Kripke defines an intuitionistic model structure as an ordered triple:

$$\langle G, K, R\rangle$$

which may be also understood as a tree model structure, where K is a set, G is an element of K, and R is a reflexive and transitive relation on K. An intuitionistic model on this structure is defined as the binary function:

$$\phi(P, H)$$

where P ranges over arbitrary proposition letters, and H ranges over elements of K. The range of this function is the set of truth-values $\{T, F\}$, and it satisfies the following hereditary condition:

$$\text{If } \phi(A, H) = T \text{ and } HRH', \text{ then } \phi(A, H') = T.$$

This condition tells us that if we already have a proof of an arbitrary formula A at the time point H, then we still have the proof of A in any later time point H'. Further conditions which satisfy the modeling function are defined by induction on the number of connectives in A. We shall only turn attention to the most important conditions which are radically different from the respective conditions in the model-theoretic semantics of classical logic.

(Atom) If A has no connectives, then it is a proposition letter P, and $\phi(P, H) = T$ or F.

(Neg) $\phi(\neg A, H) = T$ iff for all $H' \in K$ such that $HRH', \phi(A, H') = F$; otherwise $\phi(\neg A, H) = F$.

(Imp) $\phi(A \supset B, H) = T$ iff for all $H' \in K$ such that $HRH', \phi(A, H') = F$ or $\phi(B, H') = T$; otherwise $\phi(A \supset B, H) = F$.

Kripke explains these conditions in the following way.

> To assert A intuitionistically in the situation H, we need to know at H not only that A has not been verified at H, but that it cannot possibly be verified at any later time, no matter how much information is gained; to assert $A \supset B$ in a situation H, we need to know that in any later situation H' where we get a proof of A, we also get a proof of B [Kripke, 1965, 99].

This small piece of Kripke's semantical model theory for Heyting's propositional calculus is evidence that the concepts of intuitionistic 'truth' and 'falsity' are different from the respective classical concepts. Truth is not an absolute concept, but a temporal one: A proposition becomes true only when it is proved. By relativization of truth-conditions to points in time, Kripke's semantics reflect this understanding of 'truth' to some degree, although the very concept of being intuitionistically true at a certain point of time remains philosophically unclear in Kripke's semantics. An analogous observation concerns also the concept of being intuitionistically false at a certain point of time. In Kripke's semantics, both concepts are formally represented by the function which maps pairs (proposition; time point) into the set of truth-values $\{T, F\}$. The falsity of A cannot be explained by the truth of the negation of A as is the case in the model-theoretic semantics of classical logic, where the equivalence holds: A is false if and only if $\sim A$ is true. In the intuitionistic Kripke models, we can obtain:

$$\phi(A, H) \neq T \text{ and } \phi(\neg A, H) \neq T.$$

This is the case if at point H we do not have enough information to prove proposition A, but we also do not know at H that it is impossible to prove A and assert $\neg A$. Such impossibility means proving that proposition A leads to a contradiction. Note that intuitionistic logic belongs to the family of constructive logics, therefore it adopts a constructive conception of proof. At first sight, one may think that if the intuitionistic truth of A is reducible to the provability of A, then the intuitionistic falsity of A is simply the lack of such a proof of A at a certain time point. This would be something less than having the proof that A leads to a contradiction. There are those who argue for the intuitionistic non-constructive falsity as the lack of proof of A at a time point.[1] This conception of falsity says nothing about accomplishing any construction, and has nothing to do with the intuitionistic refutation. This non-constructive conception of intuitionistic falsity expresses only the possibility of refutation, and in spite of arguments proffered in its favor, it does not respect the original intuitionistic idea of falsity of A as provability of the contradictoriness of A. Kripke [Kripke, 1965, 98], mentions situations where we lack enough information to prove a proposition A, but he explains that in these situations A *has not been verified*, not that A has been proved false.

Grzegorczyk's semantics of intuitionistic logic is philosophically much more illuminating than Kripke's semantics. A. Grzegorczyk [Grzegorczyk, 1964] models intutionistic logic as a certain logic of scientific research:

> Scientific research (e.g. an experimental investigation) consists of the successive enrichment of the set of data by new established facts obtained by means of our method of inquiry. When making inquiries we question Nature and offer her a set of possible answers. Nature chooses one of them [Grzegorczyk, 1964, 596].

Accordingly, the scientific research is modelled formally as a triple:

$$R = \langle J, o, Pr \rangle.$$

J stands for the set of all possible experimental data (the information set, finite or infinite); o stands for the initial information (possibly empty), and Pr stands for the function of all possible prolongations or extensions of the information. The experimental data a is understood as ordered finite collections of atomic sentences having the forms of atomic formulas (without variables) of the classical language of predicate calculus:

$$a = (A_1, \dots A_n); A_i := \text{atomic sentence}.$$

[1] See [Shramko, 2012].

Sentences containing logical constants do not represent experimental data. The relation of extension of the information in research R : *b is an extension of a in research R* is defined inductively for a finite number of new atomic sentences. This concept is used in the main definition of Grzegorczyk's semantics; the definition of the concept of forced assertion: *the information state a in research R forces us to assert the statement expressed by the formula* χ[2]. The relation of forced assertion will be denoted here as $>$.

Definition 1 [Grzegorczyk, 1964, 597]

(Atom) $a > \chi$ iff $\chi \in a$, if χ is an atomic formula without variables.

(Neg) $a > \neg\chi$ iff $\forall b[(b$is an extension of a in $R) \rightarrow \sim (b > \chi)]$.

(Imp) $a > \chi \supset \psi$ iff $\forall b[(b$ is an extension of a in$R) \rightarrow (b > \chi \rightarrow b > \psi)]$.[3]

I omit in Definition 1 the conditions for disjunction and conjunction, as well as for existential and universal quantification. Definition 1 (with other resources) enables us to formulate and to prove the completeness theorem for Heyting's propositional calculus.

Theorem 2 [Grzegorczyk, 1964]

A formula χ (without quantifiers) is logically true in formal intuitionistic logic if and only if each information state a of every research R forces us to assert the statement expressed by the formula χ.

It is easy to notice that the Law of the Excluded Middle does not satisfy the condition of Theorem 2. It may be the case that in a certain initial information state a we do not have $Q(z)$. Therefore, the state a does not force us to assert $Q(z) \vee \neg Q(z)$. On Grzegorczyk's interpretation of intuitionistic logic, the Law of the Excluded Middle belongs to our ontological assumptions about the world, and as such it lies beyond scientific methods.

A. Grzegorczyk's completeness theorem for the intuitionistic logic of quantifiers (Theorem 2 in [Grzegorczyk, 1964]) requires a certain modification of Definition 1 proffered above. The modification concerns the clause for atomic formulas, as well as for compound ones. For example, the clause for atomic formulas takes the following form:

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[2] This concept has its origin in P. Cohen's concept of forcing.

[3] The metalanguage in Grzegorczyk's semantics is classical.

(Atom) $a > A_i$ iff irrespectively of how we continue our research R from the state a, we obtain information b such that b contains the statement A_i.

The clause formulated in (Atom) may be understood as *potential forcing* by the information state a of research R. The modification consists in adding the initial phrase: *irrespective of how we continue our research R from the state a, we obtain information b, such that* .... To formalize this phrase, A. Grzegorczyk introduces the notion of branch of the research R.

$$R \text{ is a } \textit{branch} \text{ of } R = \langle J, o, P \rangle \equiv^{Df} X \subset J \wedge o \in X \wedge \forall a(a \in X \rightarrow \text{there exists one and only one } b \text{ such that } b \in P(a) \text{ and } b \in X).$$

The idea of potential forcing, and the definition of forcing appropriate for statements expressed in the language of predicate calculus, may be now formalized in the following way:

$$X[(X \text{ is a branch of} R \wedge a \in X) \rightarrow \exists b(b \in X \wedge \ldots)].$$

Theorem 3 [Grzegorczyk, 1964]

A formula χ is provable in the formal intuitionistic logic of quantifiers if and only if the statement expressed by the formula χ is forced by each information state of every research.

The main advantage of Grzegorczyk's semantics is the absence of the truth relation and the falsity relation which are present in Kripke's model-theoretic semantics for Heyting's intuitionistic predicate calculus. Grzegorczyk's account suits well the anti-realism of philosophical intuitionism. This philosophical standpoint was noted by M. Dummett:

> Thinking of a statement as true or false independently of our knowledge involves a supposition of some external mathematical reality, whereas thinking of it as being rendered true, if at all, only by a mathematical construction does not. [Dummett, 1977, 12]

Kripke is aware of this philosophical attitude characteristic of intuitionism, and provides informal comments, as well as formal translations of his model-theoretic account into the intuitionistic discourse. Nevertheless, his fundamental semantic definition formulates truth-conditions with the help of the function ϕ which maps the pairs, each of which consists of a formula and a time point, into the truth-values T and F. But as we have already mentioned, intuitionism accepts the idea of temporal truth, according to which a proposition becomes true only when it is proved.

This idea contradicts the central idea of realism which underlies classical model-theoretic semantics and classical logic, where the Law of the Excluded Middle is accepted as a tautology. From this classical point of view, the idea of temporal truth is contentious and counterintuitive. Accordingly, there seems to be a little embarrassment in making use of such a conception of truth in any model-theory. Kripke in his [Kripke, 1963; 1965] paper does not appeal to this idea of truth, but encounters many obstacles in expressing intuitionistic semantics in a model-theoretic framework with the classical concepts of truth and falsity. In consequence, he makes much effort to make sense of his meta-theoretical evaluations in terms of classical truth and falsity applied to intuitionistic language with its intended interpretation.[4]

The solution chosen by A. Grzegorczyk for building a semantics of the intuitionistic language is better, since it avoids the clash of the intuitionistic with classical model-theoretic ideas. From a philosophical point of view, most important in Grzegorczyk's semantics is the fundamental pragmatic relation of forced assertion between an information state and a statement. This relation is defined by A. Grzegorczyk implicitly along with the definition of the connectives and quantifiers of the intuitionistic predicate language. Although formally, this relation is modelled as binary, in fact it is a ternary pragmatic relation: *the information state a forces the subject s to assert the statement expressed by the formula* χ. It is an implicit assumption of Grzegorczyk's semantics that the relation remains unchanged when applied to different subjects, and for this reason the relativization to the subject s may be omitted.

3. The Concept of Assertion in Grzegorczyk's Semantics

Many other philosophical advantages of Grzegorczyk's semantics are discussed by A. Grzegorczyk in his later papers [Grzegorczyk, 1967; 1968; 1971]. A. Grzegorczyk [Grzegorczyk, 1968] proffers a slightly different account of his [Grzegorczyk, 1964] result. The differences are of a philosophical nature. The main relation of forcing assertion by an information state is now called the relation of *strong assertion* of a sentence at a time point in a given inquiry:

[4] Cf. for example Kripke's comments in [Kripke, 1965, 95].

$$As_E(\chi, t).$$

$$\chi \in Atom \rightarrow [As_E(\chi, t) \equiv \chi \in A_E(t)].$$

$$As_E(\neg\chi, t) \equiv \forall s \in T[ts \rightarrow\sim As_E(\chi, s)].$$

$$As_E(\chi \supset \psi, t) \equiv \forall s \in T[ts \rightarrow (\sim As_E(\chi, s) \vee As_E(\psi, s))].$$

$A_E(t)$ stands for the set of atomic empirical sentences we assert in performing experiments prescribed to the time point $t \in T$ by the program of the inquiry E. A. Grzegorczyk claims that intuitionistic calculus forms the logic of the *strong assertion*. The strong assertion is compared with the *admissibility* relation: χ is admitted as supposition in the time point t in E, which may be regarded as a *weak* assertion:

$$Ad_E(\chi, t).$$

The inquiry E is understood as a triple:

$$\langle A, R, L\rangle.$$

A is the set of time points in which can be admitted χ; R is the set of time points in which can be admitted $\neg\chi$, for χ atomic; L is the conjunction of all theories accepted as background for E. The inquiry E defined as above requires discerning the admissibility conditions of atomic and negated atomic sentences, and next the proper admissibility conditions of compound and negated compound sentences. The conditions for atomic and negated atomic sentences are simple and take the following form, respectively (the index E has been omitted):

$$Ad(\chi, t) \equiv t \in A(\chi);$$

$$Ad(\neg\chi, t) \equiv t \in R(\chi).$$

The conditions for compound sentences are more complicated, and we only note that the following equivalence holds for the admissibility relation:

$$Ad(\neg\neg\chi, t) \equiv Ad(\chi, t).$$

The admissibility relation is a weak counterpart of the strong assertion which is characteristic of intuitionistic logic.[5] A. Grzegorczyk [Grzegorczyk, 1967] considers also another concept of assertion understood intuitively as *to be allowed*, meant as the relation:

$$Al_R(\chi, a).$$

[5] The admissibility relation is a formalization of the same kind of relation which occurs in Popper's logic of scientific discovery.

The intuitionistic negation is defined in terms of this assertion as:

$$Al(\neg\chi, a) \equiv \sim Al(\chi, a).$$

This is the main concept of Grzegorczyk's semantics for modal logics based on strict implication. He defines the set of theorems of the logic of strict implication as identical with the set of those formulas which are allowable by each information state of every research R. In this sense the following paradoxes of the material implication are not allowable:

$$p \rightarrow (q \rightarrow p);$$

$$p \rightarrow (\sim p \rightarrow q).$$

It turns out also that the famous Grzegorczyk formula, G, is allowable in the above sense. Formula G does not belong either to the system $S4$, or $S5$.[6]

The pragmatic act of assertion is characteristic of human cognition, while the intuitionistic and constructive logics may be regarded as logics of cognition. Accordingly, not truth, but assertion is the crucial concept underlying the intutionistic and constructive logics which one may observe in Grzegorczyk's semantic reconstruction of these logics. Intuitionistic and constructive logic remain in a close relationship with the contemporary epistemic logics, which are logics of knowledge, belief, and information. The main difference between the latter and their predecessors consists in this, that constructivism and intuitionism provide us with the logic of the process of investigation, while epistemic logics formulate the principles of an agent's knowledge or agent's beliefs. J. van Benthem [van Benthem, 1993] depicts the difference with the help of the distinction between *implicit* and *explicit* knowledge. In the topological semantics of intuitionistic logic, we have to do with the information loading of some logical constants such as negation and implication, as we have seen above. This property of the logical constants is called implicit knowledge. On the other hand, in epistemic logic, which retains the classical account of logical constants, but adds the explicit modal operator, K, we have to do with explicit knowledge. We could say that the implicit knowledge of intuitionistic logic is properly expressed by the assertibility conditions; the explicit knowledge of epistemic logic, by truth conditions describing sufficient and necessary conditions of any formula, K_χ, true in a model M and world s. J. van Benthem [van Benthem, 2009] considers embedding intuitionistic logic

[6] More about Grzegorczyk's modal logic may be found in [Maksimova, 2007].

in explicit modal-temporal theories of information processes, which enables us to define such concepts as "always in the future" and "necessarily now". He argues that epistemic logic being a result of the embedding gives an account of the rational agents in actions of observation, inference, and communication. J. van Benthem is right as to his conclusion; nevertheless the embedding of intuitionistic logic in modal logic, or in modal epistemic logic, is connected with the loss of specific semantic features of intuitionism which have been clearly proved by A. Grzegorczyk.

A deep analysis of the relation of assertion is to be found in one Grzegorczyk [Grzegorczyk, 1971] paper. A. Grzegorczyk includes in the methods of assertion the checking, deducing, and construction of an asserted sentence with semantic terms. By checking he understands applying an algorithm. The method of deducing assumes a recursive set of axioms and inference rules. A. Grzegorczyk argues that the three methods are connected with the three types of assertion:

- Classical assertion;
- Relativistic assertion;
- Constructivistic assertion.

The classical assertion is absolute, that is, non-relativistic. It does not come into degrees, and is independent of the time of assertion and the method of assertion. By this conception, the asserted sentence is identified with the true sentence being effectively justified. This conception is closest to classical logic, although, it does not imply that it is asserted either χ or non-χ, no matter whether our meta-theory is classical or not. This conception of asserting may be combined with the method of algorithmic checking if atomic sentences are formulated in the language of arithmetic. For example, if we have an atomic sentence of the form:

$$n + m = k,$$

where n, m, k are names of natural numbers, then the atomic sentence is asserted if it is justified by an algorithm. Next, the method of classical assertion tells us what it is to assert a compound sentence.

Relativistic assertion depends on many conditions: one of them may be time. On this conception, the absolute notion of assertion applied to theorems must be defined separately:

$$\chi \text{ is asserted as a theorem } \equiv^{Df} \chi \text{ is asserted in all conditions.}$$

For many collections of conditions, the set of absolutely asserted sentences is identical with intuitionistic logic.

Constructivistic assertion is a special kind of relativistic assertion relativized to method. The method is understood as an algorithm giving an expected result in a finite number of steps. In this conception, the assertion of a theorem is defined in the following way:

χ is asserted as a theorem $\equiv^{Df}$
there is the algorithm α such that α is a justification for asserting χ.

It is instructive to compare the assertibility conditions formulated for the denial of χ in the three conceptions:

- $\sim\chi$ is asserted $\equiv$ it is *not* asserted χ.
- $\neg\chi$ is asserted in the conditions $c \equiv$ there is the open interval I for c, such that for any member $i \in I, \chi$ is *not* asserted in i.
- $\neg\chi$ is asserted as justified by the algorithm $\alpha \equiv$ for any β, if χ is asserted as justified by the algorithm β, then the algorithm $\alpha(\beta)$ leads to a *contradiction*.

It is easy to notice that only the third condition is free from a tension between truth and justification. The constructivistic assertibility conditions are in fact justification conditions formulated in terms of an algorithm and the classical concept of contradiction. Neither the classical conception of assertion, nor the relativistic conception mentions how our assertion is justified. It seems that justification, and not truth, is decisive for assertion meant as a pragmatic relation. Intuitionism and constructivism rest on the following conviction:

> It does not make sense to think of truth or falsity of a mathematical statement independently of our knowledge concerning the statement. A statement is true if we have proof of it, and false if we can show that the assumption that there is a proof for the statement leads to a contradiction. For an arbitrary statement we can therefore not assert that it is either true or false [van Dalen and Troelstra, 1988, 4].

The constructivist conception of truth does not coincide with the Tarskian conception of truth, where the truth-predicate satisfies the metalogical law of the Excluded Middle, as well as the metalogical law of Non-Contradiction. In this sense, the Tarskian truth-predicate is absolute; that is, independent of time, space, and other conditions. Even if we cannot assert that a statement is either true or false, the concept of truth in the Tarskian sense enables us to believe that the statement must be either true or false. Here is how M. Dummett wrote about the intuitionistic notion of truth:

> [...] 'is true' would have to be equated with 'has been proved' and 'is false' with 'has been refuted'. On this use, any statement A that has not yet been decided is neither true nor false; but this does not preclude its later becoming true or becoming false [Dummett, 1977, 18].
>
> Such a notion of truth, obvious as it is, already departs at once from that supplied by the analogue of the Tarski-type truth-definition, since the predicate *is true* thus explained is significantly tensed: a statement not now true may later become true [Dummett, 1978, 239].

M. Dummett describes the intuitionistic notion of truth as endowed with a property usually ascribed to belief, knowledge, justification, or assertion, such as the relativisation to time. Note that justification for the *realist* has distinct properties from truth, and plays a different role, a role which consists in "binding" our beliefs with respective truths.

Assertion is regarded as a speech act taking place when the speaker makes an utterance with an *assertoric force*. This conception of assertion comes from Frege [Frege, 1918]. The same idea reappears in J. Austin's works, where the general theory of speech acts was founded, dividing them in three groups: locutionary acts, illocutionary acts, and perlocutionary acts. Acts of asserting belong to illocutionary acts. Frege and J. Austin distinguished the act of assertion from other pragmatic phenomena such as presupposition and implicature. A separate problem is connected with the relationship between truth and assertion. According to A. Tarski,

asserting *that* p is materially equivalent to asserting *that* p *is true.*

A different account of that relationship may be found in [Dummett, 1959], who identifies the act of assertion with the act which aims at truth, which is interpreted that the speaker intends to convince the hearer that he/she aims at saying something true. Note that it is usually assumed that the act of assertion may be insincere in the case of lying. Assertion may be evaluated as correct in different respects. It is commonly accepted that:

asserting *that* p is correct if and only if the speaker has good evidence that it is true *that* p.

The logic of assertion formulated by N. Rescher [Rescher, 1968] has the following principles. Let Axp stand for x *asserts that* p, then:

1. $\forall x \exists p Axp$;
2. $(Axp \land Axq) \rightarrow Ax(p \land q)$;
3. $\sim Ax(p \land \sim p)$.

The specific inference rule of this logic is called a "rule of commitment":

(C) If p implies q, then Axp implies Axq.

The principles have respectively the following intuitive meanings: someone asserts something; if it is asserted p and asserted q, then the conjunction of p and q is asserted: a contradiction is not asserted. One may say that the principles are necessary conditions of rational thinking. By contrast, the inference rule (C) is contentious. It tells us that the rational subject who asserts a certain proposition also asserts the logical consequences of the proposition. This rule is highly unrealistic, since it appeals to the conception of an ideal subject who is logically omniscient.

What is then A. Grzegorczyk's conception of assertion? First of all, A. Grzegorczyk's assertion is relativized not only to time, but also to research, or more precisely, to the programme of research (inquiry) R, accordingly

> *relation of assertion* holds between the information state, a, obtained in the research (inquiry) R up to the time point t, and the sentence χ.

The information state makes the relation of assertion founded on facts independent of our consciousness. It is assumed that the information state makes the relation of assertion correct, since it provides good *evidence* for the truth of the sentence χ. Besides, asserting the negation of the sentence, and the implication of two sentences, is what makes the relation of assertion *strong* one. To assert the negation at the point of time t, it is necessary and sufficient never later to assert the very sentence. This condition imposes on us the requirement of careful formulation of a sentence which can never be asserted. I cannot assert now many ordinary language utterances, such as "It is not the case that I have short hair" unless I am sure that I will have long hair always in the future, and I cannot assert now "It is not the case that there is a tree nearby that building" unless I am sure that there will never be any tree nearby that building. These examples merely suggest that ordinary language negation is not intuitionistic negation, but they are not an argument in favour of the view that the concept of strong assertion and that of intiutionistic negation are useless. Intuitionistic negation serves for modelling mathematical (i.e. non-empirical) discourse negation, but not natural language negation. The strong assertion is understood as an absolute assertion in a certain sense: what has been once asserted is not retracted. In other words, the relation of strong assertion is *monotonic*, since new facts do not force us to retract what has been strongly asserted before.[7]

4. Grzegorczyk's Semantics and the Informational Turn in Logic

Note that A. Grzegorczyk's relation of assertion holds between a certain *information state* and a sentence. According to the contemporary Law of Causality of Information, there is no more information received than that which has been sent.[8] In the case of empirical research, Nature sends the information which is to be partially received, and no more can be asserted than the information which has been received.

A. Grzegorczyk's work on the semantics of intuitionistic calculus was written more than a decade before the so called *informational turn* in logic, which caused a new understanding of logic as information-based. The points of evaluation, meant as information states in A. Grzegorczyk's semantics, are allowed to be incomplete, although they are always consistent[9]. At present, we consider also inconsistent information states. From the point of view of this informational trend in logic, Grzegorczyk's semantics for intuitionistic logic may be regarded as a logic of information where the relation $a > \chi$ is understood as the relation of carrying information: *the information state a carries the information that* χ. Understood in that way, Grzegorczyk's semantics is a sort of frame semantics, where the model contains the evaluation relation: $>$. Let the frame $\mathbf{F}$ be a structure containing the set of information states S, and the binary relation of extension of the information: *b is an extension of a in the research* R, which will be denoted at present as $\supseteq$. Therefore,

$$\mathbf{F} = \langle S, \supseteq \rangle.$$

A model $\mathbf{M} = \langle F, \supseteq, > \rangle$, where the evaluation relation $>$, meant as Grzegorczyk's relation of forced assertion, satisfies the respective conditions for each connective of the intuitionistic language, as well as the following monotonicity condition, a counterpart of the hereditary condition in Kripke's semantics:

$$\text{If } a > \chi \text{ and } b \supseteq a, \text{ then } b > \chi.$$

[7] The intuitionistic negation fails to distinguish between the global future absence of verification, and local falsification. Compare [van Benthem, 2009, 253].

[8] See [Pawlowski *et al.*, 2009].

[9] There are such states of information which do not force us to assert $A \vee \neg A$, while there are no such states which force us to assert $A \wedge \neg A$

J. van Benthem [van Benthem, 2009, 255] makes the important observation that intuitionistic logic registers two kinds of information:

1. Factual information about how the world is;
2. Procedural information about our current investigative process, that is, about the way we learn facts.

J. van Benthem [van Benthem, 2009] argues that branching tree-like models are closest to the intuitionistic tradition. They describe an informational process where an agent learns progressively about the state of the actual world. Surely, the evaluation conditions for the intuitionistic negation and implication in A. Grzegorczyk's semantics register procedural information, since they capture the dynamic of our investigative process.

There are many logics of information by now, each describing different aspects of this complex phenomenon.[10] A philosophically important notion in studying the phenomenon of information is the notion of information application. An interesting analysis of this notion may be found in [Sequoiah-Grayson, 2009]. The starting point of his analysis is a set of information states S with a partial order on it. The partial order is understood in a similar way as A. Grzegorczyk's relation of the extension of information. Let the partial order be denoted as $\leq$. The expression $x \leq y$ is read as: the information x is contained in the information y. The operator of the application of information holds between information states. If the operator is denoted as $\bullet$, then the expression:

$$x \bullet y \leq z$$

is understood as the result of the application of the information in x to the information in y develops into the information in z [Sequoiah-Grayson, 2009, 422]. The operator is an ontological counterpart of the binary connective, $\otimes$, occurring between formulas, which is called by S. Sequoiah-Grayson *fusion*, and the expression:

$$A \otimes B$$

is meant as the application of the information in the formula A to the information in the formula B. The evaluation conditions for this compound formula are given in terms of the model-theoretic evaluation relation, denoted by us as: $<$. Accordingly, $x < A$, is read as *x carries the information that A, or x supports A* [Sequoiah-Grayson, 2009, 410]. Thus,

$$xA \otimes B \text{ iff for some } y, z \in S \text{ such that } y \bullet z \leq x, yA \text{ and } zB.$$

[10] See [Barwise and Seligman, 1997; Floridi, 2003; Jago, 2006; Mares, 1997; Restall, 1994; Sequoiah-Grayson, 2006; Misiuna, 2012], to mention only a few.

We omit here the other connectives included to the informational language considered by Sequoiah-Grayson [Sequoiah-Grayson, 2009]. The operator of the application of information could be easily described semantically in terms of A. Grzegorczyk's semantics, augmented with the binary operator $\bullet$, in the following way:

$a > A \otimes B$ iff for some $b, c \in S$ such that $a \supseteq b \bullet c, b > A$ and $c > B$.

5. Conclusions

We have compared A. Grzegorczyk's semantics for Heyting's intuitionistic propositional and predicate calculus with the respective Kripke's semantics. Both semantics are of a topological kind, but Grzegorczyk's semantics is philosophically more acceptable than the intuitionistic Kripke models. We have argued that the main philosophical advantage of Grzegorczyk's semantics is the lack of the model-theoretic truth-relation which is present in the intuitionistic Kripke models. We have noted that A. Grzegorczyk makes a distinction between different types of assertion:

- Forced assertion [Grzegorczyk, 1964];
- Potentially forced assertion [Grzegorczyk, 1964];
- Strong assertion (compared with weak assertion; [Grzegorczyk, 1967]);
- Admissibility (weak assertion; [Grzegorczyk, 1967]);
- Relation of being allowed [Grzegorczyk, 1967].

The five types of assertion may be regarded as that of a relativistic kind. Besides relativistic assertions, A. Grzegorczyk discerns also classical and constructivistic assertions. The other peculiar property of A. Grzegorczyk's semantics, besides its philosophical insight, is its anticipatory feature. We have argued that Grzegorczyk's semantics may be regarded as a predecessor of many present semantics characteristic of the contemporary informational turn in logic.

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Krystyna Misiuna
Institute of Philosophy
University of Warsaw
Krakowskie Przedmieście 3
00–047 Warsaw, Poland
krystyna.simons@uw.edu.pl
http://www.wiedzalogikainformacja.uw.edu.pl

Roman Murawski

PHILOSOPHY OF LOGIC AND MATHEMATICS IN THE WARSAW SCHOOL OF MATHEMATICAL LOGIC*

Abstract. In the paper philosophical ideas concerning logic and mathematics developed in the Warsaw School of Mathematical Logic are considered. The views of two important representatives of this school – Alfred Tarski and Andrzej Mostowski – are analyzed in detail.

The Warsaw School of Mathematical Logic was a part of the Lvov-Warsaw School of Philosophy. It belonged to the most important centers of mathematical logic between the wars. It is natural to ask what were the philosophical views and attitudes of logicians in Warsaw towards mathematics and logic itself. One can also ask whether and to what extent those views influenced formal and technical research, whether that research had its source in philosophical considerations or was it independent of any philosophical presuppositions. Did the philosophical views bind the technical investigations or were they without meaning for them?

The attitude of Polish logicians and mathematicians towards the philosophy of mathematics can be shortly characterized as follows: they saw the mathematical and philosophical foundations of mathematics as independent although connected in a way and indispensable for understanding logical and mathematical activity. With two exceptions (Chwistek and Leśniewski) they represented a view guided by the following two principles:

- all commonly accepted mathematical methods should be applied in metamathematical investigations,
- metamathematical research cannot be limited by any *a priori* accepted philosophical standpoint.

On the other hand, logic and mathematics have their own genuine philosophical problems which should not be neglected. In particular, although

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metamathematical results do not solve philosophical controversies about mathematics and logic, yet the former illuminate the latter.

What were the sources of such an attitude? One can indicate two of them. The first one can be exemplified by Sierpiński's work on the axiom of choice (AC) and its applications in mathematics. In his French paper [1918] on the role of AC, Sierpiński distinguished two independent questions:

- philosophical controversies around this axiom,
- its place in proving mathematical theorems.

According to Sierpiński the second issue should be investigated independently of the philosophical inclinations concerning the problem whether the axiom of choice is to be accepted or not. This opinion was included in all editions of Sierpiński's books on set theory from 1923 (*An Outline of Set Theory*, 1923) to 1965 (*Cardinal and Ordinal Numbers*, 1965). In [1965, p. 94] he wrote:

> Still, apart from our personal inclination to accept the axiom of choice, we must take into consideration, in any case, its role in set theory and in calculus. On the other hand, since the axiom of choice has been questioned by some mathematicians, it is important to know which theorems are proved with its aid and to realize the exact point at which the proof has been based on the axiom of choice; for it has frequently happened that various authors have made use of the axiom of choice in their proofs without being aware of it. And after all, even no one questioned the axiom of choice, it would not be without interest to investigate which proofs are based on it and which theorems are proved without its aid – this, as we know, is also done with regard to other axioms.

This means simply that one should disregard philosophical controversies (and treat them as a "private" matter) and investigate (controversial) axioms as purely mathematical constructions using any fruitful methods.

The second source of the discussed attitude of Polish mathematicians and logicians towards philosophy was the tradition of Polish analytic philosophy originated by Kazimierz Twardowski in Lvov. According to Twardowski and his students, we must clearly and sharply distinguish world-views and scientific philosophical work. This idea was particularly stressed by Łukasiewicz, the main architect of the Warsaw school of logic. He regarded various philosophical problems pertaining to the formal sciences as belonging to the world-views of mathematicians and logicians, but the work consisting in constructing logical and mathematical systems together with metalogical and metamathematical investigations constituted for him the subject of logic and mathematics as special sciences. Hence philosophical views cannot be a stance for measuring the correctness of formal results. Yet philosophy may serve as a source of logical constructions.

One of the consequences of the described attitude of Polish logicians and mathematicians was the fact that they did not attempt to develop a comprehensive philosophy of mathematics and logic (Stanisław Leśniewski and Leon Chwistek were here the exceptions!). They formulated their philosophical opinions concerning mathematics or logic only occasionally and only on problems which just interested them or on which they actually worked. Consequently there were in Poland no genuine philosophers of mathematics. Philosophical remarks were formulated by logicians and mathematicians only on the margin of their proper mathematical or logical works (and had no meaning for the results themselves).

The current trends and views in the philosophy of mathematics, i.e., logicism, intuitionism and formalism, were of course well known (and there appeared papers discussing those tendencies, their meaning and development). But none of them was represented in the Warsaw School. Moreover, it did not represent any other trend; it had no official philosophy of logic and mathematics. This followed from the belief that logic and mathematics are autonomous with respect to philosophy. Opinions in the field of the philosophy of logic and mathematics were treated as "private" problems and philosophical declarations were made reluctantly and seldom. If they were made then it was stressed, directly or indirectly, that these were personal opinions.

Though some of the logical investigations were motivated by philosophical problems – e.g. the many-valued logics by Łukasiewicz – the formal, logical constructions were always separated from their philosophical interpretations. Another example is the investigation of intuitionistic logic carried out among others by Tarski without accepting intuitionism as the philosophy of mathematics. The programme of Janiszewski [1917] and the Polish School of Mathematics created set-theoretical foundations of mathematics in a methodological and not philosophical sense.

What were the separate philosophical opinions formulated by Polish logicians, philosophers and mathematicians? We shall answer this question considering the philosophical views of two representatives of the Warsaw School of Mathematical Logic: Alfred Tarski (1901–1983) and Andrzej Mostowski (1913–1975). Tarski belonged to the first generation of the Warsaw School; Mostowski, to the second generation.[1]

[1] For logic and the philosophy of logic and mathematics in Poland between the wars see the basic monograph [Woleński, 1989] as well as [Woleński, 1992; 1993; 1995; Murawski and Woleński, 2008; Murawski, 2004; 2011].

Alfred Tarski was interested in philosophical problems and very actively participated in the philosophical life of his time. He was convinced of the philosophical significance of his works, in particular of his work on truth [Tarski, 1933]. He described himself as [Tarski, 1944]:

> Being a mathematician (as well as a logician, perhaps a philosopher of a sort) [...]

Tarski's philosophical attitude was anti-metaphysical; he supported the idea of scientific philosophy. He accepted a programme of "small philosophy" which aims at detailed and systematic analysis of the concepts used in philosophy. Such a philosophy is minimalistic, anti-speculative and sceptical towards many fundamental problems of traditional philosophy. This attitude was inherited by Tarski from the Lvov-Warsaw School and strengthened by contacts with the Vienna Circle. He maintained also empiricism and abandoned the analytic/synthetic distinction. He stressed that logical and empirical truths belong to the same generic category. Influenced by Leśniewski and Kotarbiński he was inclined to a rather strongly nominalistic understanding of expressions. One finds many places in which he confirmed this. E.g. during a symposium organized by the Association for Symbolic Logic and the American Philosophical Association held in Chicago on 29th–30th April 1965 and devoted to the philosophical implications of Gödel's incompleteness theorems, he said (cf. [Feferman and Feferman, 2004, p. 52]):

> I happen to be, you know, a much more extreme anti-Platonist. [...] However, I represent this very [c]rude, naïve kind of anti-Platonism, one thing which I would describe as materialism, or nominalism with some materialistic taint, and it is very difficult for a man to live his whole life with this philosophical attitude, especially if he is a mathematician, especially if for some reasons he has a hobby which is called set theory.

In the biography of Tarski written by Fefermans one finds more such quotations, for example (cf. [Feferman and Feferman, 2004, p. 52]):

> I am a nominalist. This is a very deep conviction of mine. It is so deep, indeed, that even after my third reincarnation, I will still be a nominalist. [...] People have asked me, 'How can you, a nominalist, do work in set theory and logic, which are theories about things you do not believe in?' ... I believe that there is value even in fairy tales.
> [I am] a tortured nominalist.

They write also: “Elsewhere Tarski has said more specifically that he subscribed to the reism or concretism (a kind of physicalistic nominalism) of his teacher Tadeusz Kotarbiński”.

Mostowski wrote about Tarski so (cf. [1967, p. 81]):

> Tarski, in oral discussions, has often indicated his sympathies with nominalism. While he never accepted the ‘reism’ of Tadeusz Kotarbiński, he was certainly attracted to it in the early phase of his work. However, the set-theoretical methods that form the basis of his logical and mathematical studies compel him constantly to use the abstract and general notions that a nominalist seeks to avoid. In the absence of more extensive publications by Tarski on philosophical subjects, this conflict appears to have remained unresolved.

Tarski was inclined to identify mathematics with the deductive method. He maintained that there is no hard borderline between formal and empirical sciences. He admitted the rejection of logical and mathematical theories on empirical grounds. He claimed also that there is no sharp demarcation between logical and factual truth and that the concept of tautology is unclear.

One must stress that all those were his “private” philosophical views which did not influence his logical and mathematical research; in other words, his research was independent of any philosophical presuppositions. In the paper “Über einige fundamentale Begriffe der Methodologie der deduktiven Wissenschaften” [1930] he explicitly wrote:

> [...] it should be noted that no particular philosophical standpoint regarding the foundations of mathematics is presupposed in the present work.

This was typical for him and for the whole Warsaw School in logic. This independence of logical and mathematical studies and philosophical views explains the cognitive conflict and discrepancy between Tarski’s nominalistic and empiricistic sympathies and his “platonic” mathematical and logical practice. Note that his attitude enabled him to contribute to various important foundational streams without the necessity of accepting their philosophical assumptions and attempting to reconcile the philosophy and the research practice. His programme of metamathematics can be summarized by his words from the paper [Tarski, 1954] where he wrote:

> As an essential contribution of the Polish school to the development of metamathematics one can regard the fact that from the very beginning it admitted into metamathematical research all fruitful methods, whether finitary or not.

Andrzej Mostowski inherited his general philosophical attitude from Tarski. He freely used infinitary methods and strongly insisted that no formal work should be limited by philosophical assumptions. However it seems that Mostowski felt himself obliged to a more extensive and systematic treatment of his views in the philosophy of mathematics. In a review of Mostowski's *Thirty Years of Foundational Studies* [1965] published in *Studia Logica* R. Suszko characterized him as a "mathematician-logician, to whom the philosophical aspect of logic and the theory of the foundations of mathematics is not alien" [Suszko, 1968, p. 169].

In many of his technical papers and works Mostowski stressed in the introductory sections or prefaces the importance and indispensability of certain philosophical presuppositions. He discussed also the possible philosophical consequences of technical mathematical results presented there. But such comments and remarks were always reduced to a minimum and had no influence on the technical considerations.

In the *Introduction* to the monograph *Teoria mnogości* (Set Theory) written together with K. Kuratowski.[2] they wrote [1952, p. vi]:

> There exists so far no comprehensive philosophical discussion of the basic assumptions of set theory. The problem whether and to what extent abstract concepts of set theory (and in particular of those parts of it in which sets of very high cardinality are considered) are connected with the basic notions of mathematics being directly connected with the practice has not been clarified so far. Such an analysis is needed because by Cantor, the inventor of set theory, basic notions of this theory were enwrapped with a certain mysticism.

On the other hand the authors are convinced that the meaning and importance of set theory for the foundations of mathematics were demonstrated also in connection with the philosophy of mathematics.

And they declare that the most important feature of set theory is the fact that it provides a *tool* for other parts of mathematics which are directly connected with applications.

The philosophical remarks were made only in the Introduction. One finds in the book no further philosophical declarations or statements. In the whole book the authors strictly distinguish (in the spirit of Sierpiński) the philosophy of the axiom of choice and its role in mathematics and set theory itself – all theorems in which AC is used are marked by a small circle.

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[2] The discussed remarks were reprinted in the second and third Polish editions of the book.

Mostowski considered also philosophical problems in connection with Gödel's incompleteness theorems. As in the case of set theory he indicated only the philosophical problems connected with the discussed mathematical issues and showed possible solutions but avoided any fixed and definite philosophical declarations. Moreover the philosophical comments were reduced to a minimum.

He stressed that we do not have a precise notion of a correct mathematical proof. In the paper [1972, p. 83] Mostowski emphasizes that: "A mathematical proof is something much more complicated than a simple succession of elementary rules contained in the so called inference rules. [...] Therefore one must necessarily show moderation in stressing the role of logical rules in [mathematical] proofs". On the other hand the author is sure that despite the fact that the old program of formalization of mathematics has been practically waived "the collaboration of logic and mathematics was fruitful and probably will still bring important results" [p. 83].

Note also that the three trends in the philosophy of mathematics which dominated in the 20s and 30s of the 20$^{\text{th}}$ century (logicism, intuitionism and formalism) were the starting point of Mostowski's series of lectures *Thirty Years of Foundational Studies* [1965]. He stressed there that they gave rise to the development of three directions in logico-mathematical investigations: constructivism, metamathematical and set-theoretical ones. But in the main text one finds no further philosophical remarks.

So far we have shown that Mostowski was aware of philosophical problems connected with mathematics but avoided making any explicit philosophical declarations. There is however one paper by him in which he makes explicit declarations, namely the paper *The present state of investigations of the foundations of mathematics* (see [Mostowski, 1955a] and [Mostowski, 1955b]). Unfortunately there is a problem of interpretation: the paper was written in the first half of the fifties and the ideological atmosphere of that time could have had an influence on it. It is not possible now to decide to what extent outside factors influenced the paper. On the other hand the author could have restricted himself to purely mathematical issues and avoided entirely any philosophical remarks and declaration. If he did not do so we can treat his remarks as genuine.

He states there (cf. [Mostowski, 1955a, p. 42]):

> [...] An explanation of the nature of mathematics does not belong to mathematics, but to philosophy, and it is possible only within the limits of a broadly conceived philosophical view treating mathematics not as detached from other sciences but taking into account its being rooted in natural sciences, its applications, its associations with other sciences and, finally, its history.

Investigations on the foundations of mathematics by mathematical methods affect the formation of a broader philosophical view. The results obtained there confirm – according to Mostowski (cf. 1955a, p. 42]):

> the assertion of materialistic philosophy that mathematics is in the last resort a natural science, that its notions and methods are rooted in experience and that attempts at establishing the foundations of mathematics without taking into account its originating in the natural sciences are bound to fail.

Hence, Mostowski represents here an empirical point of view in the philosophy of mathematics. As mentioned above it is not quite clear what was the influence of outside factors (in particular of the then dominant ideology) on those views. Specific expressions used by him may suggest such an influence. Such statements could be, at least partially, the price that had to be paid to the official philosophy. On the other hand, note that empirical (or quasi-empirical) trends have been since the sixties of the last century still more and more vivid in the philosophy of mathematics.

Mostowski admitted in various places that constructivism (especially its aims, not necessarily its solutions) was always very attractive to him (cf. [Mostowski, 1959, p. 192]). The reason for that was the fact that (cf. [Mostowski, 1959, p. 192]):

> it wants to inquire into the nature of mathematical entities and to find a justification for the general laws which govern them, whereas platonism takes these laws as granted without any further discussion.

He stressed that constructivistic trends in the foundations of mathematics are nearer to nominalistic philosophy than to the idealistic (in the Platonic sense) one. This nominalistic character implies that constructivism does not accept the general notions of mathematics as given but tries to construct them. “This leads to the result that one can identify mathematical concepts with their definitions” [Mostowski, 1959, p. 178]. The advantage of nominalism is the fact that several important mathematical theories have been reconstructed in a satisfactory way on a nominalistic basis and those reconstructions have turned out to be equivalent to the classical theories.

*

* *

Polish logicians and mathematicians, being convinced of the importance of philosophical problems and knowing quite well the current philosophical trends, treated logic and mathematics as autonomous disciplines independent of philosophical reflection on them, independent of any philosophical presuppositions. Therefore they sharply separated mathematical and logical research practice and philosophical discussion concerning logic and mathematics. Philosophical views and opinions were treated as a "private" matter that should not influence mathematical and metamathematical investigations. On the contrary, in the latter all correct methods could and should be used. This "methodological Platonism" enabled Polish logicians and mathematicians to work in various areas without being preoccupied by philosophical dogmas. In controversial cases, as for example in the case of the axiom of choice in set theory, their attitude can be characterized as neutral – without making any philosophical declarations they simply considered and studied the various mathematical consequences of both accepting and rejecting controversial principles, and investigated their role in mathematics.

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Roman Murawski
Adam Mickiewicz University in Poznań
Faculty of Mathematics and Comp. Sci.
ul. Umultowska 87, 61–614 Poznań, Poland
rmur@amu.edu.pl

Alex Orenstein

A PEIRCIAN RECONCILIATION OF THE OLD AND THE NEW LOGIC*

Abstract. With the ascendancy of Peter Strawson's account of the categorical sentences of traditional logic, Sterling Lamprecht offered an alternative. The present paper attempts to bolster Lamprecht. It does this by distinguishing and offering different logical forms for each of the categorical sentences of traditional logic. One of the three does fit with the form now in almost all logic texts. A second is really a matter of plural quantification. However a third, suggested in Lamprecht and found in Peirce, should be dealt with in terms of restricted quantifiers when these are instantiated in a suitable way. These are instantiated in a suitable way. Quine cited Peirce to this effect. It is this restricted quantification scheme that does yield a full Square of Opposition and benefits beyond that as well. In this paper I offer a formal account thereby saving the traditional claims with special restricted quantifiers. These quantifiers have rules that parallel unrestricted ones.

Keywords: Expanding the Pierce-Quine, Account/Defense of Traditional Logic, Pierce's Defense of Traditional Logic

1. Distribution – Quine's Use of Peirce Against Geach

The doctrine of distribution is a central theme in the traditional logic of categorical syllogisms. One speaks of the subject and predicate terms as to whether they are distributed or not. Keynes says that "a term is said to be distributed when reference is made to all of the individuals denoted by

* I am particularly indebted to the Polish philosophical tradition in a personal as well as a professional way. My most important doctoral advisor was Henry Hiz. His friend George Krzywicki-Herbut was a C.U.N.Y. colleague as well as my sometime ski instructor. They served in the Polish armed forces in the early days of the second World War and met in a prisoner of war camp. Unfortunately I did not spend as much time with Professor Grzegorczyk. However, he was generous in meeting with me in Warsaw. I gained much from his conversation, and later from his correspondence and his publications. His History of Logic should be better known. It is a notable addition to the unique Polish tradition of writing on this history. Andrzej Grzegorczyk was unconventional. He maintained both the most rigorous approach in logic and an existentialist stance on broader philosophical questions.

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it ... undistributed when they are referred to only partially ...” [Geach, 1962, p. 28]. There are rules of quantity and of quality for evaluating the validity of such syllogisms. Those of quantity apply to the distribution of terms. “Illicit process” is the name for violating the rule which requires that a term distributed in the conclusion must be distributed in a premise. The fallacy of “Undistributed middle” violates the rule requiring that the middle term (the term occurring once in each premise) be distributed in one of these premises. I shall anachronistically argue that these rules constitute an algorithm, a decision procedure, for testing the validity of standard categorical syllogisms.

In a somewhat well known piece, Peter Geach claimed that the doctrine of distribution is seriously flawed. Then in a much less well known piece Quine defended Peirce’s version of the doctrine. In his review of Geach’s *Reference and Generality* [1962], Quine [1964] maintained that Geach had failed to show that the doctrine of distribution is defective. What a surprise it is to see Quine defending a doctrine of traditional Aristotelian logic. He comments that the purported flaws Geach cites are not in the doctrine itself, but in flawed accounts of it. Quine likens the situation to one in which Boolean Algebra would be condemned because of flaws in Boole’s manner of explaining it. “One might as well denounce Boolean Algebra by fastening on Boole’s mistakes and confusions” [Quine, 1964, pp. 100–1]. Geach followed a deplorable practice of reading authors, especially past ones, in a narrow unsympathetic, if not biased, spirit. The material on distribution appears in the first chapter of Reference and Generality. It had appeared as an article and was reprinted in at least one collection. Quine’s review is short. He does not spend much time on the topic and his rejection of Geach’s claim consists of citing some lines he says are derived from Peirce.[1]

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[1] I could not find this material at the place in Peirce where Quine locates it.

It is somewhat ironic that Quine should cite Peirce. Peirce was aware of the history of logic in ways that Frege, Russell, Quine himself and other proponents of the “existence is what existential quantification expresses” approach were not. Unlike Quine and his precursors, Peirce connected quantification with products and sums. His notation for the universal and the “existential” quantifier were “Π” and “Σ”. As far as I know he did not speak of the “existential quantifier” or link existence and quantification. The historian Bocheński comments on Peirce’s view as a “rediscovery” of ideas found in the Terminist Albert of Saxony [Bochenski, 1961, p. 349]. Peirce wrote on Ockham and this Terminist tradition. They treated categoricals in terms of a descent to conjunctions and disjunctions of singular sentences. This aspect of supposition theory furnished part of the motivation for adopting an all/some – and/or material adequacy condition for the quantifiers. As Albert of Saxony stated

> A sign of universality is one through which a general term to which it is adjoined is denoted to stand, in a conjunctive manner, for every one of its values (supposita).
>
> A sign of particularity is one through which a general term is denoted to stand, in a disjunctive manner, for every one of its values [Moody, 1953, p. 45]

Distribution comes to be registered by the word "every" when we paraphrase the four forms into terms of identity and distinctness thus:[2]

> All S are P: Every S is identical with some P or other.
> Some S are P: Some S or other is identical with some P or other.
> No S are P: Every S is distinct from every P.
> Some S are not P: Some S or other is distinct from every P.
> [Quine, 1964, p. 100]

Quine does not say more, but his point should be clear. The presence or absence of something like universal quantification with respect to the traditional subject and predicate terms coincides with what the doctrine of distribution claims. In other words, the logical form of the four types of categorical sentences coincide with whether the quantifications (quantifier phrases) or their equivalents are universal or particular (so-called "existential" generalizations).

2. Canonical Notation, Paraphrase, and Regimentation

A few words are in order about views on canonic notation, paraphrase and regimentation. Quine 1960, pp. 157–61 offers a distinctive view of the artificial language of first order predicate logic supplemented by sentences of English (natural language) which serve as paraphrases of the artificial notation and at the same time are supposed to improve upon, i.e., regiment other English locutions. Quine's official canonical notation consists of certain predicate logic sentences and their English paraphrases. These sentences contain unrestricted universal and particular/"existential" quantifiers, truth functional sentence connectives, predicates, individual variables (no names-names are Quinized into definite descriptions, and then Russelled away), and an identity sign. No claims are made about synonymy being a requirement for paraphrasing into the artificial symbolic notation, or for paraphrasing from non-canonical English forms to canonical ones.

There are three goals served by Quine's conception of a canonical notation:

1. as an aid to communication,
2. for deduction, and

[2] The idea is from [Peirce, 1931–1935, 2.458].

3. for delineating a philosophical set of categories ("the inclusive conceptual structure of science – called philosophical, because of the breadth of the framework concerned" – The quest of a simplest clearest overall pattern of canonical notation is not to be distinguished from a quest of ultimate categories, a limning of the most general traits of reality") [Quine, 1960, p. 161]

ad 1. As an aid to communication one function of the canonic notation is to resolve ambiguity. The ambiguity of the sentence "Dogs are friendly" can be resolved by supplying a universal or a particular quantifier.

ad 2. The notation of predicate logic and its canonic English paraphrase constitutes the content of logical theory, the science of deduction.

ad 3. A conspicuous example of the philosophical utility of this canonical notation for Quine is the use of the particular/existential quantifier to express existence claims and to help to determine ontological commitment. In one respect I would add an element to Quine's account of the aims of a canonic notation. For the purposes of communication and deduction it is necessary to include an appeal to linguistics. Linguistic differences are connected with logical differences. Moreover, this addition is in keeping with Quine's naturalistic methodology, ensuring among other things that we are conservative and do not mutilate linguistic data.

In his *Methods of Logic* [Quine, 1982, pp. 93, 95; p. 81] three types of each categorical sentence are taken as paraphrasable into predicate logic notation as unrestricted generalizations of conditionals and conjunctions. For simplicity's sake, I concentrate on the A form sentences. There is the plural form with a restricted quantifier, 'All A are B', the restricted quantifier singular copula form, 'Every A is a B', and the unrestricted quantifier conditional form, 'If anything is an A then it is a B'. To repeat, Quine does not claim that the relation between these three sentences is that of synonymy. Outside of the Geach review Quine seems to hold the view that there need not be any one right solution as to which of the three is to be preferred. Given a sentence and a context any one of these three forms may serve as canonical English versions of a predicate logic unrestricted universal generalization over a conditional. They are put on a par in *Methods of Logic*. But in the review, for the purpose of defending the doctrine of distribution, he seems to give special prominence to a slight variant, the singular copula form. This occasions a problem. Whereas on Quine's official view of canonical notation each of these English sentences is on a par, from the standpoint of the review (trying to capture the notion of distribution and its use in logic) we have to single out the singular copula restricted quantifier sentence for special consideration.

3. Resolution

Let us follow the Peirce line taken in Quine's review. We will take the singular copula form for special consideration and try to follow Quine's practice in holding that logical forms are revealed in terms of a variant of first order predicate logic translations. To reveal the traditional patterns of distribution, i.e., universality and particularity (existential quantification), I suggest regimenting Quine's four Peircean categorical sentences, such as "Every A is some B or other", into restricted quantifier singular copula claims and then representing them in predicate logic as follows:

Every A is a B	as	$(x, Ax)[(\exists y, By)x = y]$
At least one A is a B	as	$(\exists x, Ax)[(\exists y, By)x = y]$
No A is a B	as	$(x, Ax)[\neg(\exists y, By)x = y]$
At least one A is not a B	as	$(\exists x, Ax)[\neg(\exists y, By)x = y]$

We now have a closer correlation between (a more perfect paraphrase of) these quantified singular copula English sentences and their predicate logic formulations. Both the predicate logic formulations and the English sentences consist of restricted quantifiers: 'Every A', 'At least one B'/'B, and a singular form of the copula dealt with in terms of identity. This regimentation and its predicate logic expression is the one that least mutilates the natural language singular copula generalizations. It might be that the identity involved is a version of the "dreaded" identity theory of the copula (or of predication). As such, it might provide a tool for re-examining the debates surrounding that notion.

Adhering to Quine's naturalistic methodology, and in particular, to the maxim of being conservative (his maxim of minimal mutilation) when choosing between different hypotheses, we note the following. The plural 'All A are B' involves restricted plural quantification. It differs from the standard unrestricted universal quantification over a conditional form. And along the same line of reasoning, this plural form should also be distinguished from the above singular copula form. So, instead of taking all three forms on a par as Quine [1982, p. 81] and many others do, we should distinguish them. Only the last, the conditional English sentence, is best taken as a canonical English paraphrase of the predicate logic unrestricted universal generalization of a conditional, '$(x)(Ax \to Bx)$'. As far as I know in the main body of his work, Quine did not follow this policy of distinguishing the three forms of categorical sentences. He equated all three forms and deals with them in terms of unrestricted quantification.

As an aside, here are some conjectures as to why one might be led with Geach to doubt the doctrine of distribution. If we take the plural 'All A are B' or the conditional 'If anything is an A, then it is a B', forms as basic, then there is no sign in English of a special quantifier phrase associated with the predicate. Doing this makes the doctrine of distribution for the predicate term highly questionable. When we concentrate on the plural and conditional forms, it seems intuitive that the subject term is distributed. But given scope considerations we might think that the predicate is distributed as well, since the English quantifiers in these cases can be taken as having scope over the predicate position. Matters are made even worse for seeing whether a traditional predicate involves a distribution pattern, when we follow the mutilating tradition of ignoring the logical contribution of the copula. This occurs when the predicate, e.g., 'is a human', is construed holophrastically with the copula and the indefinite 'a' parts of a fused predicate, e.g., 'is-a-human', and playing no distinct roles.

As noted above, Quine's favored canonical role for quantification allows only for unrestricted quantifiers. He holds the view that where there appears to be a need for restricted quantification, it can be restated as, i.e., reduced to, an unrestricted quantification by the expedient of treating restricted universal/existential quantifications as unrestricted universal/"existential" quantifications over conditionals/conjunctions. (Quine, Set Theory and its Logic p. 235) The desired restricted A form, $(x, Ax)(\exists y, By)[x = y]$, appears in canonic predicate logic notation in terms of unrestricted quantification as

$$(x)[Ax \rightarrow (\exists y)(By \& x = y)]$$

and the restricted I form, $(\exists x, Ax)[(\exists y, By)x = y]$, as

$$(\exists x)(Ax \& (\exists y)(By \& x = y)].$$

But doing so, creates problems. To begin with, the I form would not be a logical consequent of the A form as it is in traditional logic and its full square of opposition.[3] Another problem concerns distribution. The above treatment amounts to abandoning the quest to explicate the doctrine of

[3] Some (e.g., Dorothy Edgington, Gyula Klima) propose adding the conjunct $(\exists x)Ax$ to the unrestricted version of the A form, to yield

$$(x)[Ax \rightarrow (\exists y)(By \& x = y)] \& (\exists x)Ax.$$

The A form does imply the I form, but the term A is now both distributed and undistributed. The constraints for paraphrasing enlisted below in the paper mitigate against all such uses of unrestricted quantifications in connection with the singular 'Every A is a B' form. It violates the maxim of minimal mutilation to supply the English singular copula sentence with a predicate logic representation containing elements of conditionality and conjunction.

distribution. If we try to explain distribution in terms of the unrestricted quantifier and its scope, the predicate as well as the subject is included in that scope. This conflicts with the doctrine of distribution view that the predicate is not distributed. If we say that the predicate is both distributed and undistributed, we surrender the exclusive nature that the distinction is supposed to have.

Even assuming that the rules of quantity which pertain to the doctrine of distribution are accounted for, how would one begin to give a unified account of the rule of quality and a unified account of both the rules of quantity and of quality? The singular copula approach with its own rules of derivation (see the next section) will accomplish all of this. We assume the constraints on being a categorical syllogism, i.e., having three categorical sentences with exactly three terms arranged so that the two appearing in the conclusion, appear separately, one in each premise and the third term (the middle term) appears once in each premise. The rule for illicit process records the fact that one cannot derive a universal claim from an existential one. The remaining rule of quantity, i.e., undistributed middle, and the rules of quality pertain to the element of identity. Undistributed middle insures that the terms of two different identity claims provided by the premises allow for substitutivity which will yield the identity claimed in the conclusion. Considerations concerning the substitutivity of identity require that one of the premises be an identity claim when the other premise is an inidentity claim. We can't have two negative premises, since two inidentities in these contexts will not yield a conclusion. Lastly, if one premise involves an identity claim and the other doesn't, then a valid conclusion must be an inidentity claim which results from substitution in the inidentity premise in terms of the remaining identity premise.

4. Providing a Formal System

To some extent I am tempted to try to leave the matter of supplying a formal framework open, and allow that any number of different systems might be taken to fit in with the above approach. Restricted quantification is in keeping with work done on plural quantification along the lines either of Neale [1990] or Bach's [1989] restricted quantification treatments, or in some ways with the Oxford binary quantification accounts of Wiggins [1985] or Davies [1981]. Both these restricted quantifiers and the binary ones are grounded in a theory of generalized quantifiers derived from Mostowski's paper on generalized quantifiers [1967]. It has become better known in logic and linguistics through the work of Barwise and Cooper [1981]. However, I will put aside the above suggestions and take a different approach – one which will adapt Benson Mates' method of Beta-variants. This adaptation of Mates provides us with truth conditions for my special restricted quantifiers. Here is a sketch of a system of rules of inference along the lines of the tree method and some considerations motivating these rules.

The account I favor begins by utilizing the restricted quantification notation adopted for the categorical sentences given in the previous sections. We need an account of the instances appropriate to these generalizations. A key idea is that a restricted universal generalization can be instantiated with an appropriate sentence containing a demonstrative noun phrase. For example, 'Every human is a mammal', should imply by a universal instantiation rule 'This/that human is a mammal'. It should not imply 'That horse is a mammal'. The task then is to come up with a suitable predicate logic notation, rules of inference, and truth conditions.

I start with a notation and rules for trees which will fit in with the project so long as we confine ourselves to classic Aristotelian syllogisms and the square of opposition. These will be followed by a fuller system which takes into account more complex cases and relates restricted and unrestricted quantification.

Add to the language of predicate logic restricted quantifiers such as (x, Ax), $(\exists x, Bx)$. These quantifiers are the representations in predicate logic form of English quantified noun phrases: Every A, At least one B. Placing these in front of appropriate open sentences yields well formed formulas. We need symbolic counterparts of English demonstrative noun phrases, e.g., This A, That B, which will serve as the canonical substituends of restricted quantifiers and occur in the sentences serving as canonical instances of such generalizations. Since these canonical substituends are singular terms, use lower case letters with superscripts, e.g., a^1, b^2, etc., in a special way. Just as the English 'this A' somewhat formally indicates by the presence of the same noun that it is an appropriate substituend for the restricted quantifier 'Every A' use a lower case letter (with a superscript), e.g., a^1, a^2, that is the lower case version of the capital letter occurring in the quantifier phrase, e.g., (x, Ax). In this notation, Ba^1 would be a correct or canonical substituend for the formula $(x, Ax)Bx$, but Bb^1 would not. The former corresponds to the correct inference that (this apple) is red given that (Every apple) is red, while the latter would be like reasoning that (this bird) is red since (Every apple) is red. Using the same letter of the alphabet in a lower case as the restriction on the quantifier, mimics, in our notation, the relation, in English, of the restriction on the natural language quantifier to its canonical demonstrative noun phrase. The superscript on the singular term serves to distinguish the substituends, e.g., a^2, b^1, for restricted quantifiers, from the ordinary singular terms, such as, a, b, c, x, y, which serve as substituends for variables of unrestricted quantifiers.[4]

We adopt the method of trees/semantic tableaux for unrestricted quantifiers and also adapt it by formulating rules of universal and existential instantiation and quantifier exchange for restricted quantifiers as follows:

Tree Rules

The following rules of inference applying to restricted quantifiers are added to the standard tree rules for unrestricted quantifiers.

Restricted Universal Instantiation

$$\frac{(x, Ax)\phi x}{\phi a^1}$$

(as individual constants use the lower case letter of the restriction on the quantifier with superscripts to distinguish these canonical instances for restricted quantification from instances associated with unrestricted quantification)

Particular "Existential" Instantiation

$$\frac{(\exists x, Ax)\phi x}{\phi a^i}$$

where a^i is new to the tree

Quantifier Interchange (Duality)

$$\frac{\neg(x, Ax)\phi x}{(\exists x, Ax)\neg(\phi x)}$$

$$\frac{\neg(\exists x, Ax)\phi x}{(x, Ax)\neg(\phi x)}$$

This system will do for classic Aristotelian logic. Here is an example of how it works for **Darii**: Every C is an A, Some B is a C, so Some A is a B. The tree closes:

[4] More sophisticated non-syllogistic reasoning requires a more sophisticated type of mimicking. See below **The Full System**.

	1. $(x, Cx)[(\exists y, Ay)x = y]$	
$\surd$	2. $(\exists x, Bx)[(\exists y, Cy)x = y]$	
$\surd$	3. $\neg(\exists, Ax)[(\exists y, By)x = y]$	
	4. $(x, Ax)\neg[(\exists y, By)x = y]$	3 Quantifier Interchange
$\surd$	5. $[(\exists y, Cy)b^1 = y]$	2 E.I., i.e. Existential Instantation
	6. $b^1 = c^1$	5 E.I.
$\surd$	7. $[(\exists y, Ay)c_1 = y]$	1 U.I.
	8. $c^1 = a^1$	7 E.I.
$\surd$	9. $\neg[(\exists y, By)a^1 = y]$	4 U.I.
	10. $(y, By)\neg a_1 = y$	9 Quantifier Interchange
	11. $\neg a^1 = b^1$	10 U.I.
	12. $\neg c^1 = b^1$	8, 11 Identity
	13. $c^1 = b^1$	6 Identity
	$\times$	

However, a full system also has to

a. connect restricted and unrestricted generalizations and
b. take cognizance of complex restrictions on quantifiers and reasoning involving them.

I offer the following notation and rules.

Let '$(tx^n)\Psi x^n)$' be a singular term. In quasi English it says 'this/that Ψ'.

III. The Full system

The **full** rule of **Restricted Universal Instantiation**

$$\frac{(x, \Psi x)\Phi x}{\Phi x^n(tx^n)(\Psi x^n)}$$

We also have **full Restricted Particular "Existential" Instantiation**

$$\frac{(\exists x, \Psi x)\Phi x}{\Phi x^n(tx^n)(\Psi x^n)}$$

where n is new to the tree

and **full Quantifier Interchange (Duality)**

$$\frac{\neg(x, \Psi x)\Phi x}{(\exists x, \Psi)\neg\Phi x}$$

$$\frac{\neg(\exists x, \Psi x)\Phi x}{(x, \Psi x)\neg\Phi x}$$

This will still only do the job that the notation and rules for "simple/fused" predicates did. As it stands, it does not cover reasoning with complex predicates. We arrive at a quite natural solution by focusing on the demonstrative noun claims serving as instances. To begin, note that the English expressions 'this' and 'that' serve two roles. They serve as determiners in noun phrases: '(That oak desk) is heavy', and they can stand alone as demonstratives: '(That) is heavy'. In the right circumstances, both '(That oak desk)' and '(That)' demonstrate the same item. So when standing alone, t^3, can be thought of quite naturally as being like an individual constant, a name. The following is a familiar equivalence pertaining to names.

$Fa \leftrightarrow (\exists x)(x = a \& Fx)$

So for $Hx^1(tx^1)(Dx^1)$, i.e., That desk is heavy, we put

$(\exists x)(x = t^1 \& Dx^1 \& Hx^1)$, i.e. That is a desk and it is heavy.

And in general we take as a contextual definition:

$$\Psi x^n(tx^n)(\Phi x^n) =_{def.} (\exists x)(x = t^n \& \Phi x) \& \Psi x)$$

It is quite simple showing that the reasoning from 'That brown dog is friendly' to 'That dog is friendly' is valid.

The truth conditions for this system can be a variant of Mates' method of beta variants [Orenstein, 1999; 2000]. One might also be able to provide some other method, a substitutional quantifier account or perhaps following the lines of generalized quantifiers.

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Alex Orenstein
Professor Emeritus
Queens College and the Graduate Center, C.U.N.Y.

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Andrzej Salwicki

CZEGO INFORMATYCY NAUCZYLI SIĘ OD ANDRZEJA GRZEGORCZYKA?

Abstract. The paper of Andrzej Grzegorczyk [1953] on the hierarchy of primitive recursive functions was published in 1953.[1] Till today it is one of the most frequently cited results of a Polish author in computer science literature; the number of citations is near a thousand. Moreover, the paper is cited by eminent computer scientists, e.g. [Cook, 1983; Hartmanis and Hopcroft, 1971; Meyer and Ritchie, 1967; Muchnick, 1976; Mehlhorn, 1974], quite often the laureates of prestigious prizes.

For sixty years Andrzej Grzegorczyk's works in the domain of logic have obtained many important results. These results has found applications (quite often surprising ones) in various domains of computer science [Maksimova, 2007; Ornaghi *et al.*, 2006; Görnemann, 1971; Cohn *et al.*, 1997; Wolte and Zakharyaschev, 2002; Rauszer and Sabalski, 1975].

In 2003 Andrzej Grzegorczyk found a new proof for Gödel's undecidability result. The proof constructed by Grzegorczyk omits arithmetization, which makes the proof of Gödel so difficult in understanding. His reasoning [Grzegorczyk, 2005] makes use of a much simpler notion of the discernibility of texts; the arithmetic of natural numbers has been replaced by a simpler theory of concatenation of texts.

1. Wprowadzenie

Andrzej Grzegorczyk sam nie uważa się za informatyka, por. bibliografię jego prac w [Krajewski and Woleński, 2008]. A jednak, jego prace i publikacje mają znaczenie

- dla matematycznych podstaw informatyki (dla badań) oraz
- dla wykształcenia wielu informatyków (dydaktyka).

Jego książki i monografie z monografią *Zarys logiki matematycznej* [1961] na czele, stanowią doskonałe wprowadzenie do teorii funkcji obliczalnych. Zostały przetłumaczone na kilka języków i wielu informatyków uczyło się z nich o funkcjach obliczalnych.

[1] The present text is a modified, Polish version of an earlier article [Salwicki, 2008]. The presentation of Grzegorczyk's hierarchy has been rewritten. New chapters 5 and 6 were added. The presentation of the result of Meyer and Ritchie [1967] was extended.

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Dwa wyniki Andrzeja Grzegorczyka spinają jego dorobek w dziedzinie podstaw matematyki i informatyki. Są to:

- Napisana w wieku 30 lat rozprawa habilitacyjna [Grzegorczyk, 1953] i
- opublikowana w wieku 80+ praca [Grzegorczyk, 2005] prezentująca oryginalny dowód twierdzenia o nierozstrzygalności.

Jego wyniki nauczyły nas czegoś istotnego o obliczalności i kilka pokoleń informatyków inspirowało się wynikami profesora Andrzeja Grzegorczyka. Grzegorczyk ma wyniki w innych działach podstaw matematyki: oprócz *hierarchii Grzegorczyka* można przeglądając literaturę napotkać:

- aksjomat Grzegorczyka w logice modalnej,
- logikę Grzegorczyka [Maksimova, 2007],
- indukcję Grzegorczyka [Cornaros, 1995],
- regułę Grzegorczyka [Ornaghi *et al.*, 2006],
- aksjomat Grzegorczyka w logice silniejszej niż logika intuicjonistyczna [Grzegorczyk, 1964b; 1964a; Görnemann, 1971; Rauszer and Sabalski, 1975; Gabbay, 1974; M., 1981].

Może zaskakiwać odnalezienie pracy [Ornaghi *et al.*, 2006] na temat języków obiektowych, w której autor stosuje regułę Grzegorczyka dla modelowania systemów informacyjnych.

Istnieje też praca [Bonner and Mecca, 2000], w której wyniki Grzegorczyka cytowane są w kontekście baz danych.

2. Hierarchia Grzegorczyka

W rozprawie habilitacyjnej Grzegorczyka opublikowanej w 1953 r. dopatrujemy się pionierskiego wyniku na temat złożoności obliczeniowej. Następne prace na ten temat pojawiły sie kilkanaście lat później, zobacz przeglądowa praca Hartmanisa i Hopcrofta [1971]. Dziś po 60 latach wynik Grzegorczyka jest wciąż cytowany i inspiruje wielu badaczy problemów w teoretycznej informatyce. Jest to jedna z najczęściej cytowanych prac polskiego autora w literaturze informatycznej. Co więcej cytowania te znajdują się w pracach autorów wybitnych, często laureatów nagród tak prestiżowych jak nagroda Turinga. Problemy sformułowane w tej pracy wciąż inspirują kolejne pokolenia badaczy. Ponadto, w wielu nowych gałęziach informatyki powstają prace odwołujące się do wyniku Grzegorczyka z 1953 r.

Jak objaśnić hierarchię Grzegorczyka? Mówiąc ogólnie hierarchia ta eksponuje strukturę klas złożoności w zbiorze funkcji pierwotnie rekurencyjnych $\mathcal{PR}$. Zbiór $\mathcal{PR}$ funkcji pierwotnie rekurencyjnych jest najmniejszym zbiorem zawierającym funkcję stałą 0, funkcję następnika $x + 1$, funkcje rzutowania i zamkniętą ze względu na podstawienia i rekursję prostą (por. Definicje 2.1, 2.2 poniżej).

Grzegorczyk wykazał, że w zbiorze tym istnieje hierarchia. Funkcje znajdujące się na niższym piętrze hierarchii nie rosną tak szybko jak funkcje z pięter wyższych.

Wynik pracy [Grzegorczyk, 1953] może być streszczony następująco: Istnieje pewien rosnący ciąg zbiorów funkcji pierwotnie rekurencyjnych

$$\mathcal{E}^0 \subsetneq \mathcal{E}^1 \subsetneq \mathcal{E}^2 \subsetneq \mathcal{E}^3 \subsetneq \ldots \subsetneq \mathcal{E}^n \ldots$$

taki, że

$$\bigcup_{n \in N} \mathcal{E}^n = \mathcal{PR}.$$

Każdy zbiór $\mathcal{E}^n$ jest domknięty ze względu na podstawienia i ograniczoną rekursję prostą. Jego funkcje początkowe są funkcjami pierwotnie rekurencyjnymi.

Podczas przedstawienia szkicu dowodu posłużymy się pracami przeglądowymi [Rose, 1984; Gakwaya, 1997] oraz notatkami do wykładu Kevina Kelly [2012].

Definicja 1

Zbiór podstawowych funkcji obliczalnych jest zbiorem

$$B = \{Z, S\} \cup \{p_i^n : \ 1 \leq i \ \leq n, \ n \in N\},$$

gdzie

$Z : N \to N$, funkcja stała *zero*,
$S : N \to N$, funkcja następnika,
dla każdego $n \in N$ i dla każdego $i \in \{1, \ldots, n\}$
$p_i^n : N^n \to N$, funkcja rzutowania $p_i^n(x_1, \ldots, x_n) = x_i$. □

Będziemy rozważać różne zbiory funkcji domknięte ze względu na dwie operacje *superpozycji* oraz *ograniczonej rekursji prostej*.

Definicja 2

Superpozycja $\mathcal{S}$. Niech $h_1, \ldots, h_n$ będą funkcjami r argumentowymi ($r \geq 1$, $n \geq 0$) a g niech będzie funkcją n argumentową. Powiadamy, że funkcja f jest otrzymana przez superpozycję funkcji g i funkcji $h_1, \ldots, h_n$ jeżeli dla dowolnych argumentów $\overline{x} = (x_1, \ldots, x_r))$ zachodzi następująca równość

$$f(\overline{x}) = g(h_1(\overline{x}), \ldots, h_n(\overline{x}).$$

Rekursja Prosta $\mathcal{RP}$. Niech g będzie funkcją r argumentową i niech h będzie funkcją o $r+2$ argumentach. Jeśli dla dowolnych argumentów $\overline{x} = (x_1, \ldots, x_r))$ zachodzi

$$\begin{cases} f(\overline{x}, 0) = g(\overline{x}) \\ f(\overline{x}, t+1) = h(\overline{x}, t, f(\overline{x}, t)) \end{cases}$$

to mówimy, że funkcja f została otrzymana z funkcji g oraz h przez zastosowanie schematu rekursji prostej.

Ograniczona Rekursja Prosta $\mathcal{ORP}$. Rozważmy trójkę funkcji $\langle g, h, j\rangle$, gdzie g i h mają własności wymienione powyżej, a funkcja j ma $r+1$ argumentów. Funkcja f jest wynikiem ograniczonej rekursji prostej gdy została uzyskana z funkcji g i h przez rekursje prostą i gdy jest ograniczona przez funkcję j, tzn. $f(\overline{x}, t) \leq j(\overline{x}, t)$. □

Każda funkcja pierwotnie rekurencyjna jest określona jako wynik stosowania skończoną liczbę razy operacji superpozycji oraz rekursji prostej do funkcji bazowych. Wydawać by się mogło, że hierarchię stworzą piętra – zbiory funkcji definiowane przez coraz większą liczbę zastosowań schematu rekursji prostej. Tak jednak nie jest.

Do określenia hierarchii podzbiorów zbioru funkcji pierwotnie rekurencyjnych potrzebne są:

a) „kręgosłup" na którym zbudujemy hierarchię tj. ciąg funkcji E_i o coraz większej złożoności (w tym przypadku funkcji coraz szybciej rosnących) oraz,
b) wyrzeczenie się rekursji prostej i stosowanie wyłącznie ograniczonej rekursji prostej.

W naszym przypadku określimy następujący ciąg funkcji:

$$\begin{aligned} E_1(t) &\stackrel{df}{=} t^2 + 2, \\ E_n(t) &\stackrel{df}{=} E^t_{n-1}(2) \qquad\qquad \text{dla wszystkich } n \geq 2. \end{aligned}$$

Można sprawdzić, że taki ciąg funkcji spełnia warunki (i) oraz (ii) wymienione powyżej.

Zauważmy następujące własności funkcji E_n.

Lemat 3

Dla każdego n, i dla każdego t zachodzą następujące nierówności:

$$\begin{aligned} t+1 &\leq E_n(t), \\ E_n(t) &\leq E_{n+1}(t), \end{aligned}$$

$$\begin{array}{rcl} E_n(t) & \leq & E_n(t+1), \\ (\forall m)\ E_n^m(t) & \leq & E_{n+1}(t+m). \end{array} \qquad \square$$

W dalszym ciągu potrzebować będziemy jeszcze dodawania

$$E_0(t_1, t_2) \stackrel{df}{=} t_1 + t_2.$$

Teraz określimy klasy Grzegorczyka $\mathcal{E}^n$.

Definicja 4

Zbiory $\mathcal{E}^n$ są zdefiniowane następująco:

$$\begin{array}{rcl} \mathcal{E}^0 & \stackrel{df}{=} & \langle B; \mathcal{S}, \mathcal{ORP} \rangle, \\ \mathcal{E}^{n+1} & \stackrel{df}{=} & \langle B \cup \{E_0\} \cup \{E_n\}; \mathcal{S}, \mathcal{ORP} \rangle, \quad \text{dla każdego } n. \end{array}$$

Na mocy definicji każdy zbiór $\mathcal{E}^n$ jest domknięty ze względu na operacje superpozycji $\mathcal{S}$ i ograniczonej rekursji prostej $\mathcal{ORP}$. Dla każdego zbioru $\mathcal{E}^{n+1}$ jego funkcje bazowe $\{B \cup \{E_0\} \cup \{E_n\}\}$ są funkcjami pierwotnie rekurencyjnymi. $\square$

Posługując się poprzednim lematem można dowieść:

Lemat 5 (Lemat wzrostu)

Niech $\bar{t} = (t_1, \dots t_n)$ oznacza krotkę argumentów.

- Funkcje ze zbioru $\mathcal{E}^0$ są ograniczone przez funkcję $t_i + c_f$:
  $(\forall f \in \mathcal{E}^0)(\exists i, c_f \in N)(\forall \bar{t})\ f(\bar{t}) \leq t_i + c_f$,
- Funkcje ze zbioru $\mathcal{E}^1$ są ograniczone przez funkcje liniowe:
  $(\forall f \in \mathcal{E}^1)(\exists c_0, c_1, \dots, c_n \in N)(\forall \bar{t}) f(\bar{t}) \leq c_0 + c_1 \cdot t_1 + \cdots + c_n \cdot t_n$,
- Dla $n \geq 2$, każda funkcja f ze zbioru $\mathcal{E}^n$ jest ograniczona przez odpowiednią iterację funkcji wzrostu E_n:
  $(\forall n \geq 2)(\forall f \in \mathcal{E}^n)(\exists m_f)(\forall \bar{t})\ f(\bar{t}) \leq E_{n-1}^{m_f}(max(\bar{t}))$. $\square$

Stosując argument przekątniowy można wykazać, że $\mathcal{E}^n \neq \mathcal{E}^m$, dla $n \neq m$. Dowód wykorzystuje fakt, że każda klasa $\mathcal{E}^n$ zawiera funkcję, której funkcja wzrostu rośnie wraz z indeksem n. W ten sposób otrzymuje się fundamentalny wynik:

Twierdzenie 6

Istnieje ściśle rosnący ciag zbiorów funkcji pierwotnie rekurencyjnych

$$\mathcal{E}^0 \subsetneq \mathcal{E}^1 \subsetneq \mathcal{E}^2 \subsetneq \mathcal{E}^3 \subsetneq \ldots \subsetneq \mathcal{E}^n \ldots$$

taki, że

$$\bigcup_{n \in N} \mathcal{E}^n = \mathcal{PR}.$$

□

Uwaga. Do określenia „kręgosłupa” można użyć dowolnej funkcji G o następujących własnościach:

$$G(0) \geq 2 \wedge (\forall t \in N)\ t < G(t) < G(t+1).$$

Wtedy klasy hierarchii Grzegorczyka mogą być zdefiniowane przy pomocy ciągu funkcji $\{G_i\}$, który spełnia następujące warunki:

(i) $G_1(t) = G(t)$,

(ii) $G_n(t) = G^t_{n-1}(2)$. □

Powyższy wynik znacznie wyprzedził prace na temat złożoności obliczeniowej, por. Cook [1983].

Podczas gdy argument dotyczący wzrostu funkcji doprowadził do twierdzenia 6, to nie jest oczywiste, czy klasy $\mathcal{E}^i$ definiują różne klasy zbiorów liczb naturalnych. Powiadamy, że zbiór liczb naturalnych należy do klasy $\mathcal{E}^*_i$, gdy jego funkcja charakterystyczna należy do zbioru $\mathcal{E}^i$. Udowodniono, ze dla $i \geq 2$ jest to dokładna hierarchia klas $\mathcal{E}^*_i$ [Grzegorczyk, 1953]. Następujący problem: czy poniższe zawierania są ścisłe

$$\mathcal{E}^*_0 \subsetneq \mathcal{E}^*_1 \subsetneq \mathcal{E}^*_2?$$

pozostaje. od ponad 50 lat, jednym z trudniejszych wyzwań w teorii funkcji rekurencyjnych. Uzyskano tylko częściowy postęp w rozwiązaniu tego problemu: Bel'tyukov [1982] wykazał, że $\mathcal{E}^*_1 \subsetneq \mathcal{E}^*_2$ implikuje $\mathcal{E}^*_0 \subsetneq \mathcal{E}^*_1$. Kutyłowski [1987b] udowodnił, że jeśli w definicji klas zastąpić ograniczoną rekursję prostą przez ograniczoną iterację to prowadzi to równości dwu pierwszych klas tj. $I^0_* = I^1_*$. Ponadto wykazał, że podwójna rekursja może być użyta do określenia hierarchii pomiędzy piętrami $\mathcal{E}^*_0$ i $\mathcal{E}^*_2$. Dalsze rezultaty tego rodzaju można znaleźć w pracy [Kutyłowski and Loryś, 1987].

3. Wyniki powiązane z hierarchią Grzegorczyka

W tym rozdziale przytaczamy garść wyników odnoszących się do hierarchii Grzegorczyka.

3.1. Złożoność programów

Informatycy dobrze znają i doceniają twierdzenie 6. Znaczenie tego wyniku staje się bardziej widoczne w świetle prac R. W. Ritchie [1963], A. Meyera [1965] oraz Meyera i Ritchie [1967]. Jednym z pytań motywujących A. Meyera i R. W. Ritchie [1967] było pytanie następujące: *czy można spojrzeć na program i oszacować ograniczenie górne jego czasu wykonania*? Nie ma sposobu (algorytmu) gwarantującego dobrą odpowiedź. Tym niemniej dla pewnej klasy programów $\mathcal{LOOP}$ można taki przepis podać.

Niech $\mathcal{V}$ będzie zbiorem zmiennych. Zmienne będziemy oznaczać literami X, Y, Klasa $\mathcal{LOOP}$ programów jest najmniejszym zbiorem napisów takim, że

1. napisy postaci $X := 0$, $X := X + 1$, i $X := Y$ należą do zbioru $\mathcal{LOOP}$,
2. jeśli napisy P, P' należą do zbioru $\mathcal{LOOP}$ to do tego zbioru należy też napis $P; P'$,
3. jeśli napis P należy do zbioru $\mathcal{LOOP}$ to do tego zbioru należy też napis postaci **repeat** X **times** P **end**.

Dla danego programu P i wartościowania v zmiennych, obliczeniem c nazywamy taki ciąg par $\{\langle v_i, P_i\rangle\}$, że $v_0 = v$, $P_0 = P$, i

$$\langle v_{i+1}, P_{i+1}\rangle = \begin{cases} \langle v', P'\rangle & \text{gdy } P_i = X := 0; P' \text{ i } v'(X) = 0 \\ & \qquad \text{i } v'(z) = v_i(z) \text{ dla } z \neq X \\ \langle v', P'\rangle & \text{gdy } P_i = X := X + 1; P' \text{ i } v'(X) = v_i(x) + 1 \\ & \qquad \text{i } v'(z) = v_i(z) \text{ dla } z \neq X \\ \langle v', P'\rangle & \text{gdy } P_i = X := Y; P' \text{ i } v'(X) = v_i(y) \\ & \qquad \text{i } v'(z) = v_i(z) \text{ dla } z \neq X \\ \langle v_i, R\rangle & \text{gdy } P_i = \textbf{repeat } X \textbf{ times } Q \textbf{ end}; P' \\ & \qquad \text{i } R = \underbrace{Q; \ldots; Q}_{v_i(X)\text{razy}}; P' \end{cases}$$

Para $\langle v_{i+1}, \emptyset\rangle$ jest ostatnim elementem takiego ciągu.

Fakt 7

Każde obliczenie jest skończone.

Dla czytelników książek Grzegorczyka [1961] nie jest zaskoczeniem następujące

Twierdzenie 8

Każda funkcja pierwotnie rekurencyjna jest obliczana przez pewien program $P \in \mathcal{LOOP}$. □

W dalszym ciągu autorzy wprowadzają zbiory:
- L_n – zbiór programów w których konstrukcja **powtarzaj** jest zagnieżdżona co najwyżej n razy,
- $\mathcal{L}_n$ – zbiór funkcji obliczanych przez program ze zbioru L_n.

i dowodzą, że

Twierdzenie 9

Ciąg zbiorów $\mathcal{L}_0 \subsetneq \mathcal{L}_1 \subsetneq \mathcal{L}_2 \subsetneq \ldots$ jest ściśle rosnący, tzn. stanowi hierarchię w zbiorze funkcji pierwotnie rekurencyjnych. □

Jest to więc inna hierarchia, niż hierarchia Grzegorczyka. Łatwo zaobserwować, że operacja **powtarzaj** jest blisko spokrewniona z operacją minimum ograniczonego. Zobacz także rozdział na temat hierarchii Grzegorczyka w książce Brainerda i Landwebera [1974].

Uzyskano w ten sposób pewną odpowiedź na pytanie o koszt czasowy programu z klasy $\mathcal{LOOP}$. Koszt ten jest ograniczony przez liczbę zagnieżdżonych operacji powtarzaj. Ograniczenie takie nie jest jednak precyzyjne. Łatwo wskazać programy o sporej liczbie zagnieżdżeń operacji powtarzaj, których czas wykonania jest mały lub wręcz stały. Niestety, nie istnieje metoda znajdująca najmniejsze ograniczenie górne czasu wykonania danego programu [Meyer and Ritchie, 1967].

Warto też wymienić wyniki M. Kutyłowskiego [1987a]. Autor przypomina pojęcie uogólnionej hierarchii Grzegorczyka i omawia pewne problemy wiążące się z klasami początkowymi w tej hierarchii. Ustala równości pomiędzy uogólnionymi klasami Grzegorczyka i pewnymi klasami złożoności maszyn Turinga np. P oraz P*LINSPACE. Stosując narzędzia hierarchii Grzegorczyka udowadnia twierdzenie o hierarchii dla klasy P*LINSPACE. Dla lepszego opisu niższych klas złożoności wprowadza stosowe maszyny Turinga.

S. Breidbart [1979] udowodnił taką ciekawostkę o języku programowania APL: zbiór odpowiednio ograniczonych programów w jezyku APL (tradycyjnych 1-linerów) oblicza dokładnie zbiór funkcji z klasy E_4. hierarchii Grzegorczyka (jest to klasa bezpośrednio zawierająca klasę E_3 funkcji elementarnie rekurencyjnych).

3.2. Prezentacje hierarchii Grzegorczyka

Poza książką Brainerda i Landwebera [1974] wykład hierarchii Grzegorczyka można znaleźć w: Rose [1984], K. Wagner i G. Wechsung [2001], Gakwaya [1997], i w znanej monografii Rogersa [1987]. Warto zapoznać się z notatkami K. Kelly [Kelly, 2012] dla studentów.

3.3. Zastosowania hierarchii Grzegorczyka

Shelah [1988] wykorzystuje hierarchię Grzegorczyka by uzyskać lepsze oszacowanie kosztu obliczania liczb van Waerdena. Grozea [2004][2] odkrył, że znane problemy NP zupełne np. SAT lub problem cykli Hamiltona znajdują się w bardzo niskiej klasie $\mathcal{E}^{*0}$.

3.4. Artykuły i książki na temat teorii rekursji

Spośród długiej listy prac i książek w których wspomina się lub objaśnia hierarchię Grzegorczyka wspomnijmy kilka pozycji: Schwichtenberg [1997], Wainer [1972], Axt [1963], Muchnick [1976], Cichon & Wainer [1983], Bellantoni [2000], Mehlhorn [1974].

3.5. Rozszerzenia hierarchii Grzegorczyka

Istnieje kilka prac wprowadzających różne rozszerzenia hierarchii Grzegorczyka. Muchnick [1976] studiuje wektorowe hierarchie Grzegorczyka. Zobacz także: Wainer i Cichon [1972], [1983], Weiermann [1995]. Kristiansen and Barra [2005] definiują tzw. małe klasy Grzegorczyka dla rachunku lambda z typami.

3.6. Obliczenia analogowe i obliczenia z liczbami rzeczywistymi

Wielu autorów podejmowało próby przeniesienia hierarchii Grzegorczyka do modelu obliczeń analogowych lub do obliczeń z liczbami rzeczywistymi. Pionierską pracą w tej dziedzinie była praca Grzegorczyka [1957]. W pracy [Bournez and Hainry, 2004] zaprezentowano analogiczną i niezależną od komputera algebraiczną charakteryzację funkcji elementarnie obliczalnych w dziedzinie liczb rzeczywistych. Udowodniono, że jest to najmniejsza klasa funkcji, która zawiera pewne funkcje bazowe i jest zamknięta ze względu na operacje: złożenia, liniowego całkowania i schemat prostych granic. Wynik ten uogólniono na wszystkie wyższe poziomy hierarchii Grzegorczyka. Mycka i da Costa [Mycka and Costa, 2006] udowodnili nierozstrzygalność obliczeń

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[2] Wynik ten dotyczy równoległej hierarchii klas $\mathcal{E}^{*n}$ zbiorów relacji arytmetycznych, których funkcje charakterystyczne należą do odpowiednich klas $\mathcal{E}^{n}$ hierarchii Grzegorczyka.

nad czasem ciągłym. Zobacz też prace [Campagnolo *et al.*, 2002; Downey and Hirschfeldt, 2008].

4. O innych logicznych wynikach Grzegorczyka

Informatyka, w wielu swoich badaniach posiłkuje się logika modalną lub jej odmianami. Wyniki badań Grzegorczyka w dziedzinie logik modalnych i logiki intuicjonistycznej zostały obszernie przedstawione w pracy Maksimova [2007].

4.1. Reguła Grzegorczyka

W pracy na temat języka programowania obiektowego dla modelowania systemów informacyjnych [Ornaghi *et al.*, 2006] znajdujemy zastosowanie następującej reguły Grzegorczyka.

$$\frac{\begin{array}{c}\Gamma, [\ G(p)] \\ \vdots\ \pi \\ \textsc{or}\{C(p)\ B\}\end{array}}{\textsc{or}\{\ \textsc{for}\{\tau x \mid\ G(x) : C(x)\ \}B\}}$$

Nie wiem jakie są początki tej reguły. Prawdopodobnie należy ich szukać w pracach [Grzegorczyk, 1964b; 1964a]. Grzegorczyk rozważa w nich pewną logikę pośrednią, która powstaje przez dodanie następującej formuły

$$\forall x(P \vee Q(x)) \Rightarrow (P \vee \forall x Q(x))$$

do aksjomatów Heytinga logiki intuicjonistycznej. S. Görnemann [1971] nazywa tę formułę aksjomatem Grzegorczyka i dowodzi, że sformalizowana wg Heytinga logika intuicjonistyczna wzmocniona tym aksjomatem pozwala scharakteryzować klasę wszystkich struktur Kripkego o ustalonych dziedzinach. Nazwa ta następnie przyjęła. Zobacz też interesującą pracę Rauszer i Sabalskiego [1975].

4.2. Wyniki Grzegorczyka zastosowane w geoinformatyce

W r. 1951 Grzegorczyk opublikował wynik [Grzegorczyk, 1951] o nierozstrzygalności pewnych teorii topologicznych. Niedawno wynik ten był cytowany w dwu pracach:

- pracy na temat geoinformatyki [Cohn *et al.*, 1997] i
- w pewnej pracy nt. reprezentacji wiedzy (ang. Knowledge Representation) i jej jakościowej prezentacji przestrzenno-czasowej [Wolter and Zakharyaschev, 2002].

5. Nierozstrzygalność bez arytmetyzacji

Parę lat temu Andrzej Grzegorczyk podjął się ambitnego zadania: *udowodnić twierdzenie Gödla o nierozstrzygalności nie wykorzystując przy tym arytmetyzacji badanych teorii.* Artykuł [Grzegorczyk, 2005] jest dość długi – liczy prawie 70 stron, jest wynikiem cierpliwych i starannych prac. Zamiast oprzeć, jak u Gödla, dowód na arytmetyce liczb naturalnych z dodawaniem i mnożeniem, Grzegorczyk wykorzystuje teorię konkatenacji tekstów (inaczej słów), [tablica 1] z jednym działaniem – dopisywania tekstu. Rozpatrywane teksty zawierają tylko dwa znaki. (można je uznać za bity: *zero* i *jeden* – chociaż w pracy przyjęto inne oznaczenia). Trudno o prostszą sytuację.

Tablica 1. Elementarna teoria konkatenacji

SYGNATURA TEORII

Zbiór U tekstów (*można też mówić zbiór słów*)
Działanie
$* : U \times U \to U$ – *operacja konkatenacji*
oraz dwie stałe 0 oraz 1, $\quad 0, 1 \in U$

AKSJOMATY

$$x * (y * z) = (x * y) * z \tag{A1}$$

$$x * y = z * u \to \tag{A2}$$

$$((x = z \wedge y = u) \vee (\exists w)((x * w = z \wedge w * u = y) \vee (z * w = x \wedge w * y = u)))$$

$$\neg(0 = x * y) \tag{A3}$$

$$\neg(1 = x * y) \tag{A4}$$

$$\neg(0 = 1) \tag{A5}$$

Okazuje się, że podejście takie znacznie ułatwi śledzenie dowodu nierozstrzygalności teorii konkatenacji. Napotykamy jednak pewną trudność. Na czym polega obliczalność w takiej teorii? Autor proponuje zastąpić relację obliczalności przez relacje elementarnej i ogólnej odróżnialności tekstów. Indukcyjna definicja elementarnej odróżnialności **ED** brzmi całkiem naturalnie, zobacz [tablicę 2]. Do klasy **ED** należą relacja identyczności, trójargumentowa relacja *tekst z jest wynikiem dopisania tekstu y za tekstem x*, oraz teksty atomowe 0 i 1. Klasa **ED** jest domknięta ze względu na operacje: dodawanie nowego argumentu (*bez ograniczeń na jego wartość*), identyfikowanie pierwszego i drugiego argumentu, zamiana miejscami sąsiednich argumentów, alternatywa i negacja, kwantyfikacja ograniczona do podsłów.

Po czym następuje definicja ogólnej odróżnialności **GD**, zobacz [tablicę 3]. Klasa relacji ogólnie odróżnialnych jest domknięta ze względu na operację dualnej kwantyfikacji.

W definicji operatora dualnej kwantyfikacji w pomysłowy sposób wykorzystano twierdzenie Posta i zaprzęgnięto je do pracy.

Przy tak skromnym aparacie Grzegorczyk dowodzi kluczowe twierdzenie o reprezentowalności.

Tablica 2. Elementarna odróżnialność ED

Warunki początkowe: do klasy relacji elementarnie odróżnialnych należą

1. $\{t : t = 0\} \in \mathbf{ED}$
   $\{t : t = 1\} \in \mathbf{ED}$
2. $\{\langle t, y\rangle : t = y\} \in \mathbf{ED}$
3. $\{\langle t, y, z\rangle : t = y * z\} \in \mathbf{ED}$

Definicja indukcyjna: klasa **ED** jest domknięta ze względu na następujące operacje logiczne:

a. Jeśli $R \in \mathbf{ED}$ to $\{\langle y, t_1, \ldots, t_n\rangle : R(t_1, \ldots, t_n)\} \in \mathbf{ED}$,
b. Jeśli $R \in \mathbf{ED}$ to $\{\langle t_1, t_3, \ldots, t_n\rangle : R(t_1, t_1, t_3 \ldots, t_n)\} \in \mathbf{ED}$,
c. Jeśli $R \in \mathbf{ED}$ to $\{\langle t_1, \ldots, t_n\rangle : R(t_1, \ldots, t_{k+1}, t_k, \ldots, t_n)\} \in \mathbf{ED}$,
d. Jeśli $R \in \mathbf{ED}$ to $\{\langle t_1, \ldots, t_n\rangle : nonR(t_1, \ldots, t_n)\} \in \mathbf{ED}$,
   Jeśli $R \in \mathbf{ED}$ i $S \in \mathbf{ED}$ to
   $\{\langle t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+k}\rangle :$
   $R(t_1, \ldots, t_n)$ oraz $S(t_{n+1}, \ldots, t_{n+k})\} \in \mathbf{ED}$,
e. Jeśli $R \in \mathbf{ED}$ i gdy dla dowolnych $y, t_2, \ldots, t_n$: $S(y, t_2, \ldots, t_n) \equiv \forall t_1(t_1 \angle y \rightarrow R(t_1, t_2, \ldots, t_n))$ to $S \in \mathbf{ED}$.

Znak $\angle$ oznacza relację bycia podsłowem, $t_1 \angle y$ czytamy: słowo t_1 jest podsłowem słowa y.

Twierdzenie 10

Jeśli $\mathcal{T}$ jest niesprzeczną teorią zawierającą teorię konkatenacji, to dla każdej ogólnie odróżnialnej relacji $R \in \mathbf{GD}$ istnieje formuła α, która reprezentuje relację R w teorii $\mathcal{T}$. □

Po udowodnieniu, że pewne syntaktyczne pojęcia (np. zbiór formuł, zbiór zmiennych, ciąg formuł, należenie formuły do danego ciągu, etc.) odnoszące się do teorii $\mathcal{T}$ są relacjami ogólnie odróżnialnymi, autor przeprowadza przekątniowy dowód nierozstrzygalności teorii $\mathcal{T}$.

Twierdzenie 11

Jeśli $\mathcal{T}$ jest teorią niesprzeczną i zawiera teorię konkatenacji, to jest $\mathcal{T}$ teorią *nierozstrzygalną*. □

Wynik ten ma oczywiste walory dydaktyczne – jest łatwiej dostępny czytelnikom, którzy np. nie słyszeli o chińskim twierdzeniu o resztach. Co ważniejsze, wynik ten został uzyskany przy słabszych założeniach. Teoria konkatenacji tekstów jest słabsza od arytmetyki liczb naturalnych z dodawaniem i mnożeniem. Można co prawda w tej teorii zdefiniować liczby naturalne i odpowiednik operacji dodawania, można też zdefiniować trójargumentową relację $x = y * z$, ale nie wydaje się by można było zdefiniować operację mnożenia liczb naturalnych.

Tablica 3. Ogólna odróżnialność GD

Warunki początkowe: te same jak w definicji elementarnej odróżnialności.

Definicja indukcyjna: klasa **GD** jest domknięta ze względu na operacje logiczne a.–e. z definicji elementarnej odróżnialności, a także ze względu na *operację dualnej kwantyfikacji*:
Pewna relacja R jest *ogólnie odróżnialna*, $R \in \mathbf{GD}$, gdy istnieją dwie relacje T i S takie, że $S, T \in \mathbf{GD}$ i gdy dla dla dowolnych $t_1, \ldots, t_n$ mają miejsce następujące dwa fakty:

$$R(t_1, \ldots, t_n) \text{ wttw gdy istnieje } t_{n+1} \text{ takie. że } S(t_1, \ldots, t_n, t_{n+1})$$

$$R(t_1, \ldots, t_n) \text{ wttw gdy dla każdego } t_{n+1} \quad T(t_1, \ldots, t_n, t_{n+1})$$

Zwróćmy też, uwagę na to, że teoria konkatenacji tekstów obywa się bez schematu indukcji.

6. Najnowsze pasje Andrzeja Grzegorczyka

Znaleźć takie sformułowanie podstaw logiki, które jest wolne od paradoksów formalnej implikacji. Jeden ze znanych paradoksów stwierdza: *spośród trzech dowolnych zdań, dwa zdania są równoważne*; rzeczywiście następująca formuła $(p \equiv q) \vee (p \equiv r) \vee (q \equiv r)$ jest tautologią. Ten i inne paradoksy wynikają z dwuwartościowości przyjetej przez nas semantyki.

Andrzej Grzegorczyk proponuje by na nowo scharakteryzować implikację i równoważność, a dokładniej nową, inną implikację i inną relację równego znaczenia $\cong$. Powiada on: *naszym celem jest opisanie relacji równego znaczenia przez przyjęcie odpowiednio dobranych aksjomatów.*

Grzegorczyk zaproponował układ kilkunastu aksjomatów. Jego współpracownicy zbierają się i dyskutują z nim dobór aksjomatów, ich niesprzeczność, pełność i rozstrzygalność nowej teorii. Wyniki powinny zostać niedługo opublikowane.

7. Statystyki

Trudno oszacować jak często prace Grzegorczyka były cytowane w pracach informatycznych. Wyszukiwarka Google Scholar na pytanie "*Grzegorczyk hierarchy*" znajduje ponad 1000 odpowiedzi. Jeśli odrzucimy pozycje mniej istotne to i tak pozostaje ponad 400 cytowań, rozszerzeń, zastosowań i prezentacji prac Andrzeja Grzegorczyka.

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Nasz nauczyciel wciąż pracuje, zobacz rozdział 6.

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Andrzej Salwicki
Wydział Matematyczno-Przyrodniczy UKSW Warszawa
e-mail: `salwicki@mimuw.edu.pl`

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Mieszko Tałasiewicz

PHILOSOPHICAL INSIGHTS: A CASE OF INTENTIONALITY

Abstract. Philosophical intuition may be used as a shortcut for establishing what ordinary people would say or do in certain circumstances. In such use it may be and should be replaced by proper empirical tests. This part of philosophy, in which such information is essential to arguments, should be experimental philosophy.
However philosophical intuition may be – and has been – understood differently. Namely it can be regarded as a principled, thorough, dispassionate, unbiased analysis of the fundamental traits of human cognition in the first person perspective. While science arises from a critical and systematic attitude towards the third person aspect of common sense, philosophical insights result from adopting such an attitude towards the first person aspect of common sense.
Such insights are quite rare; but if they are sufficiently deep, they can 'reflect some fundamental features of our thought about the world' (Strawson). Among instances of a philosophical inquiry aiming at formulating such insights we may count Strawson's descriptive ontology and – for another instance – Husserl's theory of intentionality. I have enumerated several observations made by Husserl on this and, by that example, have highlighted three important aspects of deep philosophical insights. One, despite their first person perspective, they are not idiosyncratic: other thinkers, even from very different traditions, arrive at very similar insights. Two, they are not dim and vague: they can be expressed precisely, even in the shape of formal calculus. Three, they are not *a priori*. They may be misguided, but their shortcomings are traceable and correctible. In particular, I show how one could think about correcting some flaws in Husserl's account of intentionality and proceeding to a new, rather naturalistic theory of it.

One of the sources of the problem with philosophical insights and the rise of experimental philosophy is the serious devaluation of the role played by intuition in philosophical arguments. Increasingly, intuition has been treated as a short cut to common knowledge, a cheap replacement for tiresome and expensive tests. A vivid example of such treatment can be found in the debate around contextualism in epistemology. Most participants in the debate (notably [DeRose, 1992]) would argue that normally people would say this or that in such-and-such situation and therefore this theory is better than that theory because the former predicts the data and the latter

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does not. The explanatory scheme is virtually identical with that typical in empirical sciences. Now, when we realize that intuition is a very *inaccurate* replacement of proper tests – and everyone is bound to realize that, once the idea of asking his or her students crosses his or her mind – the move to experimental philosophy is the only reasonable option, unless philosophers would rephrase their arguments in order to show that the appeal to common knowledge in the original version was inessential. Something like this happened to Kripke's arguments against descriptivism. They were formulated with reference to the beliefs about 'what people would say' and were challenged [Machery *et al.*, 2004] on the grounds that these beliefs were empirically inaccurate. Then Michael Devitt [2011] set out to show that these beliefs were inessential to the arguments. In other cases – where the appeal to common knowledge proved to be essential – intuition would be replaced by actual empirical tests. That is happening now in the contextualism debate (cf., for example, [Hansen and Chemla, forthcoming]).

But there is another use of intuition. It was employed, for instance, by Peter F. Strawson who, in his search for an explanation for the subject-predicate duality, assumed that the duality should

> reflect some fundamental features of our thought about the world [Strawson, 1974, 11].

His renowned, 'descriptive' ontology from the first part of *Individuals* describes precisely the fundamental features not of this world but of *our thinking* about the world:

> we are dealing here with something that conditions our whole way of talking and thinking, and it is for this reason that we feel it to be non-contingent [Strawson, 1959, 29].

We might try for the first approximation the following formulation: *Philosophical intuition in its proper understanding is a principled, thorough, dispassionate, unbiased analysis of the fundamental traits of human cognition in the first person perspective.* In this sense it is prior to any scientific enquiry and cannot be replaced by such. Priority does not mean independence. There is mutual dependence between philosophical insights and science. In one way, when we think about the world we are conscious of what science tells us about the world and about human ways of thinking (from the third person perspective); the more we know about it, the more such knowledge conditions our first person way of thinking. That is why philosophy can and should take all relevant scientific facts into consideration (so to speak: we shall naturalize as much as we can) and yet retain its specificity. Conversely, it is the first person perspective that ultimately sets the agenda for science. Science can be explanatory only on condition that it answers some interesting questions. What questions are interesting, what kinds of answers are satisfactory, why this or that enables us to understand the issue and what the issue is – these problems require some input from the first person perspective.[1]

The first person perspective alone does not yet yield philosophy. The first instance of such a perspective is just common sense or, perhaps better put, common sense is an uncontrolled mixture of a first person and a third person perspective. Indeed, common sense is sufficient to do much of the job of the first person perspective, notably to run science. We can live without philosophy, but some of us feel the need to control the first person perspective and gain deeper understanding of the foundations of our cognition in this perspective.

Philosophy answers that need. It is like implementing the spirit of the scientific method within the first person perspective. Analogously, just as science develops from a critical and systematic attitude towards a third person aspect of common sense knowledge of the world, so philosophy develops from such an attitude towards the *first* person aspect of common sense (effectively it might be as remote from common-sense as Quantum Mechanics from common sense physics). Philosophical insight is intersubjectively communicable to those who are able to develop a similarly critical and systematic attitude and may be discussed, challenged and changed – assuming that people share their cognitive foundations (not common sense 'intuitions'!). This is just as it is in science, where the outcomes are communicable to those who share the relevant background and may be discussed, challenged and changed – assuming that the object of inquiry does not change when different researchers take a look at it.

A good example of what kind of enquiry I have in mind is Strawson's descriptive metaphysics, already mentioned above. But an even better example is Edmund Husserl's work in the theory of intentionality in *Logical Investigations* [2001].

Husserl has not been well remembered in the analytic tradition because of his late phenomenological project. The project highlighted the aim of making philosophy an assumptionless, apriorical royal road to absolute certainty, which should be rightly placed next to sheer fantasy. However, a struggle for certainty is only part of the story. Phenomenology, espe-

[1] I will skip the topic of metaphysical assumptions in science although it is not entirely irrelevant here.

cially early phenomenology, has another component: a method of detailed, methodical, principled observation and categorization of the subject's own inner states accompanied by a struggle for the detection and reduction of one's biases and personal idiosyncracies. This part of phenomenology is highly respectable and it has in fact contributed much to the rise of analytic philosophy.[2]

Let us take a closer look in order to see how it works in detail.

Intentionality is rightly believed to be one of the fundamental features of our thinking and talking. I claim that the method of 'pure description', as he would call it, enabled Husserl to discover many subtle facts about human linguistic competence that the sciences (such as psychology or linguistics) were not – and still are not – properly able to describe and explain. Firstly, he started with the observation that expressions have meanings because they are founded in intentional acts whereby they appear to be directed at something:

> The concrete phenomenon of the sense-informed expression breaks up, on the one hand, into the physical phenomenon forming the physical side of the expression, and, on the other hand, into the acts which give it meaning [...] In virtue of such acts, the expression is more than merely sounded word. It means something, and in so far as it means something, it relates to what is objective [Husserl, 2001, vol. 1; 191–192].

Secondly, an intentional act, hence an expression, can be directed at nothing. Expressions must **appear** to be directed at something, but not necessarily **be** so directed. Husserl lays great emphasis on this (in opposition to his predecessors, notably Brentano and Twardowski):

> Relation to an actually given objective correlate, which fulfills the meaning-intention, is not essential to an expression [Husserl, 2001, Vol. 1; 199].
>
> If I have an idea of the god Jupiter, [...] this means that I have a certain presentative experience, the presentation-of-the-god-Jupiter is realized in my consciousness. This intentional experience may be dismembered as one chooses in descriptive analysis, but the god Jupiter naturally will not be found in it. The 'immanent', 'mental object' is not therefore part of the descriptive or real make-up of experience, it is in truth not really immanent or mental. But it also does not exist extramentally, it does not exist at all. This does not prevent

[2] The boiling pot of ideas from which analytic philosophy erupted was heated mainly by the exchange between Husserl, Twardowski and Frege, which is rightly noted in [Dummett, 1993]. All three of them, Frege notwithstanding, were in this sense phenomenologists. Frege's ideas of distinguishing sense and reference or saturated and unsaturated expressions are products of the same method.

> our-idea-of-the-god-Jupiter from being actual, a particular sort of experience or particular mode of mindedness (Zumutesein), such that he who experiences it may rightly say that the mythical king of the gods is present to him, concerning whom there are such and such stories. If, however, the intended object exists, nothing becomes phenomenologically different. It makes no essential difference to an object presented and given to consciousness whether it exists, or is fictitious, or is perhaps completely absurd. I think of Jupiter as I think of Bismarck, of the tower of Babel as I think of Cologne Cathedral, of a regular thousand-sided polygon as of regular thousand-faced solid [Husserl, 2001, Vol. 2; 98–99].

Thirdly, intentionality assumes two forms. In other words, there are two kinds of intentional acts: nominal acts and propositional acts. In language, they correspond to names and sentences (which can therefore justly be called primary semantic categories). It is not enough to say that our thinking or its verbal expression is intentional or directed at something, for our thinking and its verbal expression can be directed at it *in two different ways*.

> Nominal acts and complete judgements never can have the same intentional essence, and [...] every switch from one function to the other, though preserving communities, necessarily works changes in this essence [Husserl, 2001, Vol. 2; 152].
>
> Naming and asserting do not merely differ grammatically, but 'in essence', which means that the acts which confer or fulfil meaning for each, differ in intentional essence, and therefore in act-species [Husserl, 2001, Vol. 2; 158].

Fourthly, the distinction between nominal and propositional intentional acts (a basis for distinguishing the syntactic roles of names and sentences) crosses with the distinction between positing and non-positing acts:

> Among nominal acts we distinguish positing from non-positing acts. The former were after a fashion existence-meanings [...] refer to [an object] as existent. The other acts leave the existence of their object unsettled: the object may, objectively considered, exist, but it is not referred to as existent in them, it does not count as actual, but rather 'merely presented'. [...] We find exactly the same modification in the case of judgments. Each judgment has its modified form, an act which merely presents what the judgment takes to be true [...] without a decision as to truth and falsity [...]. Judgments as positing propositional acts have therefore their merely presentative correlates in non-positing propositional acts [Husserl, 2001, Vol. 2; 159–160].

The difference between those two distinctions is clear, but the matter requires close scrutiny because, due to obscure terminology, the distinctions may be confused. In particular, the concept of proposition (judgment) in the light of these distinctions is ambiguous. It could be understood broadly as a correlate of any propositional act, assertive or non-assertive, or understood narrowly, as a correlate of a positing propositional act (assertion). Some readers may also, wrongly, take it for a correlate of any positing act (propositional and nominal as well):

> To call all positing acts 'judgments' tends to obscure the essential distinction [...] between nominal and propositional acts, and so to confuse an array of important relationships [Husserl, 2001, Vol. 2; 166].

Fifthly, Husserl further distinguished independent from non-independent expressions, as well as complete from incomplete. Complete expression is a kind of expression which is syntactically coherent, with a unitary meaning, for example, 'a cat', 'The cat sits on the mat', or 'quite well'. Incomplete expression is an expression which lacks this internal syntactic coherence: 'the on cat quite'. Independent expressions are, for example, 'a horse', 'I've seen a ghost', 'a green cow', while non-independent expressions are, for example, 'good' or 'quite well' – they are functors which have unitary but unsaturated meanings and therefore require objects:

> Several non-independent meanings [...] can be [...] associated in relatively closed units, which yet manifest, as wholes, a character of non-independence. This fact of complex non-independent meanings is grammatically registered in the relatively closed unity of complex syncategorematic expressions. Each of these is a single expression, because expressive of a single meaning, and it is a complex expression, because expressive part by part of a complex meaning. It is in relation to this meaning that it is a complete expression. If nonetheless we call it incomplete, this depends on the fact that its meaning, despite its unity, is in need of completion. Since it can only exist in a wider semantic context, its linguistic expression likewise points to a wider linguistic context, to a completion in speech that shall be independent and closed [Husserl, 2001, Vol. 2; 57].

Let us stop the illustration here. Three points are particularly apt to be made.

One is that the observations made by Husserl are by no means idiosyncratic to him. On the contrary, we can spot very similar insights in the works of other philosophers, especially those who acknowledged the descriptive method we are talking about.

For instance, the crossed distinctions of nominal/propositional and positing/non-positing acts, mentioned above, were referred to independently by Strawson and Peter T. Geach. Strawson wrote [1950, 11]:

> [R]eferring to or mentioning a particular thing cannot be dissolved into any kind of assertion. To refer is not to assert, though you refer in order to go on to assert.

How close the approach gets to Husserl's observations is not entirely clear, because Strawson's statement can be taken as an acknowledgment of the distinction between nominal and propositional acts, but maybe it is only a way of drawing our attention to the distinction between positing and non-positing acts. Less ambiguous is a passage found in Geach [1980, section 20, p. 52]:

> A name may be used outside the context of a sentence simply to call something by name – to acknowledge the presence of the thing. This act of naming is of course no proposition [...]. It does, however [...], express a complete thought.

Geach here distinguishes naming acts as different from propositional acts although expressing complete thoughts as well as propositional acts do. In this passage he confines himself to positing acts only, but elsewhere in his book [1980, section 19] he further distinguishes propositions from assertions, which leads us to believe, with reason, that Geach, like Husserl, held both distinctions separate and considered them to be independent.

For another instance let us take the crossed distinctions of complete/incomplete and independent/non-independent sentences. *Pace* terminology, one can easily see that Husserl's non-independent expressions correspond quite well to Frege's incomplete or unsaturated ones: they exhibit a certain unitary quality of meaning, but they need to be completed in order to acquire independent meaning.[3] Husserl's incomplete expressions, or those 'containing gaps' are not Frege's incomplete, or unsaturated expressions; they are simply incoherent fragments, with no 'unitary meaning' whatsoever; they have more to do with ellipsis or with plain syntactical incoherence.

The second point is that insights like these are not loose ideas without any relevance to more rigorous accounts of language. On the contrary, they are expressible in a precise way and can be considered as postulates ready to use even in formal systems. These particular insights we have just discussed constitute the foundations of Categorial Grammar.[4] It shows that deep, careful reflection upon one's own way of thinking and talking may lead

[3] *For not all the parts of a thought can be complete; at least one must be 'unsaturated', or predicative; otherwise they would not hold together* [Frege, 1892, 54].

Statements in general [...] *can be imagined to be split up into two parts; one complete in itself, and the other in need of supplementation, or 'unsaturated'* [Frege, 1891, 31].

[4] Categorial Grammar is a highly sophisticated formal account of syntax, initiated by Kazimierz Ajdukiewicz [1935], and developed by, among others, Y. Bar Hillel, J. Lambek,

to a discovery of ideas that are shared by other distinguished thinkers in the field and the possibility of forming a framework for a universal, highly formalized syntactic theory.

The third point is that first person insights are not only shared by other thinkers, and not only expressible in a calculus, but also open to further development. Being the results of a scientific-like attitude they are not *a priori*: internal experience is experience, thought experiments are experiments. Philosophical insights resulting from them are *a posteriori*: they may be misguided in this or that, but their shortcomings are traceable and correctible.[5]

Also in Husserl's account we may find thoughts that are not thought thoroughly enough. Some problems can be spotted within Categorial Grammar, when internal difficulties with the calculus seem to reach the philosophical ground. I believe that this is the case with the syntax of quantified phrases: so far no treatment of quantification has proved entirely satisfactory in CG. But some other problems can be shown directly, from the commonsense level.

Two such charges against Husserl's account are presented by Dummett [1993].

One is that Husserl's account of the intentionality of language puts him at risk of slipping into a Humpty-Dumpty-like attitude towards meaning. Humpty-Dumpty, talking to Alice in *Through the Looking-Glass*, maintains that she cannot know the meaning of a word he has used ('glory') until he communicates his intentions in the matter. Acceptance of such attitudes is rightly believed to be absurd – if it were the case, any intersubjective communication would be impossible. Yet it seems that Husserl's insight, according to which meaning is founded in an intentional act of a subject, leads precisely to such an attitude. Another charge, akin to the former, is that in Husserl's account it is hard to imagine how language acquisition could be a social practice (which it certainly is). Husserl sounds very individualistic in that respect.

However, we can remedy that. We can try to establish and analyse further what has been left without proper treatment. I have done this in

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P. T. Geach, M. Cresswell, D. Lewis, J. van Benthem, and W. Buszkowski. It is now one of the most powerful tools for analysing natural language. For a presentation of a full-blown version see, for example, [Carpenter, 1997] or [Steedman, 2000]. For a discussion of the relation between the calculus and its philosophical background see [Tałasiewicz, 2009; 2010; 2012a].

[5] This point shows in the clearest way that the insights I am talking about here are of a different sort than the 'conceptual analysis' advocated by Kirk Ludwig [2007], which is declared to be *a priori*.

greater detail elsewhere [Tałasiewicz, 2012b], but some recapitulation seems appropriate here.

The problem with Husserl's insight that is responsible for the troubles raised by Dummett is that Husserl has not properly distinguished the perspective of the subject as a speaker (encoder) and as a hearer (recipient). He was talking generally about a 'subject' and 'subject acts' in 'solitary life' [Husserl, 2001, Vol. 1; 190], and this distinction was not clear: the subject in a sense is both the speaker and the hearer. Now, when this distinction is uncontrolled, we are apt to assume automatically that intentionality is founded in the acts of the speaker. And that is the mistake, in consequence of which we slip into the Humpty-Dumpty attitude.

Upon reflection we might choose to go the other way and deliberately adopt the hearer's perspective.[6]

That solves the problem: now meaning is formed in the acts of the recipient, when he or she perceives the expression as directed at something. The quality of this perception may be induced by those from whom the recipient is learning the language in the social practice of using language. Humpty-Dumpty imparts meaning to the word 'glory' not when he wants to subject Alice to 'a nice knock-down argument' but when he himself learns the word for the first time, and this meaning is (roughly) the same as Alice's.

Now, we might compare the outcome of our reflection in the first per-

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[6] Husserl himself, contrary to what Dummett says, was far from being obviously encoder-oriented. In many places the description of an intentional act essentially takes the receiver's perspective:

> What is involved in the descriptive difference between the physical sign-phenomenon and the meaning-intention which makes it into an expression, becomes most clear when we turn our attention to the sign qua sign, e.g. **to the printed word as such**. If we do this, we have an external percept [...] just like any other, whose object loses its verbal character. If this object again functions as a word, its presentation is wholly altered in character [...]. Our interest, our intention, our thought [...] point exclusively to the thing meant in the sense-giving act [...]. [I]ntuitive presentation, in which the physical appearance of the word* is constituted, undergoes an essential phenomenal modification when its object begins to count as an expression. While what constitutes the object's appearing remains unchanged, the intentional character of the experience alters. There is constituted [...] an act of meaning which finds support in the verbal presentation's intuitive content, but which differs in essence from the intuitive intention directed upon the word itself [Husserl, 2001, Vol. 1; 193–194].

> All objects and relations among objects only are what they are for us, through acts of thought essentially different from them, in which they become present to us, in which they stand before us as unitary items that we mean [Husserl, 2001, Vol. 1; 194].

> The meaning of the assertion [...][–] we continue to recognize its identity of intention in evident acts of reflection: we do not arbitrarily attribute it to our assertions, but discover it in them [Husserl, 2001, Vol. 1; 321 – note 5 to page 213].

> The soliloquizing thinker 'understands' his words, and this understanding is simply his act of meaning them [Husserl, 2001, Vol. 1: 321 – note 5 to page 213].

son perspective[7] to some third person scientific research, devoted to the problems of language acquisition; luckily we now have some interesting and relevant findings at hand. For instance, psycholinguists point out that fundamental for the learning of the first language is the child's ability to interpret another person's physical features as directional:

> Joint Attention Comes First [...] In a successful conversation, the two participants must agree on what is being talked about. One way to ensure this is to start with the same locus of attention. But how does one make a one- or two-year-old systematically attend to what one is saying? [...] [B]y age one, infants have become quite good themselves at checking on the adult's **gaze, stance, and physical orientation** [Clark, 2003, 32].

This in turn may lead us – by the mechanism mentioned before, that our (scientific) knowledge about the world conditions our first person perspective – to again make a first person observation that perhaps intentionality of thinking and talking is not a primary form of intentionality; that intentionality in the first place, prior to language and articulated thoughts, is a property of some physical objects, whose shape and orientation induce upon us the impression of directedness. And that concerns not only early stages of natural language, or just natural language, because, as Grzegorczyk convincingly argued over a half of a century ago, not only in natural language, but also in the language of science, 'the meaning of the name is ultimately fixed in situations involving pointing' [Grzegorczyk, 1950–1951, 307] and 'the expression "I am now pointing to ..." [...] is the primitive formula of the descriptive language [of physics]' (ibidem: 311).

Thus we arrive at a quite unexpected insight:

> Intentionality is a relative quality of certain perceived objects, in particular arrows and sticks, but also other people's gestures and glances, which exhibits itself in a relationship with a certain ability of the subject perceiving those objects. The result is that the subject perceives those objects as being directed towards something [Tałasiewicz, 2012b, 515].

According to this insight, intentionality is expressed in a simple, naturalistic way, and is similar to such relative qualities as colours.[8] Intentionality

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[7] The talk of Humpty-Dumpty and Alice, grammatically in the third person, should not cover from us the fact that epistemologically we are all the time in the first person perspective: we know what H-D or Alice would say or do because we ourselves play their role in the analysis.

[8] Compare J. J. C. Smart's definition of the latter: 'Colors [are] dispositions of physical objects to evoke characteristic patterns of discriminatory color behavior by normal human percipients in normal circumstances' [Smart, 1997, 1].

is no longer a relation between the mind and what the mind is directed at – contrary to the standard post-Brentanian account. It is rather a relation between a physical object of a certain kind and the subject who perceives this object as being directed. Human ability to perceive such things as intentional becomes a natural, congenital trait, and it is quite easy to imagine evolutionary advantages that hominids possessing such a trait would have over those lacking this ability.[9]

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[9] For further details of this insight into intentionality see [Tałasiewicz, 2012b].

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Mieszko Tałasiewicz
Institute of Philosophy, University of Warsaw
`m.talasiewicz@uw.edu.pl`

Kazimierz Trzęsicki

CHARACTERIZATION OF CLASSES OF FRAMES IN MODAL LANGUAGE

Abstract. In the paper some facts about the definability of classes of Kripke frames for tense logic are discussed. Special attention is given to the classes of frames definable by Grzegorczyk's Axiom:

$$\Box(\Box(\phi \to \Box\phi) \to \phi) \to \phi$$

as interpreted in temporal logic.

1. Temporal logic

Let us consider Kripke[1] (relational) semantics $\langle T, < \rangle$, where T is a non-empty set (of time points) and $<$ is a binary relation on T, $< \subseteq T \times T$ (the precedence relation – *earlier/later*). The flow of time $\mathfrak{T}$ is represented as $\langle T, < \rangle$. If a pair of elements (t, t_1) belongs to $<$ we say that t *is earlier than* t_1, or: t_1 *is later than* t.

$\{t : n < t\}$ is the future of n. The past of n is defined likewise: $\{t : n > t\}$.

Models of temporal logic TL are triples $\langle T, <, V \rangle$ consisting of a frame $\langle T, < \rangle$, the flow of time $\mathfrak{T}$, together with a valuation V, where V is a function assigning each propositional letter a subset of T: $V \to 2^T$. Intuitively, V to a propositional letter assigns a set of moments of time in that the letter is satisfied (true).

Besides classical propositional connectives $(\neg, \vee, \wedge, \to, \leftrightarrow)$ we will have temporal operators that are defined on $\langle W, < \rangle$.

[1] The question who invented relational semantics is a subject of discussion. Disputed is the role of Alfred Tarski [Goldblatt, 2005, p. 17]. There are some reasons to point to Leibniz as its inventor. For Leibniz the actual world is the one of the best of all possible worlds. He maintains that [Goldblatt, 2005, p. 18]:

> Not only will they hold as long as the world exists, but also they would have held if God had created the world according to a different plan.

An answer to the question who invented relational models is given by Saul Kripke [Goldblatt, 2005, p. 22].

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To each operator defined with the help of $<$ there is a symmetric-to-it operator defined with the help of the converse (inverse, transpose) relation of $<$, i.e. with the help of $>$. Temporal logic (TL) uses this possibility, distinguishing past tense and future tense operators.

Vocabulary

1. $p_0, p_1, \ldots$ – propositional letters, $Prop$;
2. $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ – propositional connectives: negation, disjunction, conjunction, implication, equivalence, respectively;
3. G, F, H, P – tense operators:
   - G – it will always be the case that ϕ, or: henceforth, ϕ;
   - F – it will be the case that ϕ, or: ϕ is true at some time in the future;
   - H – it has always been the case that ϕ, or: hitherto, ϕ;
   - P – it has been the case that ϕ, or: ϕ was true at some time in the past.

Tense logic is the study of tense operators, and of the logical relations between sentences having tense. The study of this logic has been initiated and developed by Arthur Norman Prior, e.g. [1957; 1962; 1967; 1968].

Definition 1 (well formed formula)

$$\phi ::= p_i, i \in \mathbb{N} |\ \neg\phi |\ \phi \vee \phi |\ \phi \wedge \phi |\ \phi \rightarrow \phi |\ \phi \leftrightarrow \phi$$

We will also frequently refer to the mirror image of a formula; this is simply the formula one obtains by simultaneously replacing all Hs and Ps with Gs and Fs, respectively, and vice versa. The mirror image of a formula ϕ will be denoted: $MI(\phi)$.

Definition 2 (the satisfiability of a formula at a point of time)

Let $\mathfrak{M}$ be a model $\langle T, <, V\rangle$ and let $t \in T$:

1. $\mathfrak{M}, t \models \phi$ iff $t \in V(\phi)$, if $\phi \in Prop$;
2. $\mathfrak{M}, t \models \neg\phi$ iff $\mathfrak{M}, t \not\models \phi$;
3. $\mathfrak{M}, t \models \phi \vee \psi$ iff $\mathfrak{M}, t \models \phi$ or $\mathfrak{M}, t \models \psi$;
4. $\mathfrak{M}, t \models \phi \wedge \psi$ iff $\mathfrak{M}, t \models \phi$ and $\mathfrak{M}, t \models \psi$;
5. $\mathfrak{M}, t \models \phi \rightarrow \psi$ iff $\mathfrak{M}, t \not\models \phi$ or $\mathfrak{M}, t \models \psi$;
6. $\mathfrak{M}, t \models \phi \leftrightarrow \psi$ iff $\mathfrak{M}, t \models \phi$ if and only if $\mathfrak{M}, t \models \psi$;
7. $\mathfrak{M}, t \models H\phi$ iff $\forall t_1, t_1 < t : \mathfrak{M}, t_1 \models \phi$;
8. $\mathfrak{M}, t \models P\phi$ iff $\exists t_1, t_1 < t : \mathfrak{M}, t_1 \models \phi$;

9. $\mathfrak{M}, t \models G\phi$ iff $\forall t_1, t < t_1 : \mathfrak{M}, t_1 \models \phi$;
10. $\mathfrak{M}, t \models F\phi$ iff $\exists t_1, t < t_1 : \mathfrak{M}, t_1 \models \phi$;

Kamp [1968] introduced two operators U (until) and S (since), and he showed that over the class of complete linear temporal orders, the formalism is expressively complete [Gabbay, 1981a; Gabbay and Hodkinson, 1990].

11. $\mathfrak{M}, t \models U(\phi, \psi)$ iff $\exists t_1, t < t_1 : \mathfrak{M}, t_1 \models \phi$ and $\forall t_2, t < t_2 < t_1 : \mathfrak{M}, t_2 \models \psi$,
12. $\mathfrak{M}, t \models S(\phi, \psi)$ iff $\exists t_1, t_1 < t : \mathfrak{M}, t_1 \models \phi$ and $\forall t_2, t_1 < t_2 < t : \mathfrak{M}, t_2 \models \psi$,

The mirror image of ϕ, $MI(\phi)$, is obtained by simultaneously replacing S by U and U by S, everywhere in ϕ, and other temporal operators according to the former rule of forming of $MI(\phi)$.

Definition 3 (validity of a formula in a model)

$$\langle T, <, V\rangle \models \phi \text{ iff } \forall t \in T : \mathfrak{M}, t \models \phi.$$

Definition 4 (validity of a formula in a frame)

$$\langle T, <\rangle \models \phi \text{ iff } \forall V :\rightarrow 2^T : \langle T, <, V\rangle \models \phi.$$

Definition 5 (validity in a class of frames)

Let $\mathfrak{F}$ be a class of frames. $\mathfrak{F} \models \phi$ if and only if for any $\mathfrak{T}$: if $\mathfrak{T} \in \mathfrak{F}$, then $\mathfrak{T} \models \phi$.

Less would do since actually all propositional truth functions can be defined for instance in terms of $\neg$ and $\rightarrow$; moreover P can be defined as $\neg H \neg$ and F can be defined as $\neg G \neg$. H and G can be defined with the help of U and S: $H\phi \leftrightarrow S(\bot, \phi), G\phi \leftrightarrow U(\bot, \phi)$.

2. Definability of classes of frames

Definition 6

A formula ϕ characterizes a class of frames $\mathfrak{T}$ if and only if

$$\mathfrak{T} \models \phi \text{ iff } \mathfrak{T} \in \mathfrak{F}.$$

Let $\mathfrak{F}_\phi$ denote the class of frames characterized by a formula ϕ.

If a class of frames is characterized by a formula ϕ, or – in other words – is definable by ϕ, we say that the formula expresses this class of frames. Conditions $\mathfrak{C}$ imposed on the relation $<$, $<\in \mathfrak{C}$, are expressible by a formula ϕ if and only if the class of frames $\langle T, <\rangle$, such that $<$ fulfills the conditions $\mathfrak{C}$, is characterized (definable) by the formula ϕ.

We ask what classes of frames are characterized by a formula (or set of formulas) of TL.

Lemma 7

If $\langle T, <\rangle \models \phi$, then $\langle T_1, <_1\rangle \models \phi$, where T_1 has exactly one element, and $<$ is empty or universal.

Proof.

By definition ϕ is satisfied for any valuation such that $V(p_i) = V(p_j)$, for any $i, j \in \mathbb{N}$. We do not assume that if $t_1 < t_2$, then $t_1 \neq t_2$, and there is not such an assumption in definitions of temporal operators. Hence $\langle T_1, <_1\rangle \models \phi$, where T_1 has exactly one element. In one element set $\{t\}$ there are definable only two binary relations: empty and universal, i.e. $\emptyset$ and $\{\langle t, t\rangle\}$, respectively. ■

Let us remark that the empty relation is irreflexive.

Theorem 8

Irreflexivity is not expressible in TL.[2]

Proof.

In the case of operators which are defined without any assumption about the relation $<$, by the lemma 7, any formula that is satisfied in any frame $\langle T, <\rangle$ is also satisfied in a frame $\langle T_1, <_1\rangle$, where T is a one-element set and $<_1$ is either empty or universal. If $<_1$ is universal, then $<$ is reflexive. Empty relations are irreflexive, but not all irreflexive relations are empty.

Let us consider a language whose some operators are defined by assuming that if $t < t_1$, then $t \neq t_1$. In such a case if a formula with such an operator is satisfied in a model $\langle T, <, V\rangle$, then the formula is satisfied in a model $\langle T, \leq, V\rangle$. Hence irreflexivity is not expressible. ■

Irreflexivity is not the only property which is not expressible in TL.

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[2] Irreflexivity is expressible by Gabbay's [1981b] Irreflexivity Rule, IRR:

$$\frac{q \wedge H(\neg q) \rightarrow \phi}{\phi},$$

provided that the propositional letter q does not appear in the formula ϕ.

We ask what class of frames is characterized by $MI(\phi)$ if ϕ characterizes a class $\mathfrak{F}$.

Definition 9 (converse of a binary relation)

The converse of binary relation $<$ is the relation $>$ such that:

$$t < t_1 \text{ iff } t_1 > t.$$

Let class of frames $\check{\mathfrak{F}}$ be such that:

$$\text{if } \langle T, < \rangle \in \mathfrak{F}, \text{ then } \langle T, > \rangle \in \check{\mathfrak{F}}.$$

Let us note that $\check{\check{\mathfrak{F}}} = \mathfrak{F}$.

Lemma 10

$$\langle T, <, V \rangle, t \models \phi \text{ iff } \langle T, >, V \rangle, t \models MI(\phi).$$

Proof.

The lemma is provable by the structural induction.

Let us consider only one case, namely of $S(\psi, \chi)$. By assumption we have that for any V, t:

$$\langle T, <, V \rangle, t \models \psi \text{ iff } \langle T, >, V \rangle, t \models MI(\psi),$$

and

$$\langle T, <, V \rangle, t \models \chi \text{ iff } \langle T, >, V \rangle, t \models MI(\chi).$$

By definition of MI we have $MI(S(\psi, \chi) = U(MI(\psi), MI(\chi))$.

$\langle T, <, V \rangle, t \models \psi \models S(\psi, \chi)$ by definition of S is equivalent to:

$$\exists t_1, t_1 < t : \mathfrak{M}, t_1 \models \psi \text{ and } \forall t_2, t_1 < t_2 < t : \mathfrak{M}, t_2 \models \chi.$$

It is equivalent to:

$$\exists t_1, t_1 > t : \mathfrak{M}, t_1 \models MI(\psi) \text{ and } \forall t_2, t_1 > t_2 > t : \mathfrak{M}, t_2 \models MI(\chi).$$

It is equivalent to:

$$\langle T, V > \rangle, t \models U(MI(\psi), MI(\chi)),$$

and, finally:

$$\langle T, <, V \rangle, t \models S(\psi, \chi) \text{ iff } \langle T, >, V \rangle, t \models MI(S(\phi, \chi).$$

■

By lemma 10 we have:

Theorem 11

A formula ϕ characterizes a class of frames $\mathfrak{F}$ if and only if the formula $MI(\phi)$ characterizes $\check{\mathfrak{F}}$.

The formula:

$$Gp \rightarrow Fp$$

characterizes endless time (forward seriality), $T^{\infty+}$: $\forall\ t\exists t_1 : t < t_1$.

Time without a beginning (backwards seriality), $T^{\infty-}$: $\forall\ t\exists t_1 : t_1 < t$ is characterized by the formula:

$$Hp \rightarrow Pp.$$

F-LIN, linearity in the future (forward linearity): $\forall t, t_1, t_2$: if $t < t_1, t < t_2,$ then $t_1 < t_2$ or $t_1 = t_2$ or $t_2 < t_1$ is characterized by:

$$Fp \rightarrow G(Pp \vee p \vee Fp).$$

P-LIN, linearity in the past (backward linearity): $\forall t, t_1, t_2$: if $t_1 < t, t_2 < t,$ then $t_1 < t_2$ or $t_1 = t_2$ or $t_2 < t_1$, is characterized by:

$$Pp \rightarrow H(Pp \vee p \vee Fp).$$

Some relations are such that they are equal to its converse. If a relation is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive, total, trichotomous, a partial order, total order, strict weak order, total preorder (weak order), or an equivalence relation, its inverse is, too.

The formula $Gp \rightarrow p$ characterizes a reflexive time. The converse of reflexive relation is a reflexive relation. The formula $Hp \rightarrow p$ is inferable from $Gp \rightarrow p$, and vice versa, i.e. in $\mathbf{K_t}$, the system of minimal tense logic, $G\phi \rightarrow \phi$ is mutually provable from $H\phi \rightarrow \phi$:

1.	$\neg\phi \rightarrow GP\neg\phi$	axiom of $\mathbf{K_t}$
2.	$GP\neg\phi \rightarrow P\neg\phi$	assumption
3.	$\neg\phi \rightarrow P\neg\phi$	Syll. (1,2)
4.	$\neg P\neg\phi \rightarrow \neg\neg\phi$	Trans., 3
5.	$H\phi \rightarrow \phi$	by definition of H and double negation.

The proof in the other way is similar.

It has been shown that in $\mathbf{K_t}$ the formulas characterizing transitive time:

$$FFp \rightarrow Fp$$

and

$$PPp \rightarrow Pp$$

are mutually inferable (e.g. [McArthur, 1976, p. 26]). The same is true about formulas characterizing dense time (e.g. [McArthur, 1976, pp. 31–32]):

$$Fp \rightarrow FFp$$

and

$$Pp \rightarrow PPp.$$

The empty relation and its converse are equal. Empty relation is characterized by $G(p \wedge \neg p)$. Also this relation is characterized by $H(p \wedge \neg p)$, the IM of the formula $G(p \wedge \neg p)$. Are these formulas mutually inferable in $\mathbf{K_t}$?

Theorem 12

$\mathbf{K_t} \cup \{G(\phi \wedge \neg\phi\} \not\vdash H(p \wedge \neg p)$.

Proof.

Let now:

- $\mathfrak{M}, t \models H\phi$ iff $\forall t_1, t_1 \leq t : \mathfrak{M}, t_1 \models \phi$;
- $\mathfrak{M}, t \models P\phi$ iff $\exists t_1, t_1 \leq t : \mathfrak{M}, t_1 \models \phi$;

Under this interpretation if the relation $<$ is empty, all the theorems of $\mathbf{K_t} \cup \{G(\phi \wedge \neg\phi\}$ are valid, but $H(p \wedge \neg p)$ is not satisfiable in any model.■

A binary relation $<$ is symmetric if and only if:

$$\text{if } t < t_1, \text{ then } t_1 < t.$$

The symmetry of $<$ is expressible in TL (Brouwer axiom):

$$p \rightarrow GFp.$$

The symmetrical relation is equal to its converse. The $IM(p \rightarrow GFp)$ derivable in $\mathbf{K_t} \cup \phi \rightarrow GF\phi$ [McArthur, 1976, pp. 34–35].

Some classes of relations are not characterized by any formula, e.g. – as it is stated in theorem 8 – the class of irreflexive times.[3] In particular this concerns so called negatively definable classes.[4]

The notion of definability is such that if ϕ characterizes a class of frames, then ϕ is valid in any frame of this class. To distinguish this sort of definability, we call it positive characterization.

[3] For general results about definability see [van Benthem, 2001].

[4] Venema [1993] discusses a 'negative' way of defining frame classes in (multi)modal logic. In a metatheorem on completeness he defined the conditions under which a derivation system is strongly sound and complete with respect to the class of frames determined by its axioms and rules.

Definition 13 (negatively definable class)

A formula ϕ negatively characterizes a class of frames $\mathfrak{F}$ if and only if for any $\langle T, < \rangle \in \mathfrak{F}$ for every t there is a valuation V such that $\langle T, < \rangle, t \models \neg\phi$.

Let $\mathfrak{F}_\phi$ denote the class of frames positively characterized by ϕ and $\mathfrak{F}_{-\phi}$ denote the class of frames negatively definable by ϕ. It is an interesting question about the relations between both two classes.

First of all let us remark that the classes $\mathfrak{F}_\phi$ and $\mathfrak{F}_{-\phi}$ are disjoint.

Theorem 14

For any $\phi : \mathfrak{F}_\phi \cap \mathfrak{F}_{-\phi} = \emptyset$.

We ask if both these classes are complementary, i.e. if for any $\phi : \mathfrak{F}_\phi \cup \mathfrak{F}_{-\phi} = \mathfrak{F}$, where $\mathfrak{F}$ is the class of all frames. The answer is negative: frames characterized by $Gp \rightarrow Fp$ (endless times) are not complementary with a class of frames definable negatively, i.e. the class of all frames such that the relation $<$ is empty. There are frames such that only some, but not all, points do not have a successor.

Theorem 15

Let $\mathfrak{F}_\phi, \mathfrak{F}_{-\phi}$ be both non-empty. For any $\phi : \mathfrak{F}_\phi \cup \mathfrak{F}_{-\phi} \neq \mathfrak{F}$.

Proof.

We have to show that for any ϕ there are frames $\mathfrak{T}$ such that there are moments such that for any valuation ϕ is satisfied and that there are moments such that there are valuations such that ϕ is not satisfied. These conditions are fulfilled by the following frame.

Let $\langle T_1, <_1 \rangle$ be a frame positively definable by ϕ and let $\langle T_2, <_2 \rangle$ be a frame negatively definable by ϕ. The frame is such that:

- $T = T_1 \times \{1\} \cup T_2 \times \{2\}$ and
- $< = <_1^* \cup <_2^*$, where
- $<_1^* \subset T_1 \times \{1\}$ and $(t, 1) <_1^* (t_1, 1)$ iff $t <_1 t_1$ and similarly
- $<_2^* \subset T_2 \times \{2\}$ and $(t, 2) <_1^* (t_1, 2)$ iff $t <_2 t_1$. ■

Is the class $\mathfrak{F}_\phi$ equal to $\mathfrak{F}_{--\phi}$? The formula $Gp \rightarrow Fp$ characterizes endless time. The class of frames negatively definable by this formula is characterized by the formula $G(p \wedge \neg p)$ and the class of frames negatively definable by $G(p \wedge \neg p)$ is characterized by $Gp \rightarrow Fp$.

As we see, some classes negatively definable are characterized by formulas of TL, e.g. class negatively definable by $Gp \rightarrow Fp$, and some classes are not, e.g. class negatively definable by $Gp \rightarrow p$.

3. Grzegorczyk's Axiom and classes of frames definable by it

It is well known that there is a translation of intuitionistic logic into the modal logic **S4** via provability operator [Boolos, 1993]. The fact was suggested by Gödel [Gödel,] and proved by Tarski [McKinsey and Tarski, 1948]. There is translation T [[van Benthem, 2001, p. 385] such that for each formula ϕ of the language of intuitionistic logic **INT**:

$$\mathbf{INT} \vdash \phi \text{ iff } \mathbf{S4} \vdash T(\phi).$$

Andrzej Grzegorczyk [1964; 1967] investigated relational and topological semantics for intuitionistic logic. Grzegorczyk found a modal formula grz:

$$\Box(\Box(\phi \to \Box\phi) \to \phi) \to \phi^{5}$$

that is not provable in $S4$ but the intuitionistic logic is translatable into a normal extension of **S4** by grz, **S4.Grz** [Solovay, 1976; Goré *et al.*, 1997; Goldblatt, 1978].[6] Moreover, it has been established that **S4.Grz** is the greatest normal extension of **S4** for which Gödel's translation of **INT** is still full and faithful [Esakia, 1976; Bezhanishvili, 2009], [van Benthem, 2001, Theorem 82, p. 385]. Strong provability operator "... is true and provable" provides a better model for provability than the operator "... is provable". The logic of the strong provability operator is known to coincide with Grzegorczyk logic **Grz** [Boolos, 1993].

Grzegorczyk's axiom defines the class of Kripke frames that fulfills the following conditions [van Benthem, 2001, p. 385]:

- $\forall x : x < x$ – reflexivity,
- $\forall x, y, z :$ if $x < y, y < z,$ then $x < z$ – transitivity,
- from no t is there an ascending chain $t = t_1 \leq t_2 \leq \ldots$ with $t_i \neq t_{i+1}, i = 1, 2, \ldots$ – well-foundedness.

The last condition implies antisymmetry.

The Hilbert-style axiomatic calculus **K** is composed of the classical propositional calculus, the axiom schemata:

- $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ – Distribution Axiom,

and the inference rules:

- from ϕ and $\phi \to \psi$ infer ψ – modus ponens,
- from ϕ infer $\Box\phi$ – necessitation.

S4 is defined as **K** plus the axiom schemata:

- $\Box\phi \to \Box\Box\phi$.

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[5] $\Box\phi$ can be read "formula ϕ is provable in Peano Arithmetic"

[6] For an overview of results on **Grz** and its extensions see [Maksimova, 2007].

If a logic consists of **K**, $\Box\phi \to \phi$, $\Box\phi \to \Box\Box\phi$, grz, then it is characterized by the class of reflexive, transitive and antisymmetric Kripke frames which do not contain any infinite ascending chains of distinct points. **S4** is valid in frames defined by grz. **S4** laws in $\mathbf{K} \cup grz$ were proved around 1979 by W. J. Blok and E. Pledger [van Benthem, 2001, p. 385].

Grz is characterised by a class of Kripke frames which is not first-order definable, but is decidable.

The fact that **Grz** is complete with respect to the class of upwards well-founded partially-ordered Kripke frames is provable only using some form of the Axiom of Choice [Jeřábek, 2004]. This logic is also complete with respect to Kripke frames such that T is finite and $<$ is a partial order [Segerberg, 1971], [Bezhanishvili and de Jongh, 2005, Theorem 84, p. 41].

We can ask about the class of frames negatively defined by grz or about the class of converse of the Kripke frames complete for **Grz**. We may also consider the question of Grzegorczyk's understanding of necessity in the context of temporal logic.

There are discussed different definitions of temporal modalities. E.g. $\Box\phi$ is defined as:

1. $\phi \wedge G\phi$ – Diodorean,
2. $H\phi \wedge \phi \wedge G\phi$ – Aristotelian
3. $P\phi$ – is based on the conviction that: *Quidquid fuit, necesse est fuisse.*[7]

$\mathbf{K_t}$, the minimal tense logic, is the tense logical counterpart of **K**. In temporal logic G and H are semantical (in Kripke semantics) counterparts of $\Box$. In grz the $\Box$ can be replaced by G and/or by H and grz as axiom can be added to $\mathbf{K_t}$.

System $\mathbf{K_t} \cup \{G(G(\phi \to G\phi) \to \phi) \to \phi\}$ is consistent – it has a model – the model of **S4.Grz**. The same is true in the case of $\mathbf{K_t} \cup \{H(H(\phi \to H\phi) \to \phi) \to \phi\}$. Now, as in the case of the former system, it is complete with respect to the class that instead of well-foundedness is conversely well-founded.

[7] Moreover, in [Anselm, Saint Archbishop of Canterbury, 1929, Book II, chapter XVIII (a)] we read: *Quidquid est, necesse est esse, et necesse est futurum fuisse. Quidquid futurum est, necesse est futurum esse*. In the English edition we read: Whatever has been, necessarily has been. Whatever is, must be. Whatever is to be, of necessity will be. This is that necessity which Aristotle treats of ("*de propositionibus singularibus et futuris*"), and which seems to destroy any alternative and to ascribe a necessity to all things [Anselm, Saint Archbishop of Canterbury, 1998, Book II, chapter XVIII (a)]. See http://www.sacred-texts.com/chr/ans/ans118.htm. If necessity is so conceived, the temporal possibility applies only to the future. According to Thomas Aquinas (Qu. 25, art. 4): *Praeterita autem non fuisse, contradictionem implicat* (For the past not to have been implies a contradiction). There is a Latin saying: *facta infecta fieri non possunt*; that is, what once has happened cannot become not happened.

Is consistent the system $\mathbf{K_t} \cup \{G(G(\phi \to G\phi) \to \phi) \to \phi\} \cup \{H(H(\phi \to H\phi) \to \phi) \to \phi\}$?

If we understand necessity as $P\phi$ we have:

$$P(P(\phi \to P\phi) \to \phi) \to \phi$$

we have to ask if the system $\mathbf{K_t} \cup \{P(P(\phi \to P\phi) \to \phi) \to \phi\}$ is consistent.

The formula:

$$P(P(p \to Pp) \to p) \to p$$

characterizes the class of frames such that $t:$ if $t_1 : t_1 < t,$ then $t_1 = t$. This can be understood as showing that the discussed conception of necessity is too weak from the point of view of Grzegorczyk'a axiom. Are the other conceptions of temporal necessity sufficiently strong to satisfy Grzegorczyk's Axiom in a class of all times?

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Kazimierz Trzęsicki
University in Białystok
kasimir@uwb.edu.pl

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Zygmunt Vetulani

WORDNET AND GÖDEL'S COMPLETENESS THEOREM

Abstract. We claim that Princeton WordNet-like lexical data bases (wordnets) may be considered as a natural conceptualization of the world in the form of a language-derived ontology determined by the linguistic concept of synonymy. We discuss some constraints on synonymy relations which must be satisfied in order to make sure that wordnet will behave as ontology and will reflect linguistic relations. We show a close relationship between the concept of wordnet and Gödel's Completeness Theorem whose proof is based on the fact that every consistent formal theory has a model. In particular, we show that, under some assumptions, wordnets may be generated by Henkin's algorithm of constructing such a model.[1]

1. Introduction

Using a wordnet (which encodes relations between words) to natural language processing depends on how much these relations correspond to relations between the entities in the real world. There is a general agreement to consider, after the Princeton WordNet creators, that the main wordnet organizing relations are hyponymy/hyperonymy and synonymy [Miller *et al.*, 1990]. This opinion has important consequences due to the mathematical properties of these relations: transitivity, asymmetry and irreflexivity for hyponymy and hyperonymy, as well as reflexivity, symmetry, and transitivity for synonymy.

2. Words and word-meaning pairs

Let us notice that satisfaction of these properties depends on some important prerequisites. First, due to the common (for natural languages) phe-

[1] In this paper we will use the term "wordnet" to designate lexical data bases inspired by Princeton WordNet.

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nomenon of polysemy, we assume that the above mathematical properties have to be applied not to words but to disambiguated words called word-meaning pairs in the rest of this paper. In other terms, a word-meaning pair is a word with one precise meaning. Meaning identification is indispensable to understanding text or discourse. Meaning distinction is a common practice in most dictionaries. Some of them also include information about pragmatic aspects such as frequency, register etc. As the ability to recognize the correctness/incorrectness of a sentence (abstraction making of its truth value) is a part of language competence, we will propose to use the following criterion to recognize that the given word W is used with different meanings in the given sentences A and B. This is so when the following hold:

- both sentences A and B are correct,
- there exists such a meaning of W that A is correct with this meaning while B is not, or conversely.

For instance the word "window" in sentences like "He was seen this morning watching through the window" and "You need to open the window in the top left corner of the screen".

3. Synonymy

For the word-sense pairs we define synonymy with the help of the notion of substitutivity. Already Leibnitz used this concept in his definition of synonymy of words (citation after [Vossen, 2002]):

> two expressions are synonymous if the substitution of one for the other never changes the truth value of a sentence in which the substitution is made.

According to [Vossen, 2002], Miller and Fellbaum observed that with such strong definiens, very few words will have synonyms – which is against common intuition about synonymy; they propose to modify Leibnitz's definition to some "linguistic context C" as follows [Miller *et al.*, 1990]:

> two expressions are synonymous in a linguistic context C if the substitution of one for the other in C does not alter the truth value.

Vossen [2002] remarks that [Miller *et al.*, 1990] suggest that it is enough to find one such context to apply the substitutivity criterion for synonymy. As an example of a pair of synonyms in the sense of Miller's definition, Vossen discusses "appearance" and "arrival" (see citation in Section 4, below). These words are presented as synonymous, as in some appropriate contexts C they satisfy the above substitutivity criterion of synonymy. On the other hand, comparing them respectively to "disappearance" and "departure", one observes that some kind of opposition holds between "arrival" and "departure" but doesn't hold between "arrival" and "disappearence" despite that "disappearence" and "departures" are considered both as being antonyms for, respectively "appearence" and "arrival". This means that the appropriate extension of the context can lead to the conclusion that these two words (with unchanged meaning) are no more substitutable in the extended context. (This example is also explored in the discussion of antonymy in Section 5). Applying different contexts to compare different word pairs may therefore result in a situation where synonymy will no more be transitive, which would invalidate the idea of synset as an equivalence class. It seems that a (theoretical) solution to this problem (in application to word-meaning pairs rather than to words) will consist in further generalization of the approach proposed by Miller and Fellbaum, namely in applying the substitutivity requirement with respect to some (possibly large) class of sentences (context class) selected in a way to guarantee synonymy to be reflexive, symmetric, and transitive (i.e. to be an equivalence relation).[2] For a given class Z of sentences we will define "synonymy respective to Z" by restricting substitutivity to Z. It is clear that if Z stands for the totality of sentences about the world, then synonymy with respect to Z will be identical with the synonymy in the sense of Leibnitz (for word sense pairs).[3]

Having the relation of synonymy already defined in such a way that it has the properties of an equivalence relation, we define synsets as equivalence classes with respect to this relation.

4. Wordnet as ontology

Relations holding between word-meaning pairs (disambiguated words) may in a natural way be mapped to synsets if only synonymy is congruent with these relations. By definition, the given equivalence relation is told to be congruent for the relation R if the fact that R holds/does not hold for some elements E means that, respectively, R holds/does not hold for the elements

[2] We consider this solution to be theoretical, as we are aware of the possibly large size of such a theoretical context, so that in practice a limited context will have to be used.

[3] The problems discussed above were (probably) the reasons for Vossen and others [Vossen, 2002] to make another decision for EuroWordNet. Instead of referring to the concept of substitutivity they decided to define synonymy using linguistic tests to compare the extension of words (or word sense pairs).

which are equivalent to E.[4] In a case where synonymy is congruent with respect to hyperonymy and hyponymy we say that the quotient structure (composed of synsets and hyperonymy/hyponymy mapped to synsets) is a *basic wordnet* (or briefly *wordnet*).

The basic wordnet can be considered as an ontology in which concepts (represented by synsets) are directly related to the language. The interest of considering a wordnet as an ontology for NLP applications is that it directly reflects the conceptualization of the world in the same way as does the natural language (more precisely the natural language that the corresponding wordnet is derived from), i.e. it is culturally dependent.[5] This is why wordnets are interesting as ontology candidates for natural language processing applications.

The fact that synonymy may be defined with reference to an external parameter (context class) which may be modified according to the users' needs makes it possible to extend the basic wordnet by introducing other relations mapped from the linguistic relations between word-meaning pairs, such as antonymy, metonymy, and other. In practice, introduction of new relations to the existing wordnets may be difficult because it is necessary to make sure that the synonymy is congruent with the linguistic relations we wish to be mapped to synsets. This may require redefinition of the synonymy (by modification of the context class used in the definition of synonymy) and change the wordnet granulation (refinement).

The case of antonymy is a well described example of a mapping-to-wordnet problem for a linguistic relation. Both G. A. Miller and P. Vossen, designers and developers of wordnets, articulated their doubts about the possibility to express antonymy at the wordnet level. Vossen [2002] wrote the following in the final EuroWordNet report.

> Antonymy relates lexical opposites, such as "to ascend" and "to descend", "good" and "bad" or "justice" and "injustice". It is clear that antonymy is a symmetric relation, but little more can be said, since it seems to encode a large range of phenomena of opposition, e.g. "rich" and "poor" are scalar opposites with many values in between the extremes, "dead" and "alive" can be seen as complementary opposites (...). It is also unclear whether antonymy stands between either word forms or word meanings. For instance, "appearance" and "arrival" are, in the appropriate senses, synonyms; but linguistic

[4] By "mapping to synsets" we mean the act of defining the relation R' operating on the equivalence classes to hold if and only if the relation R holds for some elements of these equivalence classes; the equivalence classes form the so called quotient structure.

[5] The EuroWordNet and other similar projects (as e.g. BalkaNet) are nice and quite successful attempts to integrate various "national", culturally-dependent wordnets.

> intuition says that the appropriate antonyms are different for each word ("disappearance" and "departure"). With respect to this, EWN (EuroWordNet) will assume the solution adopted by Miller's WordNet, that is, antonymy is considered to be a relation between word forms, but not between word meanings – namely synsets. Therefore, in the example above, the antonymy relation will hold between "appearance" and "disappearance", "arrival" and "departure" as word forms. In those cases that antonymy also holds for the other variants of the synset we use a separate NEAR_ANTONYM relation. (...)

It seems however that with an appropriate understanding of antonymy and synonymy there is no need to go so far and to resign from having antonymy defined on synsets. Let us assume that A is a set of pairwise orthogonal binary attributes. By antonymy (restricted to nouns) with respect to A we mean such a relation which holds between two word-meaning pairs if and only if there is exactly one attribute from A for which these word-meaning pairs take opposite values. The sufficient condition for synonymy to be congruent with respect to antonymy is that "antonyms of any two synonymous word-meaning pairs are synonymous to each other". To make this condition true we must further restrict the synonymy relation by considering appropriate sentences related to the attributes A as a part of the context set used to define synonymy. It follows that imposing these new restrictions to the definition of synonymy may cause further fragmentation of synsets.

5. Gödel's completeness theorem and wordnets

Considering wordnets as natural[6] ontologies in which concepts are represented by language entities appears to be compatible with the correspondence between semantic consequence (entailment) and syntactic provability in first-order logic established by Kurt Gödel [1929; 1930]. This correspondence directly follows from the so called completeness theorem which is a simple conclusion from the statement that "each consistent first order theory has a model"[7] (i.e. there exists a world in which the theory is true). The Henkins [1949] proof of this theorem shows how to construct, for a consistent theory in first-order logic, an algebraic structure which is a model for this theory. It appears that, for a given wordnet, a consistent theory T

[6] "Natural" means here "corresponding to the conceptualization shared by language users".

[7] We use the term "model" in the sense it has within the mathematical *model theory.*

may be found for which the Henkins construction will result with a model isomorphic with this wordnet. The Henkins idea of constructing the model consists in adding new constants, and then in constructing (in the extended language) a complete set of sentences[8] consistent with the given theory. In particular among the new individual constants the model construction algorithm selects individual names for entities whose existence is postulated by existential sentences belonging to the constructed complete set of sentences. The idea of the construction of a model of the theory T follows the following lines. First we extend the language L of the theory T to the language L' introducing a new countable infinite set C of individual constants. Then we build an infinite sequence $\langle S_n \rangle_{n=0,1,...}$ of set of sentences of the language L'. Let $P = \langle P_n \rangle_{n=0,1,...}$ be the sequence enumerating all prenex-normal-form formulas of the language L'. We construct successive consistent extensions S_n of T by first putting T as S_0 and then considering one by one the formulas from the sequence of sentences P. At each step we check whether the considered formula belongs to $Cn(S_n)$ or not.[9] If the negation of the formula P_n considered in step n belongs to $Cn(S_n)$ (i.e. is a logical consequence of S_n), then we add it to S_n and obtain S_{n+1}, otherwise we consider two subcases. If the formula P_n is not an existential sentence, then we simply add it to S_n to obtain S_{n+1}. Otherwise, together with this existential sentence we add to S_n the sentence in which the existential quantifier is omitted and the variable bounded by the quantifier in the considered sentence is replaced by a new (i.e. still not used in this extension procedure) constant from C (this sentence is a constructive witness of the existence of an entity with the required property). We denote by S the union $\bigcup_{n \in N} S_n$ of all successive consistent extensions of T. The resulting set of formulas S is consistent and complete, i.e. for each sentence ϕ of the extended language L' either this sentence ϕ or its logical negation $not(\phi)$ belongs to S. Then we define in the set C of individual constants the following relation $\sim$ which turns out to be an equivalence relation. Specifically, we define

$$s_1 \sim s_2 \leftrightarrow_{df} \text{`} s_1 = s_2 \text{'} \in Cn(S)$$

This relation can be extended to the whole Herbrand universe U of the language L' (here the Herbrand universe consists of all terms with no

[8] By a *complete set of sentences* we mean any set S of sentences that for any given sentence φ, either ϕ belongs to S or its negation $not(\phi)$ belongs to S.

[9] $Cn(S_n)$ is the set of all logical consequences of S_n, i.e. the set of all sentences provable from S_n.

free variables). This relation is a congruence with respect to all relations $\Re$ defined in a natural way through S for all R as follows:

$$\Re(t_1, t_2, \ldots, t_k) \leftrightarrow_{df} `\Re(t_1, t_2, \ldots, t_k)' \in Cn(S)$$

The set of equivalence classes over U with induced relations forms the quotient structure. The rest of the proof of Gödel's theorem consists in showing that this quotient structure is a model for S, and therefore for T.[10]

Our further considerations depend on the assumption following Montague's famous conjecture that natural language may (with some limits) be considered as a formal language and, therefore, that Tarski's truth concept is applicable. In particular we will also consider word-meaning pairs instead of words. The validity of considerations will be limited to the fragment of natural language which, in accordance to the ideas of Richard Montague, may be considered as equivalent to the language of first-order predicate logic.[11] With this assumption, the set of true sentences about the real world may be considered as a theory of the natural model (the natural model being the conceptualization of the real world). Clearly, the theory of the natural model is consistent.

The relation of synonymy, introduced above in our paper with reference to the idea of substitutivity, may be formalized in the following way. By an *NL context* (or *context*, for short) we mean a text with a variable X occurring one or more times. By $K(s)$ we will denote the context K with all occurrences of X substituted by (the appropriate form of) s. Let us consider the finite class of contexts $K = \{context_i : i = 1, 2, \ldots, k\}$ with the property of distinguishing all word meanings for any given word. Let us consider the relation $\approx$ in the set of word-meaning pairs of the natural language defined as follows:

$$s_1 \approx s_2 \leftrightarrow_{df} \text{the sentence } `\forall_{i=1,2,\ldots,k}\ (context_i(s_1) \leftrightarrow context_i(s_2))' \\ \textit{is true in the natural model}$$

The relation $\approx$ is an equivalence relation in the set of word-meaning pairs and – according to our terminology – the corresponding equivalence classes are synsets.

Let us observe that the correct use of synonyms by natural language speakers do not entail contradictions (provided that the speakers correctly distinguish the word-meanings). This means that we may assume that

[10] For more details the reader may consult the Grzegorczyk [1974] or Shoenfield [1967] textbook on the foundations of logic.

[11] c.f. the paper by R Montague [1970] on English as formal language.

the word sense pairs satisfy the additional *equality axiom for synonyms* in the form of the following formula:

$$\text{`}\forall_{i=1,2,\ldots,k}(context_i(s_1) \leftrightarrow context_i(s_2))\text{'} \rightarrow s_1 = s_2$$

In other terms, the equality axiom for synonymy is consistent with the theory of the natural model (for the correctly chosen set of contexts K). This last remark allows us to claim that Henkin's model constructed to prove Gödel's conjecture, when applied to natural language, is built out of natural language synsets (the relations $\sim$ and $\approx$ are in fact identical).

6. Conclusions

The above considerations allow us to notice a close connection between two, at first sight very different from each other, scientific ideas. The first one is that wordnets conceived as networks of connections between natural language words appear to be natural ontologies whose concepts are directly linked to language entities (which means that these ontology concepts may be represented in computers in a way that eases their application in natural language processing). We have discussed some essential theoretical problems related to the theoretical foundations of the concept of wordnet, mainly those connected with the nature of the relation of synonymy and we have presented the algebraic structure of the wordnet(s), as well as some fine problems connected with the operation of mapping the linguistic relations to the universe of synsets. The second of the two is the idea of the constructive proof of Gödel's completeness theorem which contributes to a better understanding of the relationship between syntactic consequence (entailment) and semantic consequence. The key element of this proof is a procedure to construct the model for a consistent theory. This model is built out of terms of a (formal) language. We have shown (under some assumptions) that a model constructed according to this procedure for a consistent set of true sentences about the world is equal to a wordnet. This means that the model postulated by Gödel's theorem corresponds to the natural conceptualization of knowledge about the world represented in natural language. Our final message is that one may safely claim that Kurt Gödel's work was an early portent for the idea of a wordnet as a natural ontology whose concepts are directly linked to words.

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Zygmunt Vetulani
Adam Mickiewicz University in Poznań
`vetulani@amu.edu.pl`

Jan Woleński

NATURALISM AND THE GENESIS OF LOGIC*

Abstract. This paper investigates the genesis of logic as a philosophical problem treated from a naturalistic point of view. Logic is defined via the consequence operation *Cn*. This operation is a kind of closure operation similar to that studied in topology. Since logical competence (the skill to use *Cn*) is a manifestation of logic, the main problem can be framed as the question: How the consequence of operation emerged in biological organisms, particularly the human one. Various data from microbiology suggest that organisms have various devices protecting information from its dispersion. One can even say that sequences of DNA have some topological properties. The main thesis is that *Cn* is superstructured on such properties.

It is traditionally accepted that we differentiate between *logica docens* and *logica utens*, that is, between theoretical logic (logic as theory) and applied or practical logic. Both can be defined with the use of the notion of logical consequence. The first is a set of consequences of an empty set, symbolically $LOGT = Cn\emptyset$, provided that the operation Cn satisfies the well-known general Tarski's axioms, i.e. denumerability of the language (a set of sentences) L, $X \subseteq CnX$ (the inclusion axiom; X, Y are sets of sentences of L), if $X \subseteq Y$, then $CnX \subseteq CnY$ (monotonicity of Cn), $CnCnX = Cn$ (idempotence of Cn) and, if $A \in CnX$, then there is a finite set $Y \subseteq X$ such that $A \in CnY$ (Cn is finitary). Cn is a mapping of the type $2^L \to 2^L$, that is, transforming subsets of L into its subsets. In order to make things simpler, I assume that Cn is based on classical logic. $LOGT$ can also be defined as the only common part of the consequences of all sets of sentences.

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Otherwise speaking, logic is the only non-empty intersection of the family of all subsets of L. What follows from this is that logic is included in the consequences of each set of sentences, which underlines its universal character. If $CnX \subseteq X$, then $X = CnX$ due to the inclusion axiom. Moreover, if $CnX \subset X$, we say that that X is closed by the consequence. This is the definition of a deductive system (a deductive theory). Thus, in the case of deductive systems, Cn does not extend X beyond itself. The concept of logical consequence belongs to the syntax of language. The notion of logical following (entailment) is a semantic counterpart of Cn. The properties of both of these notions are such that if $A \in CnX$ and X consists of true sentences, the sentence A also must be true as well. If X is a theory and the set CnX coincides with a set of true sentences (specifically: true in a determined model M or relevant class of models) of X, this theory is semantically complete.

The statement that Cn closes sets of sentences as long as $CnX \subseteq X$, suggests some analogies with topology, since certain properties of this operation satisfy Kuratowski's axioms for topological spaces. Let Cl denote the closure operation of a topological space, and X, Y – any subspaces (subsets); I intentionally use the same letters for denoting sets of sentences and sets investigated by topology. Then [Duda, 1986, p. 115]:

1. $Cl\emptyset = \emptyset$;
2. $X \subseteq ClX$;
3. $ClClX = ClX$;
4. $Cl(X \cup Y) = ClX \cup ClY$.

Operations Cn and Cl differ from each other as far as the matter consists of axioms 1 and 4, because, in the case of logic, set $Cn\emptyset$ is non-empty and $CnX \cup CnY \subseteq CnY$ (but the reverse inclusion does not hold). The first difference is founded on the specific definition of logic, which does not possess a clear topological sense (I will return to this question below), while the other one indicates a partial analogy between closed sets in the topological sense and deductive systems in the logical sense, because 4 does not hold for arbitrary sets of sentences. Thus, "logical" closure is weaker than a topological one. The set of theses of logic is for sure non-empty and it is a system. It can be treated as a specifically closed topological space, with individual theorems as its points.

Topology $\{\emptyset = Cl\emptyset, X\}$ is minimal (see [Wereński, 2007, p. 124]) in the sense that the smallest one cannot be examined. Next, $Cl\emptyset \subseteq ClX$, since for each X, $\emptyset \subseteq X$. Let us agree (this is a convention) that $Cl\emptyset$ is a topological equivalent of logic. Motivation for this convention consists in taking into consideration that a proof of logical theorems does not require any assumption. The evident artificiality of this convention can be essentially weakened by the acknowledgment that closing an empty set produces any theorem of logic. It can be shown that if A and B are theses of logic, then $Cn\{A\} = Cn\{B\}$, which means that any two logical truths are deductively equivalent. Let us assume that X (this time as a set of sentences) is consistent and consists of the set X' of logical tautologies and a set X'' of theorems outside logic. Thus, $X' = Cn\emptyset$ and $X'' \subseteq J \setminus X'$. Sets X' and X'' are disjoint and constitute mutual complements in the set (space) X. Since set X' is closed, its complement, i.e. X'' is an open set. The introduced convention about $Cl\emptyset$ allows one to "topologize" the properties of sets of theses; in particular it makes it possible to treat the set X (of theses) as a clopen set. From the intuitive point of view, the operation of logical consequence encodes inference rules for deriving some sentences from other sentences; that is, a deduction of conclusions from defined sets of premises. Deduction, at the same time, is infallible; that is, it never leads from truth to falsity.

What is applied logic or *logica utens*? When X is any non-empty set of sentences, then applied logic $LOGA(X)$ of this set can be associated with operation Cn applied to X. This is applied logic in a potential sense. This understanding of *logica utens* is, however, decidedly unrealistic, since its user applies only these rules that he needs, independent of whether or not he does so in a conscious way. In other words, real applied logic of a given set is the stock of those logical laws (or rules) that are used in a concrete inferential work. This circumstance makes it impossible to give an abstract definition of real applied logic. It is worth observing that Cn can be based on a non-classic logic, e.g. intuitionistic, many-valued or modal logic. Furthemore, we can neglect the monotonicity condition in order to obtain a non-monotonic logic. These remarks point to the fact that non-classical logics are similarly definable as the classical system. Since applied logic operates on closed-open sets, they, by this assumption, contain extra-logical sentences beside theorems of logic; the inclusion condition decides that logic can be deduced from any set of sentences. This fact has serious methodological importance. If deduction within closed sets 'leads' to accumulation points in the topological sense, adding new extralogical sentences can be executed in an extra-deductive manner. This corresponds to the definition of an open set as such that includes all of its neighborhoods. To put it in a different way, the transition to neighborhoods of sentences as points in spaces in the set X" – that is, extension of this set – can be non-deductive. The above considerations suggest that there was *logica docens* 'at the beginning' and it became *logica utens* through application. According to this image, logic is thus applied like already ready mathematics in a concrete physical theory, e.g. Euclid's geometry in classical mechanics or non-Euclidean geometry in a specific theory of relativity. This circumstance makes a naturalistic interpretation of logic difficult, and even impossible, because, generally speaking, laws of logic are considered abstract to the highest degree and as such are thought to belong to Plato's world of forms.

A contemporary follower of Plato says that naturalism is helpless with respect to the domain of abstracts for two reasons. Firstly, because the naturalistic view acknowledges the existence of temporal-spatial objects as the only ones (there exclusively exist temporal-spatial and changeable objects), while the logical realm exists out of time and space. According to Platonism, this is the main ontological difficulty of naturalism. Secondly, the naturalist also faces an epistemological problem, since as a genetic empiricist with respect to sources of cognition he or she cannot elucidate the genesis of the genuine universal and infallible knowledge represented by logic and mathematics. In particular, the follower of Plato adds that no empirical procedure is able to generate logical theorems as true, independent of empirical circumstances. Platonism is – as a matter of fact – a historical and metaphorical label on the above remarks. From the systematic point of view, it is much better to use transcendentalism (or anti-naturalism) as the opposition against naturalism, since every criticism of naturalism (sooner or later) makes references to transcendental arguments in the sense of Kant. It is in this way that, for example, criticism of psychologism (as a version of naturalism) was executed by Frege and Husserl; one could say the same about Moore's arguments against the reductibility of axiological predicates to non-axiological ones. In general, transcendentalists reproach naturalists with what Moore defined as a naturalistic fallacy on the occasion of his criticism of reduction of moral values to utility. Dualisms of facts and values or logical and extra-logical theorems are not the only ones which naturalism has difficulties with. Other oppositions, from which – in the transcendentalists' opinion – naturalists are cut off in the sense of the impossibility of their satisfactory explaining are, for example, the following: physical information – semantic information or quantity – quality.

It, of course, is fairly true that naturalism must meet various difficulties. Criticism of this view, however, overlooks problems of anti-naturalism, which Moore had already drawn attention to. The arguments he used were that super-naturalistic (Moore used this qualification) grounding of morality as rooted in the supernatural world is a similar error to that of reduction of axiological predications to ones definable in purely natural categories. Another problem of transcendentalism arises in connection with the so called Benacerraf argument indicating the enigmatic character of cognition of mathematical objects provided that each cognition consists in causal interaction on the part of the object of epistemic acts, while numbers – on the power of their nature according to Platonism – do not interact in a causative way on people. How, then, can an anti-naturalist explain the genesis of logic? He or she can either assume – as Plato did – that the world of abstract forms is eternal, or argue – like Descartes – that certain ideas are inborn, or still – like some theists – that man obtained logic as a gift from God when he was created as *imago Dei*. The Platonic and Cartesian paths are *ad hoc*, whereas that of the theists is based on extra-scientific premises. In any case, the situation of an anti-naturalist is not to be envied as it must resort to secret beings (souls, spirits, ideas) and secret kinds of cognition (intellectual intuition, etc.). The naturalist can paraphrase the title of Hoimar von Ditfurth's book *Der Geist fiel nicht vom Himmel* (*The Ghost Has Not Fallen From Heaven*) by saying that logic has not fallen from the other world, Platonic or other (see [Ritchie, 2008] for a general discussion about naturalism; as regards defense of naturalism in other contexts compare [Woleński, 2006; 2010a; 2010b; 2011]; see [Papineau, 1993] for a defense of philosophical naturalism).

For a positive naturalist's account of the genesis of logic it is indispensable to combine the *dychotomy logica docens – logica utens* with the notion of logical competence, modeled on grammatical competence in Chomsky's sense. Both abilities play a similar role. The grammatical competence generates the right usage of linguistic devices, while the logical competence determines the application of logical rules in inferential processes. Nevertheless, the analogy is not complete, at least according to my own concept of the question. Much as Chomsky defines grammatical competence simply as grammar, the distinction which I am going to use differentiates logic, both theoretical and applied, from logical competence. The last category refers to a determined disposition of the biological organisms which are able to perform mental functions (compare further comments below). Speaking more precisely, logical competence is the ability to use operation Cn. A logical theory is not thus logical competence, but its articulation. The dispositional character of logical competence does not settle whether each element of logic as a theory finds its coverage in its natural generator. By the way, a negative answer is rather obvious as the development of logical theories was and is heavily dependent on nature and the need for communicative interactions within human society. Further considerations in this paper will be devoted to the genesis of logical competence. They refer to the genesis of logic inasmuch as without the possibility to create and apply rules of logic, there would not appear logic in either of the two distinguished senses. In other words, *logica docens* and *logica utens* are derivatives (precisely speaking – one derivative) of logical competence. This circumstance justifies the title "Naturalism and the genesis of logic". Anyway, one of the main theses of this paper says that logical competence is not eternal; it appeared in the Cosmos at some time and is rooted in the biological structure of organisms. However, I have to make it clear at once that I do not treat my comments relating to the biological question as empirical. My cognitive interest is of a philosophical nature and remains within evolutionary epistemology. Yet, however, in compliance with my metaphilosophical convictions, I have to take into consideration the output of empirical sciences, biology, in particular, in the analysis of philosophical problems. Speaking otherwise, philosophical analysis, though somehow speculative in its character, is superstructured on empirical knowledge.

In accordance with the above explanations, logical competence precedes logic, both theoretical and applied; nevertheless, there is also a feedback because theoretical reflection on logic and its applications to concrete questions can enhance the logical competence. Everything points to the fact that logical theory required prior application of rules of logic and the development of language. In the case of Mediterranean culture, applied logic appeared, for sure, with Greek mathematicians and philosophers. When Anaximander said that there does not exist the principle of closeness, since it would create a boundary of *apeiron* which is boundless, he made use of a rule similar to *regressum ad absurdum*. Pythagoras proved the existence of irrational numbers and his reasoning was a proof by reduction in the modern sense. Various paradoxes formulated by the Eleats were of a similar character. The first logical theory, that is Aristotle's syllogistics, originated much later, although on the basis of extensive practical material accumulated earlier. It was also, in a vital way, linked to the structure of sentences of the Greek language. One cannot, however, say that carrying out logical operations requires knowledge of language because they are typical of infants (see [Langer, 1980]), and the latter do not have linguistic material at their disposal yet.

The question of the sense of understanding in animals, other than humans, is controversial, yet the following example (which can be treated as an anecdote) is only too suitable in this place (see [Aberdein, 2008]). In 1615, in Cambridge, there was held a debate devoted to dog logic, which was attended by King James I. The problem concerned the question whether hunting dogs which were used for locating game during hunting, applied logic, in particular the so-called law of disjunctive syllogism in the form "A or B, thus if *non*-A, then B" (this question had already been considered by Chrysippus). Let us suppose that a hound reaches a fork in the road. The trailed game runs away to the right or to the left. The hound establishes that it is not to the left, and therefore runs to the right. The debate had a very serious character and a truly academic form. John Preston (a lecturer of Queens' College) defended the thesis that dogs apply logic, whereas his opponent, namely Matthew Wren (of Pembroke College) argued that hounds are directed solely by scent and it is the only reason why they choose the right direction. The role of moderator was played by Simon Reade (of Christ's College). When the latter acknowledged Wren to be right, the King, himself a great enthusiast of hunting and relying on his own hunter's experience observed that the opponent, however, should have a better opinion of dogs and lower of himself. Wren, very skillfully managed to get out of the tight situation by saying that the King's hounds – in contrast to others – were exceptional, since they hunted upon the ruler's order. This compromising solution is said to have satisfied everybody. After all, even if hunting dogs do apply disjunctive syllogism occasionally, they certainly do not do this making use of a language.

The debate held in the presence of the King of England is a good illustration of a certain difficulty as regards the analysis of the genesis of logic. There appears the question of what evidence could help here. The debaters in Cambridge considered dogs' behavior and drew conclusions from that. In the case of humans we can observe signs of inferential processes in people or base ourselves on the written evidence of the past. Anyway, the empirical base is greatly limited. Not much can be inferred from the inscriptions found on walls of caves inhabited by our distant predecessors. All the information through which human logical competence manifested itself has been recorded in a language developed to such a degree that it made it possible to encode the deductions carried out factually, even if it did not suffice to formulate a logical theory. In this respect, the genesis of logic appears to be more mysterious than the appearance of mathematics (see [Dehaene, 1997]) or language (see [Botha, 2003; Johansson, 2005; Larson *et al.*, 2010; Talerman and Gibson, 2012]). In both mentioned domains, especially in the latter one, there have appeared a host of works. In particular, the question relating to whether animals can count and make use of a language, at least of a protolanguage (see [Hauser, 1998; Bradbury and Vehrencamp, 1998]).

Studies in the origins of logic are limited to research into the logical competence of children going through their pre-language period, or that of people living in primitive societies. This provides solely epigenetic and ontogenetic material, whereas phylogenetic only to the extent in which the traditional and strongly speculative Haeckel's assumption that ontogenesis reproduces phylogeny is accepted. Nevertheless, considerations concerning the origins of the genesis of calculating competence and language competence are important also to the discussion on the origins of logic. This concerns, in particular, the concept of the origin and development of grammar and sign systems (see [Heine and Kuteva, 2007; Hurford, 2012]). According to a fairly common conviction, signs were the earliest to appear, especially expressive ones, then iconic signs, followed by symbols. This corresponded to the evolution of grammatical structures from nominal through sentential-extensional to sentential-intensional. Thus, the development of language progressed on the basis of transition from a-semantic or little-semantic objects to fully-semantic (intensionality symbolism). The origin of language has always been an object of animated interest on the part of philosophers (compare [Stam, 1976] for a review of earlier theories). In 1866, the French Linguistic Society decided that considerations on this subject should be excluded from the sciences. As a matter of fact a renaissance of studies of the appearance and evolution of language was observed beginning with the middle of the 20$^{\text{th}}$ century. One can speculate that works which treat of the genesis of logic, if they had been written on a mass scale in the first half of the 19$^{\text{th}}$ century, would have shared the fate of linguistic dissertations on the origin of language as too speculative.

It is not without significance to model microbiological and neurological processes, for instance through cell automata (see [Ilachinski, 2011]) or even with the help of advanced mathematical techniques (see [Bates and Maxwell, 2005]) and computational ones (see [Lamm and Unger, 2011]). These enterprises indicate that the organisms themselves and whatever is happening inside them possess properties which can be formulated mathematically. However, far-fetched methodological carefulness is indispensable. The title of one of the quoted books runs as follows *DNA Topology*. It can be understood in a dual way: firstly, it suggests that, for instance DNA in certain circumstances has a looped structure; secondly, this can be understood in a weaker manner, i.e. in such a way that the topological notion of a loop models certain properties of DNA. Reading the book by Bates and Maxwell inspires to conclude that the authors make use of both meanings. My opinion on this problem consists in recommending another sense of modeling. It is assumed here only (or as much as that) that the world is mathematizable (that is describable mathematically) due to its certain properties, yet is not mathematical. Works in the field of the evolution of language and those devoted to modeling biological phenomena, as a rule, accept naturalism, silently or explicitly. An expression of that is the appearance of biosemiotics (see [Barbieri, 2011; Bar, 2008; Hoffmeyer, 2008]), cognitive biology (see [Auletta, 2011]; this author declares the theistic *Weltanschauung*, yet suspends it in his book) or the ever more popular physicalization of biology (see [Luisi, 2006; Nelson, 2008]). Since syntheses of biology and semiotics or biology and cognitive science can be conducted, there is no reason why we should not link logic to biology.

The only advanced attempt at naturalistic grounding of logic that I am familiar with derives from William Cooper (see [Cooper, 2001]), who considers the following sequence: ($\star$) mathematics, deductive logic, inductive logic, theory of decision, history of life strategies, evolution theory. The relations between elements ($\star$) are such that from evolution theory to mathematics we deal with implication, whereas reduction proceeds in the opposite direction. As far as deductive logic is concerned, it is directly implied by inductive logic and reduces itself to the latter. The evolution theory is the ultimate basis, both for implication and reduction. Briefly speaking, deductive logic arose at a certain stage of evolution (Cooper does not make it precise in detail) through natural selection and adaptive processes. Cooper's schema leaves a lot to be desired. Omitting the lack of a more detailed definition of 'production' of logic through the process of evolution, which was indicated earlier, the notions of implication and reduction are not clear in Cooper's model. Since deductive logic (that is $Cn\emptyset$) is implied by any set of sentences, the role of inductive logic (I neglect here disputes relating to its existence; however, see below) is not specific. In consequence, reduction of deductive logic to inductive logic appears to be highly unclear. Moreover, the phrase 'logic as part of biology' (the subtitle of Cooper's monograph) is ambiguous. It may mean that *logica docent* is a part of biological theory (more precisely: evolution theory) or, also, that deductive competence (Cooper does not use this name) is an element of the biological equipment of a human being. Indeed, in compliance with the well-known maxim of Theodosius Dobzhansky, nothing has sense in biology if it is not considered in the context of evolution, but this does not mean that everything can be explained on the basis of evolution theory. Cooper, in his analysis, ignores genetics completely which is, perhaps, the most serious deficiency of the model.

A purely evolutionistic classical approach towards the establishment and development of human mental competences such as the ability to use a language or reasoning is – in outline – as follows (it can be found in countless publications dealing with the theory of evolution and its application to different specific problem areas; compare for instance [Lieberman, 2005; Tomasello, 2010]). The Universe appeared about 15 billion years ago (all the dates here are given in approximation). The age of our Earth is 4.5 billion years. The first cell appeared a billion years later, and multicellular organisms after the next 2.5 billion years. Plants have been around for 500 million years, reptiles – for 340 million years, birds – 150 million, and apes – for 7 million years. The species of *homo* appeared two million years ago, *homo erectus* – from 1 million to 700 thousand, and *homo sapiens* – 200 thousand years ago. The cultural-civilization evolution marked by language (in the understanding of our modern times), the alphabet, and writing began 8,500 years ago. Three and a half billion years from the moment of the appearance of the first cell to that of the appearance of civilization and culture was completely sufficient to form a mind capable of performing typical intellectual activities, in particular, to carry out a logical operation. *Homo sapiens* must have been able to do so much earlier; maybe it happened at the beginning of this species. It cannot be ruled out that the rudiments of logical competence had already been available to *homo erectus*. Establishing the date of the appearance of logical competence in the course of evolution, and also attributing it to other organisms than human seems here not particularly important. A safe evolutionist hypothesis in this respect can claim, for instance, that inferential ability appeared by way of randomly acting mutation, and because it proved to be an effective adaptive tool, it was developed by *homo sapiens*, also thanks to available and more and more perfect linguistic instruments. The logical theory appeared as the final product of a long evolution process. This is an adaptation of the classical concept of the evolution of language (compare, however, the conclusions at the end of the paper).

Neo-Darwinian evolutionism connects the appearance of life and its further evolution with entropic phenomena (see [Brooks and Wiley, 1986; Küppers, 1990]). This perspective leads to the need for indicating anti-entropic phenomena, i.e. mechanisms which maintain the stability of organisms and their internal order, and thereby determine the continuance of their existence (see [Kauffman, 1993]). The decisive event to enhance a serious revision of evolution theory was the discovery of DNA structure by Crick and Watson in 1953 (the model of the double helix), as well as further research into genetic encoding. Those results demonstrated the necessity of more profound linking of evolution with genetics. The notion of genetic information and the manner of its transfer became key instruments of a new biological synthesis, obviously, while keeping suitably modified classical categories of evolution theory. Notice that formal analogies between information and entropy have caused biologists and philosophers of biology to become interested more closely in relations between the first notion and the course of biological processes since as early as the 1920s (see [Wereński, 2005]).

Several facts established by molecular biology are significant from the point of view of this paper (for a while I mention them without a 'metalogical' commentary; I entirely omit the physical-chemical questions, likewise the mechanism of hereditariness). Firstly, passing genetic information is directed from DNA through RNA (more precisely: mRNA – the letter 'm' denotes that RNA is in this case a messenger, that is an agent passing information) to proteins. This observation makes the so-called main dogma of molecular biology. There are, as a matter of fact, certain exceptions in this respect (e.g. in the case of viruses), but at least in the so-called eukaryotic organisms (humans belong to this biological group) transmission of information is in compliance with this dogma. Secondly, genetic information is passed in ordered, linear and discrete, and sequential a manner. Thirdly, DNA particles are subject to replication (copying) and recombination (regrouping). Fourthly, the intracellular information system encodes and processes information, which causes the encoding in question to be interpreted as a computational system and to be modeled accordingly. Fifthly, the passing of genetic information is not deterministic but random in its very nature, thanks to which there may appear genetic novelties. This last fact is vital from the point of view of evolution theory because it explains the way in which mutation appears on the microbiological level.

The view that genetic information is of a linguistic character is only too tempting. Indeed, it is very often that we can see it treated as a language. And thus, we can speak about alphabets, words, syntax, codes and encoding, or about translation (in the sense of transfer from the genetic language into another one); this is done especially by representatives of biosemantics, who – in the genetic information – detect a semantic dimension or, at least, its germs. Such an approach is, however, very debatable (compare the discussion in [Kay, 2000; Sarkar, 1996] which rejects the notion of genetic code, but this solution seems too radical). In true fact, technical elaborations of genetics avoid comparing the genetic code with a language (see, for example, [Klug *et al.*, 2006]). Independent of the applied language, for instance, some authors write about 'words' as components of the genetic code, surely using quotation marks to indicate a certain metaphorical investing of genetic information with the linguistic dimension, while others do so about words; we can easily find here the problem of relation of physical information as something quantitative to the semantic information as qualitative. The mathematical theory of information concerns the former, and only indirectly relates to the latter. Shannon's well-known statement on the capacity of channels of transmitting information and limiting so-called information noise has sense only with reference to its quantitative understanding. The genetic information is a kind of physical, not semantic information. On the other hand, processing the former, i.e. quantitative, into the other, i.e. qualitative, is a notorious fact; for example, reading a book – as long as we understand the language in which it has been written, we rapidly process the physical stimulus into semantic units, i.e. such that we understand them according to their linguistic sense. For the time being we do not know the mechanism of this transformation and it makes the biggest anthropological puzzle (see [Hurford, 2007]). Perhaps, the properties of the genetic information lie at the foundations of, so to say, the semiotization of mental processes, yet this is a fairly speculative assumption from the biological assumptions, though one could consider it as philosophically justified to some degree.

At first sight, if the genetic information were a language in the full sense or even if only in an approximate one, we could look for the genesis of logical competence directly on the microbiological level. After all, the properties of the genetic code, whatever it is, stand far from those that can serve to define operation Cn. Nevertheless, these properties can be tied to logic in their understanding of today. Before I pass on to essay to show this relation, I will draw attention to certain theoretical questions. Kazimierz Ajdukiewicz (see [Ajdukiewicz, 1955]) divided inferences into deductive, increasing probability (inductive in a broad sense), and logically worthless. The first are based on operation Cn which holds between the premises and the conclusion (the conclusion results logically from the accepted assumptions), the second ones increase the probability of the conclusion on the basis of the premises, and the third ones are devoid of any logical relation between the links, e.g. "if Krakow lies on the Vistula, then Paris is situated in France". Logic, in this context is understood in a broader way than at the beginning of the paper, since it includes also induction rules. We can, too, extend respectively the notion of logical competence, still I do not wish to consider such a generalization. Treating the thing from the information point of view (see above), while deduction does not broaden the information included in the premises (although it does not allow it to be lost), the conclusion is false, which disperses (in the sense of entropy) the information acquired earlier; whereas logical inference that is worthless is redundant from the information point of view.

The infallibility of the rules generated by operation Cn derives from the fact that they correspond to theorems of logic, i.e. to sentences (formulas) that are true in all circumstances. One of the axioms of probability calculus is the assumption that there exists an event whose probability obtains the value 1 (the whole space on which the probability measure is defined constitutes this event). An interesting interpretation of this axiom consists in acknowledging that it prevents the leveling (dispersing) of probabilities ascribed to particular occurrences; that is, subsets of the whole space. In other words, this axiom saves the differences in the amount of information which condition its flow. Thus, it performs the anti-entropic function, i.e. blocks dispersion of information: it protects it in this way. Operation Cn can be understood also as an instrument for protecting information from its dispersion, since it prevents formation of false information on the basis of true information. As I have already mentioned, the logic of induction is debatable, yet – on the other hand – nobody contradicts the fact that at least certain induction rules, e.g. those of statistical induction, are rational. It is true that they do not exclude dispersion of information, but still are able to somehow normalize its flow and in this way save or control it. Inferences that are logically worthless do not play any role in the processes of information protection.

Saving information (obviously it is not problems of a legal or moral nature that I mean here) both physical and semantic, appears as a vital function of all organisms which operate with a given type of code. Since we treat operation Cn as an information-protective instrument, saving the possessed content (in the sense of information content, not necessarily meaningful in the sense of intensional semantics), then – at least from the naturalistic point of view – the logical consequence has a biological rooting. With relation to this, I will return to the properties of the genetic codes and genetic information mentioned earlier, this time in the metalogical context. Here are the features of the genetic code (see [Klug *et al.*, 2006, p. 307]; I keep abstracting from the nature of elements of the code with one exception only amino acids due to the comprehensiveness of certain formulations):

1. genetic code is written in a linear form;
2. if we assume that mRNA consists of ‘words’, then each such word has three ‘letters’;
3. each three-letter group, that is, the codon, determines another element in the form of an amino acid;
4. if the code is unambiguous, it delineates one and only one amino acid;
5. if the code is degenerated, the given amino acid can be determined by more codons;
6. the genetic code includes the initial signal and the terminal one in the form of codons initiating and finalizing the processes of passing the genetic information;

7. the genetic code does not contain punctuation characters (commas);
8. elements of the genetic code do not overlap, that is, a concrete 'letter' can be a part of only one codon;
9. the genetic code is nearly universal, i.e. apart from a few exceptions, the same 'dictionary' of encoding serves all viruses, procariotic, and eucariotic organisms.

Completing the remarks offered earlier, I would like to add (see [Klug *et al.*, 2006, pp. 264–265]) that replication of DNA (forming two new strings of the primary helix) can be semi-conservative (each replicated particle of DNA has one old string and one new one), conservative (the parent string is conserved as a result of synthesis in two new strings), or dispersed (old strings are dispersed in new ones). The most frequent is the case of semi-conservation. Nevertheless, the genetic information that exists earlier is inherited by helixes formed by way of replication.

It follows from properties 1–9 that the 'syntax' of the genetic code is rigorous. It is based on a detailed specification of simple elements ('letters') and their combinations (codons). The lack of commas points to the fact that it is a series of concatenations. A code is unambiguous inasmuch as it is not degenerated. This property can be likened to syntactic correctness, while degeneration to a lack of it. 'Letters' are atoms in the same sense as a simple expression, which is non-decomposable any further. Transformation of a codon into an amino acid is a function, unless the code is degenerate. The beginning and the end of the procedure realized by the code is clearly marked with separate 'words'. I have marked some expressions with letters so as not to suggest treating the code as a language. The linguistics-oriented terminology could easily be avoided through speaking about configurations and their elements. Genetic codes treated in this manner can be and are similar to electric nets or cellural automata, which – as a matter of fact – is underlined by the above-mentioned modeling of genetic phenomena. The essence of things relies on the idea that the outline 'syntax' is of an effective character and is trivially recursive, since operations realized by the codes are of a terminal character.

Although there hold similarities between the structure of the genetic code and the syntax of formal languages, there is no reason why we should see the genetic concatenation as a result of the action of operation Cn. On the other hand, if we were to consider the configuration determined by codons, it is clopen in the topological sense, which is also characteristic of the space of sentences on which the logical consequence operates together with non-deductive rules of organizing semantic information. The semi-conservative character of the most typical replication of DNA corresponds to this. Closing of a part of this space protects the information accumulated earlier, and the fact that it includes also open subsets secures the appearance of new information. Sometimes this is said (see [Kauffman, 1993, p. 2003, pp. 447–449]) about channeling of processes of genetic regulation through extensional (Boolean) functions. Let x be an active element in such a process, and object (x or y) a regulated element. Then, the object (x or y) is also active. The procedure, in this case, is analogous to that applied in the synthesis of electric networks. In the terminology used in this paper, the channeling (in the sense of Kauffman) is a partial objective case of logical consequence. A general conclusion which can be derived from the registered analogies is as follows: the genetic code is the biological foundation of logical competence. Since speculation becomes a philosopher, the thing can be framed as follows: topological or proto-topological properties of the 'genetic space' directed the biological evolution in such a direction that it developed – perhaps by way of relevant mutations, with the appearance of dispositions to operate with logical consequence.

It is not a feasible thing to establish various vital details. It is not known when logical competence appeared in its fullest beauty, so to speak, and what its scope is with reference to other species than ours. Putting it differently, it is not known whether the logical ability is only granted to humans, or – maybe – is also available to other species, or even to what extent real human logical competence corresponds to its abstract image formulated in logical theory. Perhaps evolution theory could add something relevant in this matter. It seems, for instance, that species which invest in a lower number of offspring (birds and mammals) have been 'forced' to create stronger means of protecting genetic information than insects, reptiles and fish, in which dispersion is compensated with a great number of potential specimens. Be it as it were, a naturalist claims that, to repeat once again, logical competence has not fallen from another world but came into being on the planet Earth. This makes, similarly to other mental operations, a realization of dispositions determined by the genetic equipment and the course of the evolution. If it is inborn, then it is phylogenically, not ontogenically. Independently of how the proposed approach is general, it somehow contributes to (in order to use Andrzej Grzegorczyk's phrase; see [Grzegorczyk, 1997]) understanding logic as a human affair. If, as Grzegorczyk claims, logic displays human rationality, both, logic and rationality, are deeply rooted in our biological equipment.

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Jan Woleński
Professor (Emeritus) of Philosophy
`wolenski@uj.edu.pl`

Urszula Wybraniec-Skardowska

ON LIFE ACCORDING TO THE LOGIC OF GIFT, TOIL, AND CHALLENGES

Abstract. The present essay deals with certain questions in the field of humanistic philosophy, ethics and axiology, discussed in the light of still newer and newer challenges of our changing times. It highlights the significant role of Professor Andrzej Grzegorczyk in solving and overcoming problems encountered in human life, which is based on his natural logic and incessant efforts aimed at preservation of fundamental moral values, as well as at shaping the principles of individual and social life. The views held by Andrzej Grzegorczyk, which are outlined in the work, form a certain rationalistic vision of the world and mankind.

> Life is a trial, an examination and a judgement.
>
> A. Grzegorczyk: *The Philosophy of the Times of Trial*

Speaking about life, we mean here human existence: man's life and activity in changing times and social, political, and cultural conditionings. Each human life is a peculiar gift, a gift of nature, or – to people who believe in God – a gift obtained from God. It is unrepeatable, a fundamental value.[1] A life, thus, is something of great value, sincerity and uniqueness, and as such has for centuries now made a subject of philosophical enquiry into the following: How to perceive this value of life? How to realize it? How to live so as the life should be invested with a sense that marks out a value in itself? What individual or general aims delineate the sense of life? They are questions which pertain to philosophy and which reach the very roots of humanity themselves.

These questions are not only an object of interest on the part of broadly understood philosophical anthropology (theoretical and practical), as well as the philosophy of human being, axiology, ethics, and religion. After all, they touch each of us, especially when we ponder the sense of all our deeds or actions to date. And in each human action and endeavor there are intellectual-

[1] This is declared by *The Constitution of the Republic of Poland*, art. 38.

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cognitive elements and those desire-based ones, connected with our will, which are coupled, so to say, and penetrate and complement each other. In each of us, there functions both a *logic* that stems from our mind and a *desire* that is not always rational. This is the essence of human nature and its material and spiritual needs. An individual or collective life which is devoid of contact with the leading logic of reason, cut off from reality, from the truth about the world, forfeits its essential sense. A practical life, be it individual or collective, private or public, directed solely towards pragmatism, inclined towards consumerism and benefit, the 'logic' of profit, not being able to find proper cooperation with intellectual life, forfeits indeed whatever makes the nature of man as a reasonable creature that can:

- think logically,
- act rightly in compliance with universal ethical principles, as well as
- look for the truth.

Should we think that the highest values of life are the vital forces in it, the vital instinct, and the want of benefit, instead of a reasonable will?

All people long for unrestricted development of their natural potential, and are willing to delight in beauty, long for happiness. While some have the knowledge of where and how to look for these values, others are not able to direct their lives to achieve them. Raising awareness in people so that they could possess these values is connected with realization of what ancient Greeks called *logos*. Socrates, the founder of dialectics, understood it as the art of sober discussion or verbal argumentation, saw the sense of his life in comprehending and materializing the *logos*. Plato and Aristotle also referred to *logos*, the latter, the founder of formal logic – as did Marcus Aurelius, a stoic, in his famous *Meditations*. The last regarded life as a string of duties, and their fulfillment as the road to happiness. "You can always live a happy life if you follow the right road and want to think and do well," he writes in his *Meditations* [1984, Księga V 34]. Such a life is connected with toil, yet also is one lived in compliance with nature – as Marcus Aurelius [1984, Księga I 17] states, accepting this fundamental principle which constitutes the supreme good for stoics. *Meditations* is a praise of mind, the *logos* which penetrates the whole Universe; it is a course-book on the art of living. And even though there is no mention in it of logic, whose beginnings – in today's understanding of it – are sought for just in stoics, the considerations included in it display the beauty of Aurelius' personal natural logic. We will soon make reference to the notion of the *natural logic* of man. The Greek word *logos*, from which – etymologically – the contemporary word *logic* derives, has a number of meanings. For our purpose, however, we shall adapt its meanings by which it denotes *mind* that directs the will of man, *thought* that is a creation of the mind, as well as *word* that is a representation of thought so that the latter could be transferred in communication-cognition processes; the word also means *science*. Without entering into a discussion concerning the history of logic as a domain of science, nor reminding of what scientific logic deals with, let us only draw attention to the fact that it did grow out of the *natural logic* of man.[2] Natural logic is unquestionably a feature of human beings. It is also called inborn or innate logic. It constitutes the basis of human life logic. It can also be called the *gift logic* of nature or the Creator, since it is a natural disposition of the mind of each human being, which is linked to the nature of *homo sapiens* – the capability of correct and reliable thinking and reasoning without knowledge of the laws that govern this correctness, the ability to utter true sentences without realizing the fact that proving their truthfulness is possible just thanks to this logic. Does natural logic satisfy the needs of our everyday lives and actions? Or is the knowledge of logic as science indispensable? In a regular life, common, instinctive logical effectiveness – natural logic – does suffice very often. There are common people who are not familiar with scientific logic, yet whose reasoning is correct, sometimes even surpassing that of those who were formally learning logic. There are people whose innate logic is developed in a peculiar way, for whom this inborn ability of their mind is a special gift. We shall call it here *gift logic*. It is typical of geniuses or those displaying unique intellectual powers. Their lives are often marked with *toil* and drudgery, intellectual effort, related to satisfying spiritual, non-material needs. There are, among them, individuals whose creative work is dedicated to a special service done to others and society as a whole, for whom the sense of life means rising above their regular duties and problems, recognizing *challenges*, following the challenges of life and of the changing times in which they happen to live. Their logics, talents, toils and challenges are marked with their belief in the mind which makes sense of their lives and, moreover, are what I would like to call the *logic of gift, toil and challenges*. Life, according to this logic, is determined by the popularly accepted universal values: *Truth, Good*, and *Beauty*. They are what builds up life, investing it somehow with a shape, setting goals, assigning new tasks and challenges in the transforming world, contemporarily dominated by money and socioeconomic as well as religious conflicts. Their lives become then a gift of themselves. They pay off the debt of the gift of living. The first logicians made references to natural logic in order to codify the knowledge about what was merely instinctive or not made conscious; they

[2] See: [Kowalewski, 1959, pp. 20 and 21].

endeavoured to find general rules governing the mind in the form of scientific laws. Scientific logic arose on the basis of the so-called fundamental laws of thinking: *the law of contradiction, the law of identity, the law of excluded middle*; until the 19$^{\text{th}}$ century it had been regarded as part of cognitive psychology, as a science establishing its laws as inductive generalization of laws governing the mind. Abandoning of the trend – called psychologism – followed in the 20$^{\text{th}}$ century. The *anti-psychologism* of the 20$^{\text{th}}$ century assumed that the laws of logic are objective as they say not about how the mind works by reasoning, but how it should work properly. These two currents, in our century, seem not so much to compete with as complement each other. Scientific logic as a theoretical science cannot be torn away from life. Throughout all the years of its existence and development it has always been considered to be *a tool* (Greek: *organon*; Aristotle's logical works were collected and entitled *Organon*). Logic was, then, and still is of an *instrumental* nature, utility oriented in relation to other domains of science and the needs of living and man. In this sense, logic is a universal science of a service rendering character, which, on the one hand, provides the basis of each science, while on the other it proves useful in each walk of life and reality. It serves, especially, the needs of linguistics, and computer science – Artificial Intelligence and mathematics (deduction systems). It is of value to everyone who wishes to correctly, clearly, and precisely express thoughts, think and reason in a correct manner, seeing that through deepening both *natural logic* and the acquired one in practice it makes it possible to avoid making mistakes in various situations of life, including both logical language errors and those of reasoning. It allows improving innate logical abilities:

- perfecting the usage of language for communication-cognition purposes,
- setting appropriate forms of reasoning against erroneous ones,
- working out skills of independent, correct, and at the same time critical thinking, and justifying theorems.

Thus, logic, in the broadest aspect, is to serve human beings. Practiced professionally, in the spirit of services rendered to man, it fulfills a glorious role – it testifies to the appreciation of the dignity of human beings and their intellect which needs cultivating and 'rearing'. A crisis of mental culture is undoubtedly related to a crisis of logical culture of society. The life logic of man and of the community the former lives in, can be shaped only by perfecting logical skills, both innate and the ones acquired through life experience. Masters of logic are indeed only those logicians and thinkers who – on the basis of their *logic of gift, toil, and challenges* – have not cut off, as scholars their contacts with reality and people, making their activity a peculiar service to man and society.

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Not many thinkers have managed to achieve the goals which they set earlier already in their lifetimes, only a few have been acknowledged to be celebrities already during their lifetimes, owing to their published works and personal activity, as well as their force of exerting an influence on society. Consequently, those who have managed to gain recognition not only because of their output, but due to the attitude assumed in their lives and a peculiar kind of service rendered to people and society, ought to be valued in a special way. The unceasing, multi-directional writer's activity of Andrzej Grzegorczyk,[3] who writes in continuation of the rationalistic traditions of the Lvov-Warsaw School, constantly evolving, and addressed to all people capable of logical thinking, an activity in which not only vital scientific problems of mathematical logic, philosophy, and ethics, but also important views of a typical life-related nature have been raised in a clear and transparent way, the views making a unique message and moral duty towards society as they are marked out by the independence of expressing one's own opinions in the face of changing totalitarian systems and a variety of political trends, offers a challenge to all those for whom life makes a truly unrepeatable value and provides a sense of fulfillment at the same time. The answers to the questions: How to understand this challenge of life? How to realize its values in changing times, full of conflicts, threats, and in the light of various socio-political conditions? can be found in many of Andrzej Grzegorczyk's publications in the fields of humanistic philosophy, ethics, and axiology. I shall limit myself to referring to a few of them which are connected with the subject matter of this essay, showing a particular dimension of the intellectual life of their author, according to principles of certain logical order that delineate the creative, uncommon activity of a man of science.[4]

A. Grzegorczyk writes about the exceptionality of human life in the course of history: "[...] each life in each epoch can be called a time of trial". We read the words in the Preface to his book *Filozofia czasu próby* (*The Philosophy of the Time of Trial*) [1979],[5] which – following a few failed attempts to have it published in Poland – came out in Paris in 1979, already

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[3] The characteristics of this activity are included in the work by S. Krajewski and J. Woleński [2008].

[4] I omit in this way, in particular, the well-known achievements of A. Grzegorczyk in the field of mathematical logic.

[5] The motto of this essay comes from the last sentence of the book. The quotes of Andrzej Grzegorczyk's utterances will be a free translation from the Polish language.

in the time of declining communism. The work contains considerations that are a bold answer to contemporary worldwide and human problems related to the value and sense of life, and – simultaneously – makes an appeal to every man, recommending to him to form certain moral attitudes and respect certain principles of each individual life, based on one's own effort and honesty, criticism, respect for every man, and propagation of truth. By advocating the indeterminism of human fate, the author opposes Marxist ideology, and – as a follower of the *non-violence movement* – all forms of violence and tactics for subordinating society. This is connected with a display of some encumbrance of life in general, and life as a fight for justice and a new, better tomorrow. Stressing the significance of rational thinking and the cognitive role of science in the rational search for, among others, solving conflicts and displaying the world of human values, A. Grzegorczyk points also to the role of science to serve the whole of society. From people of science, one can expect a proper reaction to problems of the world. Shaping life attitudes in the form in which life has been offered to us, with all the duties which it imposes, with all hardships and misfortunes it brings, is – as the author writes – a basic quality that characterizes man's attitude towards the world, which we will count on. He expresses his personal attitude towards fate in the following manner,

> The philosophical image of man's fate and the essence of humanity can be thus perceived as a certain personal call to realize the ideal of humanity. Man, contrary to animals, is a creative being, able to actively change the conditions of his existence and his own lifestyle. Events of life can be perceived not only as something that touches us and what we suffer from, but also as something that sets a goal and in this way invests our existence with a sense, as something that constitutes a call and a challenge to make an effort and to fight. Indeed, the greatness of man consists in the fact that he is able to creatively react to his own fate. The metaphysical basis of the creativity is human metaphysical freedom. [...] Man always has the possibility of making a choice. [...] As we are granted this mysterious freedom of decision, it seems proper to treat it as a chance, a call for investing our lives with a deeper sense, a certain value. [Grzegorczyk, 1979, p. 128]

Further on, we can read, "[...] as long as we are ready to fully accept our existence, if we accept reality as it is given to us, [...] then we will always find a good number of tasks," [Grzegorczyk, 1979, p. 133] "A hard life, full of tasks, becomes a trial and a judgment to us" [Grzegorczyk, 1979, p. 134]. Professor Grzegorczyk calls this attitude of acceptance of one's own existence as "humility towards reality". "The world is given to us so that in the sea of man's toil and pain one could find his own difficult task for himself, which – however – does not alter the architecture of this world" [Grzegorczyk, 1979, p. 140]. Assuming the attitude of acceptance of reality and existence means also acceptance of another man like oneself, therefore devoid of any elements of dominance, violence, subordination, exploitation, but holding another man in respect (even in a conflict situation), as well as respecting the existential value of the latter, being concerned for the moral good.

A. Grzegorczyk presents also a globalist vision of social reality and humanity

> which gains a minimal number of conditions necessary for a compatible survival through a few successive hundreds of years, humanity saved from threatening cataclysms and driven to the state of stability as regards all the elements of life which fill with concern [Grzegorczyk, 1979, p. 175].

Such a vision means a transformation of the world and human life through recognizing extended spiritual needs, influencing people's spiritual experiences in compliance with the principles of justice and equality, respect for human and nations' rights through

> abiding by determined forms of life, conscientiousness, dutifulness, not neglecting even the small elements of the order that compose the general order, [Grzegorczyk, 1979, pp. 180–181]

and also spreading free-from-violence (non-violence)

> culture of persuasion, understanding, and – if a need arose – even co-suffering, patience and coordination, [Grzegorczyk, 1979, p. 183]

as well as anticipating possible conflicts in order to avoid potential hazards. In this "small utopia" (as the author called it), there is a place for preserving individual national character and guaranteeing a compatible co-existence between nations, which consists in mutual helping one another; the conciliatory policy allows avoiding misunderstandings and conflicts. Attaining the above-presented vision is to be possible through common education of techniques of coordinating and anticipating possible conflicts. A. Grzegorczyk underlines here the importance of the relevant rearing of youth, alterations in the educational system and the significance of a rational effort connected with the spread of knowledge and global consciousness; these factors would lead to realization of the humanistic concept of transforming the social reality and humanity, which is outlined here. A rational effort for the good of humanity is connected with new tasks permitting us to meet the challenge posed to humanity. A. Grzegorczyk calls this challenge *a challenge to a new* *moral attitude*. It can be formulated in the form of the following sentence [Grzegorczyk, 1979, p. 34]:

> Let us share our potentials of survival; let every nation take account of the desire for survival of other nations.

A positive reaction to this challenge offers also a chance of one's own survival that is neither more nor less vital than that of any human being; this is in compliance with the Christian principle of love for others.

> It is not the fight for existence, but emphasizing an all-human moral sense of uniting with each human being, [Grzegorczyk, 1979, p. 236]

the sense of all-human solidarity, concern for everybody, can eliminate the situation in which humanity finds itself – catastrophic, full of threats and adversity. It is not existence that invests life with its sense, A. Grzegorczyk writes in another place, but evidence of brotherhood [Grzegorczyk, 1979, p. 227]. A feature of our existence is *fairness in survival*. This feature of life is defined by its quality, its realization of moral values. Possibilities of choosing a path of life and overcoming unexpected circumstances are two versions of the *moral trial of life*; the other one is *a test of life*, our readiness to serve given ideals.

The course of the sociopolitical events in Poland and Europe at the turn of the 1980s and the 1990s allowed Professor Grzegorczyk, in his new book bearing the meaningful title *Życie jako wyzwanie* (*Life as a Challenge*), [1995] to update, deepen, develop, and logically systematize the concept of a vision of the human world and human life which he had presented 15 years earlier, by investing it with the direction of *rationalism open to values* towards current problems of life. A logical, penetrating justification of the rationalistic condition and European Rationalism is contained, in particular, in another book by A. Grzegorczyk under the title *Logic – a Human Affair* [1997], which was published some time later[6] than *Life as a Challenge*. The book published in Polish, Ukrainian and Russian, is designed for a wide circle of readers and performs – apart from its scientific function – also an educational one. It is aimed, among other things, at raising society's moral level and eliminating certain negative ethical attitudes popularly accepted. The book provides not only theoretical knowledge relating to the very world of values itself, by pointing to their oppositions, but also the

[6] The book includes the idea that the desired form of our rationalism should be **rationalism open to spiritual values**.

practical knowledge necessary to realize all-human values such as: *Respect for everybody, Justice, Kindness*. Realization of these values, constituting here the basic element of life, is an action which contributes to forming valuable, precious, spiritual human experience. A. Grzegorczyk developed his concept of vision of the human world here with the aid of a creative construction of notions that serve the purpose of an intellectual search for a new realization of accepted values and collective behaviours, with close abiding by requirements of logic, precision of systematization, and clarity of presentation. The proposition is a creative concept for shaping spiritual values, as superior to those vital ones, and shows a peculiar gift of the author, sensitivity to spiritual values, and richness of theoretical and practical knowledge in various domains of science. It is characterized by the attitude of rendering services to others. The author follows, at the same time, a specific logic of toil, which he describes as overcoming different difficulties that are sometimes hard to foresee, an inner discipline and psychic effort. Here are a few of the author's thoughts:

> If somebody intends to pass a value to somebody else and this message requires toiling, then resignation from this toil can be comfortable for the doer, yet it is harder to interpret it as a service rendered to others,

and further,

> Effective serving others usually demands making a serious effort at attention and concentration to others' needs. [...] Experiencing the toil of human action can be regarded as a reflection in the sphere of psyche of a certain fundamental feature of human existence. [Grzegorczyk, 1997, p. 135]

Thus, realization of the virtues acknowledged to be valuable requires toil. For a collective life, one that is not suppressed (*non-violated*) within the structure of a state (a state as a structure with a higher spiritual degree of organization) the good of its citizens is valuable. Without toil, intellectual effort and support from intellectual elites, true social good cannot arise. With reference to the pathology of social thinking (especially in relation to German Nazism and Soviet Communism) the sentence written on the cover of the book under discussion attracts attention. It reads:

> Attaining a structure that realizes spiritual values and sustaining it require a spiritual activity and effort, without which there follows deviation or disintegration.

A. Grzegorczyk means here, in particular, deviations of political totalitarian structures, sick and unjust, as well as an effort connected with realization of universal spiritual values, an effort of perseverance and an uncompromising attitude in whatever is valuable.

> Good cannot arise without an effort. Good depends on a highly organized social matter. This organization must simply be created by someone. Without an effort there follows merely a disintegration of highly organized structures,

A. Grzegorczyk [1997, p. 191] asserts very firmly on degeneration of the structure – as he adds in another place. It is toil that is connected with the realization of spiritual values, a creative intellectual effort, full of sacrifice, and morally appropriate spiritual activity, as well as the physical effort and risk of suffering related with this activity. Not undertaking this effort in order to realize values, and producing subjectively accepted effects instead of this realization, leads to cognitive deviation, mendacity, and – in consequence – to deviation of acting. If the deviations penetrate the system of governing, they become dangerous to free intellectual thinking and to the creative activity of citizens. Opposing such a situation is always a *challenge* to contemporaries. Rational opposing of such a situation can be aided by a well organized civic debate and propagation of logical culture, therefore an attitude of criticism towards disseminated ideas, on the basis of a reliable observation and analysis of facts, logical argumentation and *respect for others* – a spiritual value which ought to be realized straight after satisfying basic vital needs, and which – in the opinion of the author of the book – serves the purpose of conciliation.

The proposition of conceptualization of the world and human life, which is being discussed here, as a certain concept of *rationalism open to values*, sets the direction towards looking at human life, especially the author's own life, from the perspective of the subject matter of this essay. The individual character of the axiological vision of the world and human life is reflected in the very title of the book by A. Grzegorczyk. He believes in the attainability of the proposed vision and summons us to realize it, and this as early as on the cover of the book, where we can read the following words:

> Let us contribute to that everybody should experience precious states spiritually: learning the truth, respect, justice and understanding shown towards others, and also acceptance of human fate and belief in its sense.

A significant element of the challenge which life brings along is to A. Grzegorczyk – at the same time – meeting every man,

> [...] so as to invest the contact with this man with a certain vital sense, [...] I accept that being put on my way, he poses a challenge to me to create, just with him, a certain new quality,

as we can read in the final part of the book. Professor Grzegorczyk, through his indefatigable intellectual creativity, proves somehow the attainability (at least to some extent) of his vision of repair-oriented change of the world. In the light of the transformations which took place at the end of the previous century and at the beginning of the present one, A. Grzegorczyk draws attention to the new challenges of contemporary times. He continues his considerations on the challenges and social problems in other publications, too. In the paper entitled "Czasy i wyzwania" [2002/2003] (Times and challenges), as a keen and penetrating observer of changes connected with, on the one hand, new techniques, and on the other, new forms of human activity, he also perceives new threats in the countries of this part of Europe and – primarily – in Poland itself. He draws attention to deviational actions in achievements of civilization, intensified development of consumption oriented attitudes and also exempting oneself from the inner discipline of truth and effort for the benefit of shaping universal values. Analyzing well known cases of violence (occurring not only in the past) and the experience of history, he concludes that "the created or established *tools* in which we often place our trust, do not lead to social good on their own, but require *long-lasting effort, consciousness, constant control from the viewpoint of values*" [Grzegorczyk, 2002/2003, p. 10]. Broadly understood, violence is the subject matter of challenges for the contemporary young generation, the young intelligentsia. The moral challenge posed to the young intelligence of the 21$^{\text{th}}$ century is one to cross over what in the Marxist vision was called determinism, and thus a challenge towards non-determining our will.

In another work, A. Grzegorczyk presents certain guidelines concerning the challenges of globalization, commonwealth of humanity and development of dispositions towards forming it, beginning with small communities within the framework of larger ones, not resigning from the fight for common education of moral values, a fight for a commonwealth with moral principles.[7] A. Grzegorczyk ties the concern for repairing humanity to underlining the role of logic and philosophy in educating begun on the lowest levels. He opposes all forms of freeing the human mind from the correct logical thinking. In the paper "Naprawianie świata. Pożytki filozofii" [2010] (Mending of the world. The advantages of philosophy), he writes:

> There is a need, in the present condition of mankind, to rehabilitate the natural mind which is sincerely searching for the truth about the whole of our human fate.

[7] I mean here the paper "Globalizacja i jej wyzwania" [2009] (Globalization and its challenges).

He highlights also that

> Thanks to words, grammar, and logic, the human being is able to deliver general convictions which make the basis of feeling the sense of life.

We owe the gift of logical thinking and broadening of knowledge to the development, perfecting, and using of language, which is a special gift, and drawing from which invests our existence with a sense. The role of logic consists in, among others, fulfilling by language the cognitive-communicative function. Human speech is a tool to realize the project of all-human solidarity, the perseverant realization of accord and cooperation. Challenges of the last decade have been connected with the growing consumerism-oriented lifestyle initiated by the progress of civilization and desisting from making an intellectual effort to the advantage of pleasure and comfort of living. These challenges are also in opposition to the 'logic' of fight and hatred. For the further intellectual development of each human commonwealth, not only the European one, it is of paramount importance to apply correct argumentation: logical argumentation for the benefit of truth in discourse and public debate. "In the times of commercialization, affecting also the intellectual life, one should defend the basic conditions of truth," A. Grzegorczyk writes in his work "Dekalog rozumu" (The Decalogue of the mind). He formulates in it 10 norms relating to the culture of social debate.[8] Respecting them is the fundamental basis of the culture of each real intellectual discussion, whose aim is to strive for truth or – at least – for working out a unanimous standpoint on a given issue. They expose critical aspects of social life and make a peculiar challenge posed to people responsible for the life of a given community or pretending to a dominance of the commonwealth. They refer also to people displaying the adaptive group characteristic based on the current economic situation. A. Grzegorczyk's deontological 'commandments' play a peculiar educative function, and this not only with reference to the young generation. They teach respect for other people's beliefs, respect for those who think differently. Professor Grzegorczyk, in his coursebook of logic[9] and in some texts available to the author of this essay, formulates certain suggestions concerning the shaping intellectual and moral attitudes of human beings, as well as further challenges for the human condition. I am going to quote, highlight or summarize a few utterances or texts by A. Grzegorczyk.

[8] The work is included in the yearbook of the PTU [Grzegorczyk, 2006/2007].

[9] The course-book has not been published yet.

> Knowledge means that we are capable of presenting our arguments in a clear and precise manner, and that we can understand others' arguments when they are clearly and precisely laid out; this knowledge offers all of us a hope for a more agreeable life, for removing conflicts and eliminating quarrels on the basis on mutual recognition of basic values [...] In order that justification of one's views should play this socially useful role within mankind, the culture of justifying and the very searching for reasonable justifications and care for their quality must be propagated within the framework of regular school-based education of next generations.

A substantial argument must be supported not only by knowledge in the fields which it covers, but should rest on a foundation provided by logic – the basis of the structure of knowledge. Logic – in the opinion of A. Grzegorczyk – should be placed in the very centre of man's life as the *Basic intellectual discipline of steering the whole of one's life*. This view of A. Grzegorczyk is exposed already in the title of his earlier book *Logic – a Human Affair*,[10] in which he opts for turning the philosophy of logic towards a certain kind of psychologism. The very logic itself is conceived in it as the most general ontology; the laws of logic are about the world, providing knowledge about the world described here in a reistic style. Knowledge is an ingredient of the formed human ability to adapt, and adaptation of human individuals which is understood in a broader way, within the philosophical perspective, means – as A. Grzegorczyk concludes – *Adapting of our abilities to conduct ourselves to CONDITIONS AND CHALLENGES OF THE WHOLE HUMAN CONDITION*. Logic is, at the same time, one of the significant trends of the cultural development of mankind, and contributes – as far as Professor Grzegorczyk sees it – to

> Perfecting the skill of language-based describing of the fragment of the reality which is being studied; that is, **perfecting collective knowledge formed and consolidated by means of the language**.

This logic contributes to the enrichment of knowledge and intellectual development of world populations.

> The whole of deductive formal logic makes a natural stage in the development of human intellectualism and is a transition from spontaneous steps of the natural development of cognition to conscious creation of methods of conduct which imitate and consciously perfect the same steps that the nature of our life keeps offering us.

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[10] Op. cit.

Logic orders simple and trivial steps of thought and imagination, and allows achieving an intellectual construction, whose sense, and significance, as well as applications, unexpectedly step beyond triviality – adds A. Grzegorczyk. Logic is, at first sight, a mine of simple obvious tautological truths, yet – in the full system of logic – out of tautological truths and with the help of obvious procedures of proof, one can obtain very intricate theorems which are far from being obvious. This happens so on the basis – A. Grzegorczyk claims – of the fact that man is capable – by means of simple tools of thought – of building very complicated constructions. So as to obtain tautological truths, which are – at the same time – not that obvious and which render something significant (though they derive from obvious truths), one needs to repeat the procedures and arrange them in a very complicated and revealing manner.[11] The applied logic makes an important way of utilizing the possessed cognitive, hence practical abilities, since – as A. Grzegorczyk states – logic offers a certain useful set of tools of reasoning, that is:

> Logic is a tool for enlarging our cognition of the world **by means of thinking and knowledge already in our possession**.

Logic has played and still does its service-rendering role as a tool applied in different disciplines of knowledge. Today – as Professor Grzegorczyk underlines – almost the whole of collective communication is based on direct transfer and information technology, whose foundation is a description of situations established by basic laws of logic relating to conjunction, alternation, and negation. He states that

> Today, one can say that the whole civilization of the world is taking part in an experiment of ordering logical behaviours which regulate our conscious conduct.

[11] I would like to observe that Andrzej Grzegorczyk, through his psychologistic approach to semantics, applying formal-logical tools, solved in his *Logic – a Human Affair* the problem of semantic paradoxes and proposed the system of universal formal syntax, on the ground of which he gave proof of the Adequacy Theorem for the classical conception of truth (the proof of this theorem was given in Tarski's famous work on the notion of truth, which was translated into many languages). N.b. Grzegorczyk, in his essay *Prawdziwość cecha ważna, łatwa do określenia, trudniejsza do osiągnięcia* (Truthfulness – an important feature, easy to define, more difficult to attain) published in *Felieton filozoficzny*, most likely in 2010 (to which, unfortunately, I have had no access) somehow 'removes the spell' from Tarski's definition of truth, showing that it does not go beyond the triviality of Aristotle's explication.

> The thought can develop towards different directions. **Logic takes care solely of its correctness**. Everybody marks out the direction of their mental searching for themselves in dependence on their own interests. [...] Nevertheless, some substantial coherence of the thought being developed can count into logical values of human knowledge.

There is a need, then, for a logical mental discipline as well. The knowledge alone of principles of thinking does not suffice to manage thought and good activity. There is a need for toil, an effort directed towards the development of intellect, and logical thinking so that they should have a proper effect, an influence on the whole of the life of an individual and a community. The whole effort aimed at understanding the surrounding reality, inquisitiveness of the truth of the world – the road to reach the truth – does not run along commonly accepted paths, but it requires the toil of moving step after step towards discovering the construction of the World. From the above cited quotations or summaries of Professor A. Grzegorczyk's works which have been presented during the few recent months or expressed on different occasions, there emerges a certain general view advanced by the Professor, which I can formulate below, using his own words that I found in a letter sent to me:

> In the present situation created by the civilization of the species of homo sapiens it becomes indispensable to have a very rigorous logical discipline of thought, without which the whole social life would fall into a complete chaos.

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New times are bringing along still newer and newer technical possibilities, new forms of activity, but also new threats to the intellectual life of each of us, to the life of the community in which we live, and even to the whole of mankind. Hence, there are new tasks and new challenges posed to us, especially to the young intelligentsia. A young man, the young generation, generally, needs thus a master, a mentor, a guide, somebody who is wiser than we are, somebody who is able to create a relevant intellectual atmosphere and – at the same time – encourage us to ponder over philosophical anthropology. Scientific knowledge and knowledge about life, its sense, about morality, are inseparable components of the intellectual development of man and a good life. Competent passing of fundamental values of life, logical and righteous conduct to new generations, educating, wise and beautiful reference to the ideals of *Paidea*, they all make a peculiar challenge of life – worthy, filled with creative intellectual toil – the life of Andrzej Grzegorczyk.

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Urszula Wybraniec-Skardowska
Group of Logic, Language and Information
Opole University
Matejki 5/21, 45-055 Opole, Poland
uws@uni.opole.pl